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WHAT IS THE TYPICAL NILPOTENT LIE ALGEBRA?

Eugene M. Luks

1. Introduction

We must remark at the outset that we shall not attempt an exact definition of 'typical' much less a precise answer to the title question. Rather we shall describe some computer assisted studies of what may be thought of as 'random samples' of nilpotent Lie algebras. The motivation for these efforts was evidence that some counterexamples should, if they exist at all, be quite abundant. In fact, it is suggested that the use of the computer searches would have facilitated the answers to questions on the existence of a number of interesting classes of algebras. Actually, the answers, where they had already been found, had required laborious hand computations and considerable time.

We discuss nilpotent Lie algebras (Section 4) and, separately, the subclass of metabelian Lie algebras (Section 3). In each instance we describe a procedure which, in some sense, corresponds to the choice of a random algebra. Since there does not appear to be a natural probability measure to assign to these classes, we are guided somewhat by intuition in establishing the sampling process. We neither insist nor desire that all algebras be chosen with equal likelihood, only that, theoretically, none are specifically excluded. For this purpose, we describe certain subvarieties of the variety Lie(X) of Lie multiplications on a vector space X (see, for example, [10]). These subvarieties cut across all relevant isomorphism classes. However, unlike the full variety, their structure is suggestive of a selection procedure.

It is no surprise, in each instance, that the procedure turns out to be prejudiced in favor of the points in Lie(X) which give rise to the smallest derivation algebras, for having many independent derivations is an algebraic condition on the coordinates of the point. Indeed, this observation inspired both the nilpotent and metabelian studies since, in each case, we wanted to uncover algebras with 'few' derivations.

2. Notations and Preliminaries

For a Lie algebra, L, we denote by Der(L) the derivation algebra of L. If L has integral structure constants we often consider the algebra, L_p, obtained by passing to the finite field Z_p. Since the rank of the system of linear equations satisfied by derivations cannot increase upon passage modulo p we have the elementary but quite useful

Proposition 2.1. $\dim(\mathrm{Der}(L)) \le \dim(\mathrm{Der}(L_p))$.

Typically, we need to compute bases of derivation algebras and so we restrict our attention to algebras with integral structure constants. However, we still face computational difficulties since the integral row reduction procedures are very slow and they may lead to integer overflows on the computer. If we pass to Z_p these problems are eliminated. Fortunately, it is often the case that we can already find $\dim(\mathrm{Der}(L_p))$ independent derivations of L, either by inspection of L itself or by pulling back elements of $\mathrm{Der}(L_p)$. In such instances the proposition assures that these derivations are a basis of Der(L).

All vector spaces and algebras are assumed to be finite dimensional. Unless otherwise indicated $V \oplus W$, for subspaces V, W of an algebra, shall denote the direct sum as vector spaces.

3. Metabelian Lie Algebras

3.1 Computing Derivations

In this section we describe a particularly effective algorithm for computing the derivations of metabelian Lie algebras. It makes strong use of the known structure of the derivation algebras and allows one to deal with larger algebras than would be otherwise tractable.

If L is a metabelian Lie algebra (i.e., $L^3 = 0$) and U is any linear complement in L of the derived algebra, then $L = U \oplus V$ where $V = L^2 = U^2$ and the multiplication in L is determined by the map $\mu: U \wedge U \to V$ such that $\mu(u_1 \wedge u_2) = [u_1, u_2]$. There is an injection

$$\mathrm{Hom}(U,V) \xrightarrow{i} \mathrm{Der}(L)$$

in which $i(f)(u) = f(u)$ for $u \in U$ and $i(f)(V) = 0$. The image of i is an abelian ideal in Der(L) over which Der(L) splits. Namely

$$\mathrm{Der}(L) = \mathrm{Der}_U(L) \oplus i(\mathrm{Hom}(U,V)) \tag{3.1}$$

in which we have denoted by $\mathrm{Der}_U(L)$ the subalgebra of derivations which stabilize U. Thus we only need to compute the elements of Hom(U,U) which extend to derivations of L. For this purpose, we fix bases $\{u_1, u_2, \ldots, u_m\}$, $\{v_1, v_2, \ldots, v_n\}$ of U,V, respectively. The multiplication μ is represented by n skew-symetric $m \times m$ matrices, $A^k = (A^k_{ij})$, $k = 1,2,\ldots,n$, where

$$\mu(u_i \wedge u_j) = \sum_k A^k_{ij} v_k.$$

We let $\mathcal{A}$ denote the linear span of $A^1, A^2, \ldots, A^n$. Let $\tau \in \mathrm{Hom}(U,U)$ and suppose T is the matrix of τ relative to $\{u_i\}$. Then as noted in [6, Proposition 4.7], τ extends to a derivation of L if and only if $\mathcal{A}$ is stabilized by the map

$$X \to TX + XT^t. \tag{3.2}$$

The next step is to translate into some computable form the condition that a subspace is stabilized. One procedure is to choose a reasonable complement to $\mathcal{A}$ within the space, $\mathcal{Sk}$, of all $m \times m$ skew-symmetric matrices and express the *linear* condition that $TA^i + A^iT^t$ have no component within that complement. For this purpose, we use the Killing form $\langle , \rangle$ of the natural representation of $gl(m)$, i.e., $\langle X,Y \rangle = tr(XY)$. We assume henceforth that the characteristic of the base field is different from 2. This assures, in particular, that the restriction of $\langle , \rangle$ to $\mathcal{Sk}$ is non-degenerate. Let $\mathcal{B}$ denote the orthogonal complement of $\mathcal{A}$ in $\mathcal{Sk}$ relative to $\langle , \rangle$. Then, for $A \in \mathcal{A}$, $TA + AT^t \in \mathcal{A}$ if and only if $\langle TA + AT^t, \mathcal{B} \rangle = 0$. However, for $A,B \in \mathcal{Sk}$, $\langle TA + AT^t, B \rangle = tr(TAB) + tr(AT^tB) = tr(TAB) + tr(BTA) = 2tr(TAB) = 2\langle T,AB \rangle$. Hence $\mathcal{A}$ is stabilized by the map (3.2) if and only if $\langle T,AB \rangle = 0$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$. We have

<u>Proposition 3.1</u> Suppose L, U, $\mathcal{A}$, $\mathcal{B}=\mathcal{A}^{\perp} \cap \mathcal{Sk}$ are as above. Then $(\mathcal{AB})^{\perp}$ is a subalgebra of $gl(m)$ isomorphic to $Der_U(L)$ (where $\perp$ denotes the orthogonal complement in $gl(m)$ relative to $\langle , \rangle$).

Thus, given structure constants (A^k_{ij}) of the metabelian Lie algebra L, the computation of Der(L) involves two steps:

(I) Compute a basis $B^1, B^2, \ldots, B^{\binom{m}{2}-n}$ of $\mathcal{B}$. This involves solving a system of n homogeneous linear equations in $\binom{m}{2}$ unknowns (the coefficients being, e.g., the above-diagonal entries of the A^i).

(II) Compute a basis of $(\text{span}\,\{A^iB^j\})^{\perp}$. This involves solving $n(\binom{m}{2}-n)$ equations in m^2 unknowns.

Note that a direct computation of Der(L) would require solving a system of $\binom{m+n}{2}(m+n)$ equations in $(m+n)^2$ variables. Even if one restricts one's attention to $Der_U(L)$, the derivations stabilizing U and V, direct computation involves solving a system with m^2+n^2 unknowns.

3.2 The 'Typical' Metabelian Lie Algebras

Before indicating the method for drawing an element at random from the urn of metabelian Lie algebras, it is useful to discuss what we were hoping to find therein.

Suppose $L = U \oplus V$, $\mathcal{A}$, $\mathcal{B}$ are as in Section 3.1. Since tr(IAB = tr(AB) = 0 for $A \in \mathcal{A}$, $B \in \mathcal{B}$, $(\mathcal{AB})^{\perp}$ always contains the identity matrix, I. From this, or directly, one sees that the identity map of U extends to a derivation of L. In [6] and [7] the author and G. Leger considered the class of metabelian Lie algebras in which this derivation actually spans $\mathrm{Der}_U(L)$. We shall say that $L = U \oplus V$ is of type $\mathcal{I}(m,n)$, where $m = \dim(U)$, $n = \dim(V)$, (or, simply, of type $\mathcal{I}$ if it is not necessary to explicate m,n) if $\dim(\mathrm{Der}_U(L)) = 1$. One of the reasons that such algebras appeared to be of interest is that their holomorphs seem to forget the multiplication of the algebra. Recall that the holomorph, Hol(L), of a Lie algebra, L, is the semi-direct sum Der(L) + L. It is shown in [6] that Hol(L) for L of type $\mathcal{I}$ takes a remarkably simple form. In particular,

Proposition 2.2 (Leger-Luks [6, Section 4]). If L_1 and L_2 are both of type $\mathcal{I}(m,n)$ then $\mathrm{Hol}(L_1) \cong \mathrm{Hol}(L_2)$.

This proposition provided the key to the resolution of the question of whether nilpotent Lie algebras are determined by their holomorphs. The problem had been suggested in [11] where it was shown that the free nilpotent algebras are so determined. However, examples are exhibited in [7] of two metabelian Lie algebras over Q of type $\mathcal{I}(6,4)$ which are non-isomorphic even under extensions of the base field. We shall return to this point in Section 3.3.

Initially, instances of metabelian Lie algebras of type $\mathcal{I}$ did not seem to be easy to find. In retrospect, the problem was that, because of the difficulties of hand computation, we tended to try algebras whose structure constants were too simple and these did not exhibit the right properties. In fact, it now appears that truly arbitrary choices of algebras would invariably uncover examples of type $\mathcal{I}$. To elaborate, we consider what may be called the varieties of metabelian Lie multiplications.

We assume vector spaces U, V are fixed and set $L = U \oplus V$. Denote by $\mathrm{Meta}(U,V)$ the variety of skew multiplications μ on L for which $\mu(L,L) \subseteq V$ and $\mu(V,L) = 0$. For such μ, the corresponding algebra, L_μ is a metabelian Lie algebra and $L_\mu^2 = U^2 \subseteq V^2$ (we do not insist the inclusion be an equality). When the spaces do not require specification, we denote $\mathrm{Meta}(U,V)$ by $\mathrm{Meta}(m,n)$ where $m = \dim(U)$, $n = \dim(V)$. Note that $\mathrm{Meta}(U,V)$ may be identified with $\mathrm{Hom}(U \wedge U, V)$ and so $\mathrm{Meta}(m,n)$ is an affine space of dimension $\binom{m}{2}n$.

By equation (3.1), if $\mu \in \mathrm{Meta}(m,n)$ then L_μ is not of type $\mathcal{I}(m,n)$ if and only if the dimension of $\mathrm{Der}(L_\mu)$ exceeds $1 + \binom{m}{2}n$. Since that is an algebraic condition on μ, we have

<u>Proposition 3.3.</u> Suppose the base field is infinite. If the class of algebras of type $\mathcal{I}(m,n)$ is non-empty then $\{\mu \in \mathrm{Meta}(m,n) \mid L_\mu$ is of type $\mathcal{I}\}$ is Zariski-open and dense in $\mathrm{Meta}(m,n)$.

The proposition suggests, in effect, that if algebras of type $\mathcal{I}$ exist, then they are almost all that exist. Thus some random sampling of elements of $\mathrm{Meta}(U,V)$ ought to either reveal their existence or provide strong evidence for their non-existence. The computer implementation of this admittedly imprecise notion was irresistible.

Now, Proposition 3.3 also guarantees the existence and density of rationally defined algebras of type $\mathcal{I}$ if they ever appear in characteristic 0. Hence we may, with confidence, restrict our attention to algebras with integral structure constants A_{ij}^k (see Section 3.1). Of course, it is not immediately apparent how to select random integers. Furthermore, we know that if they get too large, or even if they are small but m and n are large, the row-reduction might "blow up". With this in mind, we agree in advance that we shall be passing to Z_p after generating the algebra and hope that the remarks of Section 2 can be used to determine Der(L). Indeed Proposition 2.1 and equation (3.1) yield

Proposition 3.4. Suppose L is a metabelian Lie algebra with integral structure constants. Then, if L_p is of type I, so is L.

One expects further, that if p is very large, L_p is likely to be of type I when L is. Thus, although Z_p is finite and so the probability of success each time is no longer 1, if algebras of type I exist then we still ought to find them. Hence, we *pick* our "random" A_{ij}^k from the urn: $\{0,1,\dots p-1\}$.

This procedure was used to search for examples employing several primes up to 46337. (That limit was imposed to guarantee that integral computations such as rs + t would not result in numbers exceeding the single precision ceiling of 2^{31}). Before reporting the result we remark that we already knew:

Proposition 2.5 [6, Theorem 4.9]. There are no algebras of type $I(m,n)$ unless

$$m = 5 \quad \text{and} \quad 4 \le n \le 6$$

or

$$m \ge 6 \quad \text{and} \quad 3 \le n \le \binom{m}{2}-3.$$

The computer search has suggested:

Algebras of type $I(m,n)$ *do exist for* (m,n) *as above with the exception of* (6,3) *and* (6,12). *(Values of* m *up to* 15 *were tried).*

We remark that, although we have not succeeded in *proving* that algebras of type $I(6,3)$ and $I(6,12)$ do not exist, the computer has given us some confidence in the conjecture.

3.3 The Abundance of Nonisomorphic Metabelian Lie Algebras With Isomorphic Holomorphs

It is worth noting that the very existence of algebras of type $\bar{I}(m,n)$ could have sufficed to establish the existence of nonisomorphic metabelian Lie algebras with isomorphic holomorphs. For, by Propositions 3.2, 3.3, if metabelian Lie algebras were determined by their holomorphs then the isomorphism class of any algebra of type $\bar{I}(m,n)$ is dense in Meta(m,n). However, one sees that, for most U,V, the isomorphism class of any $\mu \in \mathrm{Meta}(U,V) \equiv \mathrm{Hom}(U \wedge U,V)$, i.e., the orbit of μ under the action of

$$G = GL(U) \times GL(V)$$

cannot get that big. We assume, for the moment, that the base field is algebraically closed and of characteristic 0. Then, since the stability group of G corresponding to μ (equivalently $\{\sigma \in \mathrm{Aut}(L_\mu) \mid \sigma(U) = U\}$) has dimension $= \dim(\mathrm{Der}_U(L))$,

<u>Proposition 3.6.</u> The dimension of the orbit of μ is

$$(\dim U)^2 + (\dim V)^2 - \dim(\mathrm{Der}_U(L)).$$

We remark that his proposition was suggested by results in [4, especially Section 7] where an equivalent formulation of varieties of metabelian Lie algebras is presented.

The proposition implies, in particular, that the isomorphism class of any $\mu \in \mathrm{Meta}(m,n)$ lies in a subvariety of dimension $\leq m^2 + n^2 - 1$. Thus, by Proposition 3.3,

<u>Corollary 3.7.</u> Suppose $m^2 + n^2 - 1 < \binom{m}{2}n$. Then, if there exist algebras of type $\bar{I}(m,n)$, there exist infinitely many nonisomorphic algebras in Meta(m,n) with the same holomorph.

Now it is further noted in [4, Theorem 7.8] that, for the values of (m,n) given in our Proposition 3.5, the inequality of the corollary holds except when $(m,n) = (5,4)$ or $(5,6)$. One interpretation, then, of this discussion is that: For many values of (m,n), if *two* elements of Meta(m,n) are chosen at random then

(i) With probability 1, they are nonisomorphic.

(ii) With probability 1, their holomorphs *are* isomorphic.

In other words, holomorphs are particularly *ineffective* invariants for metabelian Lie algebras.

We remark, finally, that in the exceptional cases (5,4) and (5,6) algebras of type $\mathcal{I}(m,n)$ do exist. Two examples of algebras of type $\mathcal{I}(5,4)$ were given in [6, Section 4] which we originally had hoped were nonisomorphic (they *are* nonisomorphic over Q). It is interesting that, although we have been unable to exhibit the actual isomorphism, the fact that both have dense orbits implies that they *are* necessarily isomorphic over C.

3.4 An Answer to a Question on the Existence of Dense Orbits

We conclude the discussion of metabelian Lie algebras with a computer-inspired answer to a question of Gauger (see [4, remark (1) on page 326]).

In the setting of the present paper, the question is whether the existence of an element μ whose orbit under $GL(U) \times GL(V)$ is dense in Meta(U,V) implies that the total number of orbits (i.e., isomorphism classes) is finite. A search for a counter-example was inspired by the result [4, Theorem 7.10] that there are an infinite number of isomorphism classes of algebras in Meta(m,2) for $m \geq 8$. Can one, nevertheless, be dense?

Noting Proposition 3.6, we search for an element of Meta(U,V) = Meta(m,2) with $\dim(\mathrm{Der}_U(L)) = m^2 + 2^2 - \binom{m}{2}2 = m + 4$. Again, a "random" choice ought to find such an element, if it exists. For $m = 8$, the search was futile, but for $m = 9$, the computer found one every time. With this assurance, we went on to look for one with 'simple' structure constants. For example, the following $\mu \in \mathrm{Hom}(U \wedge U, V)$ works:

$$\mu(u_1 \wedge u_2) = \mu(u_3 \wedge u_4) = \mu(u_5 \wedge u_6) = \mu(u_7 \wedge u_8) = v_1,$$

$$\mu(u_1 \wedge u_9) = \mu(u_2 \wedge u_3) = \mu(u_4 \wedge u_7) = \mu(u_6 \wedge u_8) = v_2$$

with $\mu(u_i \wedge u_j) = 0$ for $i < j$, otherwise.

4. Nilpotent Lie Algebras

4.1 Generating 'Random' Algebras

We indicate, in this section, the procedure for generation of nilpotent Lie algebras. Recall first that a nilpotent Lie algebra, L, of dimension n has a central composition series

$$L = V_0 \supset V_1 \supset \dots \supset V_n = 0$$

of ideals V_i, i.e., $\dim V_i = n-i$ and $[V_i, L] \subset V_{i+1}$. Then L/V_{i+1} is a central extension of L/V_i by a one-dimensional ideal and so it is determined by L/V_i and an element of $Z^2(L/V_i)$, that is, a 2-cocyle with trivial coefficients.

With the above in mind, we suppose now that

$$\mathcal{V}: 0 = V_n \subset V_{n-1} \subset \dots \subset V_0$$

is a fixed tower of vector spaces with $\dim V_i = n - i$. Set $L = V_0$ and denote by $\mathrm{Nilp}(\mathcal{V})$ the variety of Lie multiplications μ on L for which $\mu(V_i, L) \subset V_{i+1}$. Our selection of an arbitrary point in $\mathrm{Nilp}(\mathcal{V})$ is accomplished by choosing, for each i, a multiplication on L/V_{i+1} which extends that previously determined on L/V_i. These multiplications are determined by successive choices of 2-cocycles. Thus, at each stage, we choose a 'random' solution to the system of linear homogeneous equations determined by the cocycle condition. This is done by row-reducing the system and assigning random values to the free variables. (When working over the integers, it may be necessary to mutiply the solution by an appropriate least common denominator to clear the fractions in the bound variables).

Note that the selection, by chance, of a coboundary would result in $L/V_{i+1} = (x) \oplus L/V_i$ (algebra direct sum) and we do not specifically exclude the possiblity. However, it is known that, for N nilpotent of dimension ≥ 2, $H^2(N) \neq 0$ (see e.g., [1]) and so, if the base field is infinite, the selection of coboundaries ought to occur with probability 0. Indeed, except in the case of small finite fields, coboundaries were never chosen, with the result that the nilpotent L always had two generators. To study, realistically, the nilpotent algebras with

$$\dim(L/L^2) = r > 2$$

one may specify the first r cocycle selections to be 0.

We further comment that there are undoubtedly many other important classes of algebras against which our procedure is biased. For example, in dimensions ≥ 7 we never seemed to find algebras of maximal nilpotency index, dim L. However, even though such multiplications form an open set in Nilp(V), the result is not very surprising for there is evidence that these lie in components of relatively small dimensions (see [12, Introduction]).

Remark. We observe that the generation procedure can be generalized to select super-solvable algebras, i.e., those having composition series with one-dimensional quotients. The only difference is that one does not assume the successive extensions are central. Thus, given $K = L/V_i$, we first choose a random one-dimensional representation, ρ, of K, that is, an element of $(K/K^2)^*$, and then a random element of $Z^2(K,\rho)$.

4.2 Characteristically Nilpotent Lie Algebras

As in the metabelian case, we had some idea of the properties to expect in the typical nilpotent algebra although the situation is not quite so clear cut.

Now in [5], Jacobson showed that over a field of characteristic 0, if a Lie algebra has a nonsingular derivation then it is necessarily nilpotent. He then asked whether, in fact, every nilpotent Lie algebra actually has a nonsingular derivation. Quite to the contrary, it was found

Proposition 4.1 (Dixmier-Lister [2]). There exists an 8-dimensional nilpotent Lie algebra with only nilpotent derivations.

Dixmier and Lister called algebras with this property *characteristically nilpotent* and they remarked that the study of these "might prove more tractable" than that of the larger class of general nilpotent Lie algebras. The object of our computer search was to see how large the subclass was. In fact, because the typical algebra should have the fewest derivations we expected to get characteristically nilpotent algebras very often, at least in dimensions ≥ 7. (Morosov's classification [9] demonstrates that no characteristically nilpotent Lie algebras exist in characteristic 0 through dimension 6. On the other hand there is one of dimension 7 [3]). To be sure having *few* derivations is not the same as having only nilpotent derivations but there is evidence of some correlation between the two properties. For example, any linear transformation of L mapping L to $L^2 \cap \mathrm{Center}(L)$ and L^2 to 0 is a nilpotent derivation. Indeed, Dixmier and Lister showed that the outer derivation algebra of their example is already spanned by the images of such derivations. This inspires their remark that the algebra "has as few other derivations as possible." In a sense, the metabelian situation lends further evidence. Of course, metabelian Lie algebras cannot quite be characteristically nilpotent since, by virtue of the relation $L^3 = 0$, Der(L) inherits a derivation which induces the identity map on L/L^2. However, by Proposition 3.3, the 'typical' metabelian Lie algebra comes as close to characteristic nilpotence as it can get.

Thus motivated, we generated nilpotent Lie algebras according to the procedure in Section 4.1 and computed their derivation algebras. The size of the algebras was limited (we went to dimension 15) by the need to solve equations in $(\dim L)^2$ unknowns for the derivations. We found

The algebras of dimension ≥ 8 were always characteristically nilpotent.

(We comment on the dimension 7 case in Section 4.4).

There was no difficulty in verifying the nilpotence of all derivations since the computed basis of the derivation algebra was always strictly triangular with respect to the given basis (i.e., $\{x_1, x_2, \ldots, x_n\}$ where $\{x_{i+1}, \ldots, x_n\}$ is a basis of V_i) of L. For computational convenience, once again, most of the algebras were generated over large finite fields. However, it was reasonable, having confidence now in their abundance, to seek characteristic 0 examples on the computer. Although we had to take care now to pick only 'small' solutions to the cocycle equations, we often succeeded. We present here one of the computer generated examples: L has basis $\{x_1, x_2, \ldots, x_8\}$ and multiplication determined by

$$[x_1,x_2] = x_3 - x_4 + x_5 + x_7$$

$$[x_1,x_3] = x_4 + x_5 - x_6 - x_7 - 2x_8$$

$$[x_1,x_4] = -x_5 - x_6 + x_7$$

$$[x_1,x_5] = x_6 + x_8$$

$$[x_1,x_6] = x_8$$

$$[x_1,x_7] = -2x_8$$

$$[x_2,x_3] = -x_5 - 2x_6 + x_8$$

$$[x_2,x_4] = -3x_6 - 2x_7 + x_8$$

$$[x_2,x_5] = x_6 + x_7$$

$$[x_2,x_6] = -2x_8$$

$$[x_2,x_7] = 2x_8$$

$$[x_3,x_4] = x_6 + x_7 - 3x_8$$

$$[x_3,x_5] = x_8$$

with $[x_i,x_j] = 0$, for $i < j$, otherwise. To verify characteristic nilpotence, it was not necessary to solve the system of derivation equations (in 64 variables) over Z. We found that $\dim(\mathrm{Der}(L_5)) = 10$. It sufficed then to exhibit 10 independent and strictly triangular derivations of L. One sees easily that any linear transformation τ: $(x_1,x_2) \to (x_6,x_7,x_8)$ extends to a derivation, $\bar{\tau}$, of L with $\bar{\tau}(x_3) = [x_1,\tau(x_2)] - [x_2,\tau(x_1)]$ and $\bar{\tau}(L^3) = 0$ (Note that (x_6,x_7,x_8) is the transporter of L to Center(L)). The space of these is 6-dimensional and intersects the 7-dimensional space of inner derivations in a 3-dimensional subspace. Therefore $\dim(\mathrm{Der}(L)) = 10$ and L is characteristically nilpotent.

Note that, in one sense, the above 'typical' algebra does not have as 'few' outer derivations as the Dixmier-Lister example since the images of central derivations (i.e., derivations mapping to Center(L)) do not already span the outer derivation algebra. However, the Dixmier-Lister example has a derived algebra of codimension 4 and, as we already remarked, our random searches never produce such an example. This inspired a computer search amongst those algebras with $\dim(L) = 8$ and $\dim(L/L^2) = 4$; and this time the images of the central derivations did span the outer derivation algebras.

4.3 Characteristically Nilpotent Derived Algebras

Dixmier and Lister also showed that their example answered in the negative the stronger question of whether every nilpotent Lie algebra was a derived algebra of some Lie algebra. They then asked whether a characteristically nilpotent Lie algebra can ever be a derived algebra. With some effort we were able (see [8]) to construct by hand an 18-dimensional algebra with a 16-dimensional characteristically nilpotent derived algebra. However, it seemed natural recently to ask the computer the question. Again generating random examples, we found

> *The algebras of dimension* ≥ 13 *always had characteristically nilpotent derived algebras.*

Here too, it was possible to find examples in characteristic 0. In fact, the following algebra is an extension of the example in Section 4.2. The algebra, L, has basis $\{x_1, x_2, \ldots, x_{13}\}$ and multiplication

$$[x_1, x_2] = x_3 - x_4 + x_5 + x_7 - x_9 - x_{10} + 2x_{11} - x_{12} - x_{13}$$

$$[x_1, x_3] = x_4 + x_5 - x_6 - x_7 - 2x_8 + 2x_9 + 2x_{13}$$

$$[x_1, x_4] = -x_5 - x_6 + x_7 - x_9 + x_{10} - x_{11} - x_{12} + x_{13}$$

$$[x_1, x_5] = x_6 + x_8 + x_9 + x_{10} + x_{11} - 2x_{12} + 2x_{13}$$

$$[x_1, x_6] = x_8 + 5x_9 - 2x_{10} - 4x_{11} + 54x_{12} - 2860x_{13}$$

$$[x_1, x_7] = -2x_8 - 2x_9 - x_{10} + x_{11} - 2x_{12}$$

$$[x_1, x_8] = x_9 + x_{11} + x_{12} + x_{13}$$

$$[x_1, x_9] = x_{11} + 4x_{12} + 194x_{13}$$

$$[x_1, x_{10}] = 2x_{11} + 5x_{12} + 10x_{13}$$

$[x_1, x_{11}] = -5x_{12} - 5x_{13}$

$[x_1, x_{12}] = -15x_{13}$

$[x_2, x_3] = -x_5 - 2x_6 + x_8 + x_9 - x_{10} + x_{11} - 2x_{12} + x_{13}$

$[x_2, x_4] = -3x_6 - 2x_7 + x_8 + 2x_9 - 5x_{10} + 7x_{11} - 79x_{12} +$

$2441\, x_{13}$

$[x_2, x_5] = x_6 + x_7 - x_9 + x_{11} - 2x_{13}$

$[x_2, x_6] = -2x_8 + 5x_9 - 4x_{10} + 4x_{11} + 213x_{12} - 9780x_{13}$

$[x_2, x_7] = 2x_8 + x_{10} + 2x_{11} - x_{12} - 2x_{13}$

$[x_2, x_8] = -x_{10} + x_{11} + x_{12} - 2x_{13}$

$[x_2, x_9] = -x_{11} - 20x_{12} - 160x_{13}$

$[x_2, x_{10}] = -40x_{12} + 435x_{13}$

$[x_2, x_{11}] = 175x_{13}$

$[x_3, x_4] = x_6 + x_7 - 3x_8 - 3x_9 + 2x_{10} + 6x_{11} + 149x_{12} -$

$2170x_{13}$

$[x_3, x_5] = x_8 + x_9 + x_{10} - 3x_{11} - 55x_{12} + 5176x_{13}$

$[x_3, x_6] = -2x_9 + x_{10} - 51x_{12} + 3372x_{13}$

$[x_3, x_7] = 2x_9 - 2x_{10} - 13x_{12} - 1064x_{13}$

$[x_3, x_8] = -x_{11} + 10x_{12} - 95x_{13}$

$[x_3, x_9] = 5x_{12} + 55x_{13}$

$[x_3, x_{10}] = 250x_{13}$

$$[x_4,x_5] = 3x_9 - x_{10} + 5x_{11} + 76x_{12} - 2047x_{13}$$

$$[x_4,x_6] = x_{11} - 43x_{12} + 1847x_{13}$$

$$[x_4,x_7] = -4x_{11} + 48x_{12} - 67x_{13}$$

$$[x_4,x_8] = -125x_{13}$$

$$[x_4,x_9] = -75x_{13}$$

$$[x_5,x_6] = 5x_{12} - 1290x_{13}$$

$$[x_5,x_7] = -20x_{12} + 950x_{13}$$

$$[x_5,x_8] = -75x_{13}$$

$$[x_6,x_7] = 150x_{13}$$

with $[x_i,x_j] = 0$ for $i < j$, otherwise. Again, making use of Proposition 2.1, we determined that $\dim(\mathrm{Der}(L_7^2)) = 27$ and then managed to identify 27 independent, strictly triangular elements of $\mathrm{Der}(L^2)$.

4.4 The Nilpotent Lie Algebras of Dimension 7

The computer never found a characteristically nilpotent Lie algebra of dimension 7 even though G. Favre [3] had shown that one exists. However, like Favre's example the generated algebras did have derivation algebras of dimension 10. One suspects then that there are "characteristically nilpotent points" within the open set of multiplications yielding minimal derivation algebras but such points do not themselves form an open set. Furthermore, this suggests that it ought to be possible to deform the characteristically nilpotent example to non-characteristically nilpotent ones. Indeed, we find the following family, L_t, of nilpotent Lie algebras on the 7-dimension space with basis $\{x_1, x_2, \ldots, x_7\}$: The multiplication in L_t is given by

$$[x_1, x_i] = x_{i+1} \qquad \text{for } 2 \le i \le 6$$

$$[x_3, x_4] = x_7$$

$$[x_3, x_2] = x_6 + tx_5$$

$$[x_4, x_2] = x_7 + tx_6$$

$$[x_5, x_2] = (t+1)\, x_7$$

$$[x_i, x_j] = 0 \qquad \text{for } i + j > 7.$$

The algebra L_0 is Favre's example which is characteristically nilpotent. However, in general, L_t has the derivation δ_t where

$$\delta_t(x_1) = 4t^2 x_1 + 2x_2$$

$$\delta_t(x_2) = 8t^2 x_2 + (6t+2)x_3 - (5t+1)x_4$$

$$\delta_t(x_3) = 12t^2 x_3 + (6t+2)x_4 - (5t+1)x_5$$

$$\delta_t(x_4) = 16t^2 x_4 + (4t+2)x_5 - (5t+3)x_6$$

$$\delta_t(x_5) = 20t^2 x_5 + (2t+2)x_6 - (5t+5)x_7$$

$$\delta_t(x_6) = 24t^2 x_6$$

$$\delta_t(x_7) = 28t^2 x_7.$$

Thus L_t is *not* characteristically nilpotent for $t \neq 0$.

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Department of Mathematics
Bucknell University
Lewisburg, Pennsylvania 17837