Name: ________________________________

CIS 624, Fall 2014, Final Examination
9 December 2014
Please do not turn the page until everyone is ready.

Rules:
• The exam is closed-book, notes are allowed.

• Please stop promptly at 16:45.

• You can rip apart the pages, but please write your name on each page.

• There are 165 points total, distributed unevenly among 7 questions. A perfect score is 100 points, any extra points over 100 will be added to your midterm score (if it was less than 100).

• Most questions have multiple parts. You will receive points for any parts you complete.

• You are not expected to complete all questions.

Advice:
• Read questions carefully. Understand a question before you start writing.

• Write down thoughts and intermediate steps so you can get partial credit.

• The questions are not necessarily in order of difficulty. Skip questions you are not confident about and if you have time, come back to them later. Remember you only need 100 points total.

• If you have questions, ask.

• Relax. You are here to learn.

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Simply Typed Lambda Calculus with pairs

\[ s ::= \text{skip} | x := e | s | \text{if } e \text{ s } | \text{while } e \text{ s} \]
\[ e ::= c | x | e + e | e \ast e \]
\((c \in \{\ldots, -2, -1, 0, 1, 2, \ldots\})\)
\((x \in \{x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots, \ldots\})\)

\[ H \vdash e \downarrow c \]

\[ \begin{array}{llll}
\text{CONST} & \text{VAR} & \text{ADD} & \text{SEQ1} \\
H \downarrow c & H \downarrow H(x) & H \vdash e_1 \downarrow c_1 & H \vdash e_2 \downarrow c_2 \\
\text{MULT} & H \vdash e_1 \downarrow c_1 & H \vdash e_2 \downarrow c_2 & H \vdash e \downarrow c_1 + c_2 \\
& & H \vdash e_1 \downarrow c_1 & H \vdash e \downarrow c_1 \cdot c_2 \\
& & & H \vdash e \downarrow c_1 \cdot c_2 \\
\end{array} \]

\[ H_1 ; s_1 \rightarrow H_2 ; s_2 \]

\[ \begin{array}{lll}
\text{ASSIGN} & \text{IF1} & \text{IF2} \\
H \vdash e \downarrow c & \text{if } e \downarrow s_1 s_2 \rightarrow H ; s_1 & \text{if } e \downarrow s_1 s_2 \rightarrow H ; s_2 \\
H ; x := e \rightarrow H, x \rightarrow c ; s & H ; \text{skip} ; s \rightarrow H ; s & H ; \text{skip} ; s \rightarrow H ; s \\
\end{array} \]

\[ \text{WHILE} \]
\[ H ; \text{while } e \rightarrow H ; \text{ if } e (s ; \text{while } e \rightarrow s) \text{ skip} \]

Simply Typed Lambda Calculus with pairs

\[ e ::= \lambda x. e | x | e | e | c | (e, e) | e.1 | e.2 \]
\[ v ::= \lambda x. e | c | (v, v) \]
\[ \tau ::= \text{int} | \tau \rightarrow \tau | \tau \cdot \tau \]

\[ e \rightarrow e' \text{ and } \Gamma \vdash e : \tau \text{ and } \tau_1 \leq \tau_2 \]

\[ \begin{array}{llllll}
\lambda x. e \rightarrow e[v/x] & e_1 \rightarrow e_1' & e_2 \rightarrow e_2' & e_1 \rightarrow e_1' & e_2 \rightarrow e_2' \\
e_1 e_2 \rightarrow e_1' e_2 & v e_2 \rightarrow v e_2' & (e_1, e_2) \rightarrow (e_1', e_2) & (v_1, e_2) \rightarrow (v_1, e_2') \\
\end{array} \]

\[ \begin{array}{llllll}
\frac{e \rightarrow e'}{e.1 \rightarrow e'.1} & \frac{e \rightarrow e'}{e.2 \rightarrow e'.2} & (v_1, v_2).1 \rightarrow v_1 & (v_1, v_2).2 \rightarrow v_2 \\
\end{array} \]

\[ \begin{array}{llllll}
\Gamma \vdash c : \text{int} & \Gamma \vdash x : \Gamma(x) & \Gamma, x : \tau_1 \vdash e : \tau_2 & \Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2 & \Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 & \Gamma \vdash e_2 : \tau_2 \\
\end{array} \]

\[ \begin{array}{llllll}
\frac{\tau_3 \leq \tau_1}{\tau_1 \rightarrow \tau_2 \leq \tau_3 \rightarrow \tau_4} & \frac{\tau \leq \tau}{(\text{S-Refl})} & \frac{\tau_1 \leq \tau_2}{\tau_2 \leq \tau_3 \leq \tau_4} & \frac{\tau_1 \leq \tau_3}{\tau_1 \leq \tau_3} & \frac{\tau_1 \leq \tau_3}{(\text{S-Trans})} \\
\end{array} \]
• If \( \cdot \vdash e : \tau \) and \( e \rightarrow e' \), then \( \cdot \vdash e' : \tau \).
• If \( \cdot \vdash e : \tau \), then \( e \) is a value or there exists an \( e' \) such that \( e \rightarrow e' \).
• If \( \Gamma, x : \tau' \vdash e : \tau \) and \( \Gamma \vdash e' : \tau' \), then \( \Gamma \vdash e[e'/x] : \tau \).

**System F (syntax)**

\[
\begin{align*}
e &::= e | x | \lambda x. e | e e | \Lambda \alpha. e | e[\tau] \\
\tau &::= \text{int} | \tau \rightarrow \tau | \alpha \mid \forall \alpha. \tau \\
v &::= e | \lambda x. e | \Lambda \alpha. e
\end{align*}
\]

\[
\Gamma ::= \cdot \mid \Gamma, x : \tau
\]

\[
\Delta ::= \cdot \mid \Delta, \alpha
\]

**System F: \( e \rightarrow e' \) and \( \Delta; \Gamma \vdash e : \tau \)**

\[
\begin{array}{cccc}
\frac{e \rightarrow e'}{\Delta; \Gamma \vdash e \rightarrow e'} & \frac{e \rightarrow e'}{\Delta; \Gamma \vdash e \rightarrow e'} & \frac{e \rightarrow e'}{\Delta; \Gamma \vdash e \rightarrow e'} & \frac{(\lambda x : \tau. e)v \rightarrow e[v/x]}{\Delta; \Gamma \vdash \tau \rightarrow \tau} & \frac{(\Lambda \alpha. e)[\tau] \rightarrow e[\tau/\alpha]}{\Delta; \Gamma \vdash \tau \rightarrow \tau}
\end{array}
\]

\[
\begin{array}{c}
\Delta; \Gamma \vdash x : \Gamma(x) \\
\Delta; \Gamma \vdash e : \text{int} \\
\Delta; \Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \\
\Delta; \Gamma \vdash e_2 : \tau_1 \\
\Delta; \Gamma \vdash e \rightarrow \forall \alpha. \tau_1 \\
\end{array}
\]

Simple System F examples: Let \( \text{id} = \Lambda \alpha. \lambda x. \alpha. x \). Then \( \text{id} \) has type \( \forall \alpha. \alpha \rightarrow \alpha \); \( \text{id} \) has type \( \text{int} \rightarrow \text{int} \); and \( \text{id} [\text{int} * \text{int}] \) has type \( (\text{int} * \text{int}) \rightarrow (\text{int} * \text{int}) \).

**Sum types, iso-recursive types**

\[
\begin{align*}
e &::= \ldots | A(e) \mid B(e) \mid (\text{match } e \text{ with } A.x. e \mid B.x. e) \mid \text{fold}_\tau e \mid \text{unfold } e \\
\tau &::= \ldots | \tau_1 + \tau_2 \mid \mu \alpha. \tau \\
v &::= \ldots | A(v) \mid B(v) \mid \text{fold}_\tau v
\end{align*}
\]

**match A(v) with A.x. e_1 | B.y. e_2 \rightarrow e_1[v/x]**

\[
\begin{array}{c}
\frac{e \rightarrow e'}{A(e) \rightarrow A(e')} \\
\frac{e \rightarrow e'}{B(e) \rightarrow B(e')}
\end{array}
\]

**match B(v) with A.x. e_1 | B.y. e_2 \rightarrow e_2[v/y]**

\[
\begin{array}{c}
\frac{e \rightarrow e'}{e \rightarrow e'} \\
\frac{e \rightarrow e'}{e \rightarrow e'} \\
\frac{e \rightarrow e'}{e \rightarrow e'} \\
\frac{e \rightarrow e'}{\text{fold}_\mu \alpha. \tau e \rightarrow \text{fold}_\mu \alpha. \tau e'} \\
\frac{e \rightarrow e'}{\text{unfold } e \rightarrow \text{unfold } e'}
\end{array}
\]

\[
\begin{array}{c}
\Delta; \Gamma \vdash e : \tau_1 + \tau_2 \\
\Delta; \Gamma \vdash x : \tau_1 \vdash e_1 : \tau \\
\Delta; \Gamma \vdash y : \tau_2 \vdash e_2 : \tau \\
\Delta; \Gamma \vdash \text{match } e \text{ with } A.x. e_1 | B.y. e_2 \vdash \text{match } e' \text{ with } A.x. e_1 | B.y. e_2
\end{array}
\]

\[
\begin{array}{c}
\Delta; \Gamma \vdash e : \tau_1 \\
\Delta; \Gamma \vdash A(e) : \tau_1 + \tau_2 \\
\Delta; \Gamma \vdash B(e) : \tau_1 + \tau_2 \\
\Delta; \Gamma \vdash \text{fold}_\mu \alpha. \tau e : \mu \alpha. \tau
\end{array}
\]

\[
\begin{array}{c}
\Delta; \Gamma \vdash e : \mu \alpha. \tau \\
\Delta; \Gamma \vdash \text{unfold } e : \tau[(\mu \alpha. \tau)/\alpha]
\end{array}
\]
1. **(12 points)** For each of the following OCaml definitions, does it type-check in OCaml? If so, what type does it have? If not, why not?

(a) let a = 3 in (fun f -> (fun x y -> x) (f a) (f true))

(b) let b = (fun f -> (fun x y -> x) (f 1) (f (f (f 5)))))

(c) let c = (fun x y z -> x y z) (fun p q -> p * q) 5 10

(d) let d = (fun f -> (fun x y -> y) (f 3) (f (-10)))

**Solution:**

(a) Does not type-check: The type-inferencer will conclude that `g` must be a function takes an `int` and a function that takes a `bool`, and these cannot both hold.

(b) Type-checks: `(int -> int) -> int`

(c) Type-checks: `int`

(d) Type-checks: `(int -> 'a) -> 'a`
2. (30 points) We want to extend IMP (defined on p. 2) with case conditional of the form

```
case e of
c1 : s1;
c2 : s2;
...;
cn : sn
dcase
```

where case, of, and encase are new keywords; e is an arithmetic expression, each \( c_i \) is an integer constant, and each \( s_i \) is a statement. This program is executed by first evaluating the expression e to obtain a constant \( c \); if the first occurrence of \( c \) in the list \( c_1, \ldots, c_n \) is \( c_i \) (duplicates are allowed in the list \( c_1, \ldots, c_n \)), then the statement \( s_i \) is executed. If \( c \) does not occur in the list \( c_1, \ldots, c_n \), then the program immediately terminates (i.e., is equivalent to skip).

(a) (10 points) Give a BNF definition of the syntax of case conditionals by extending the current IMP definition of statements, s. It can be helpful to (optionally) use a separate metavariable CaseList for the list of cases between of and endcase.

```
CaseList ::= c : s | c : s; CaseList
s ::= skip | x := e | s ; s | if e s s | while e s
```

(b) (10 points) Give small-step operational semantics for case statements (you should have at least two new rules).
(c) (10 points) What is the value of $x$ at the end of the program below? Show a correct sequence of steps, specifying the small-step judgement rule(s) used in each step.

```plaintext
x := 3;
case 2 * x of
    3: x := -1;
    6: x := x + (-1);
    5: x := x + 1;
    6: x := 0
endcase
```

Solution:

(a)  

```
CaseList ::= c : s | c : s ; CaseList
s ::= skip | x := e | s ; s | if e s s | while e s
     | case e of CaseList endcase
```

(b)  

```
CASE1
\[ H ; e \Downarrow c_i \]
\[ \rightarrow H ; \text{case } e \text{ of } c_1 : s_1; \cdots; c_i : s_i; \cdots; c_n : s_n \text{ endcase } \rightarrow H ; s_i \]

CASE2
\[ H ; e \Downarrow c \quad c \notin \{c_1, \cdots, c_n\} \]
\[ \rightarrow H ; \text{case } e \text{ of } c_1 : s_1; \cdots; c_i : s_i; \cdots; c_n : s_n \text{ endcase } \rightarrow H ; \text{skip} \]
```

(c)  

\[ H = \{\}; x := 3; \text{case } 2 * x \text{ of } 3 : x := -1; \ 6 : x := x - 1; \ 5 : x := x + 1; \ 6 : x := 0 \text{ endcase} \]
\[ \rightarrow H = x \rightarrow 3; \text{case } 2 * x \text{ of } 3 : x := -1; \ 6 : x := x - 1; \ 5 : x := x + 1; \ 6 : x := 0 \text{ endcase} \]
\[ [\text{Seq2, Assign}] \]
\[ \rightarrow H = x \rightarrow 3; x := x - 1 \ [\text{Case1}] \]
\[ \rightarrow H = x \rightarrow 3, x \rightarrow 2; \text{skip} \ [\text{Assign}] \]
3. (20 points) Define a list encoding using the simply-typed lambda calculus with functions, and integers as considered in class. A non-empty list can be represented as $\lambda s. s\ h\ t$ where $h$ and $t$ are the head and tail of the list.

You can (optionally) use the definition of booleans and pairs from lecture, or other helper expressions.

```
"true"    $\lambda x. \lambda y. x$
"false"   $\lambda x. \lambda y. y$
"mkpair"  $\lambda x. \lambda y. \lambda z. z\ x\ y$
"fst"     $\lambda p. p\ \lambda x. \lambda y. x$
"snd"     $\lambda p. p\ \lambda x. \lambda y. y$
```

Define the lambda functions for each of the following operations. You can use previously defined shortcut names (e.g., “mkpair”, “true”, etc.).

(a) Create an empty list:

(b) Check if a list is empty:

(c) Create a non-empty list containing the integers 1, 2, and 3 (you can use numbers directly, no need for Church encoding).

(d) Get the last element (tail) of the list containing 1, 2, and 3 (you can use numbers directly, no need for Church encoding).

Solution:
(a) “emptylst” = “mkpair” “false” “false” (each node represented by a pair whose first element is head of the list, and second element is the tail; “false” as the first element of a pair designates the empty list)

(b) “isempty” = “fst” = λp. p(λx. λy. x) = λz.λp. p(λx. λy. x)(λp. p(λx. λy. y)z)

(c) λh. λt. “mkpair” “true” (“mkpair” h t)

(d) “tail” = λz. “snd” (“snd” (“snd” z)) = λz.λp. p(λx. λy. y)(λp. p(λx. λy. y)z)
4. (26 points) This problem uses System F with pairs and extended with integer pairs and a new operation, pair subtraction. For example, with pair subtraction \((3, 4) - (1, 3)\) should result in \((2, 1)\). Note that the answers to all parts should be brief.

(a) Define a *large-step operational rule* for subtraction of expressions of the form \(e_1 - e_2\) where \(e_1\) and \(e_2\) can be reduced to values that are pairs of integer constants.

\[
\text{E-SUB} \\
\hline
\quad e_1 - e_2 \downarrow
\]

(b) Give the appropriate System F *typing* rule for subtraction of expressions of the form \(e_1 - e_2\) where the types of \(e_1\) and \(e_2\) are pairs of ints.

\[
\text{T-SUB} \\
\hline
\Delta; \Gamma \vdash e_1 - e_2 :
\]

(c) Consider a typing context where:

- There are no type variables in scope.
- \(x\) is the only term variable in scope and it has type \(\forall \alpha. \alpha \rightarrow \alpha\). **Correction:** The type should have been \(\forall \alpha. \alpha \rightarrow \alpha \rightarrow \alpha\); answers satisfying either type will be accepted without any point deduction.

i. What does \(\tau\) need to be for the program fragment

\[
x \ [\tau] \ (\lambda y: \text{int} \times \text{int}. \ \lambda z: \text{int} \times \text{int}. \ y - z) \ (10, 2) \ (5, 2)
\]

to typecheck? (Recall application — of types or terms — associates to the left.)

\(\tau\) is

ii. Given your choice for \(\tau\) above, what is the type of

\[
x \ [\tau] \ (\lambda y: \text{int} \times \text{int}. \ \lambda z: \text{int} \times \text{int}. \ y - z) \ (10, 2) \ (5, 2)
\]
(d) If \( v \) is an arbitrary value such that
\[
v[\tau] (\lambda y : \text{int} \times \text{int}. \lambda z : \text{int} \times \text{int}. y - z) (10, 2) (5, 2)
\]
type-checks (notice \( v \) is a \textit{value} and no longer polymorphic), then:

i. What type does \( v \) have? (Hint: it’s different from the answers to part c).

ii. What might the following expression evaluate to?
\[
v (\lambda y : \text{int} \times \text{int}. \lambda z : \text{int} \times \text{int}. y - z) (10, 2) (5, 2)
\]

Solution:

(a)
\[
\begin{align*}
  e_1 \downarrow (c_1, c_2) & \quad e_2 \downarrow (c_3, c_4) \\
  e_1 - e_2 \downarrow (c_1 - c_3, c_2 - c_4)
\end{align*}
\]

(b)
\[
\frac{\Delta; \Gamma \vdash e_1 : \text{int} \times \text{int} \quad \Delta; \Gamma \vdash e_2 : \text{int} \times \text{int}}{\Delta; \Gamma \vdash e_1 - e_2 : \text{int} \times \text{int}}
\]

(c) i. \( \tau \) must be \( \text{int} \times \text{int} \rightarrow \text{int} \times \text{int} \rightarrow \text{int} \times \text{int} \)
ii. \( \text{int} \times \text{int} \)

(d) i. \( (\text{int} \times \text{int} \rightarrow \text{int} \times \text{int} \rightarrow \text{int} \times \text{int}) \rightarrow (\text{int} \times \text{int} \rightarrow \tau_1) \) for any \( \tau_1 \).
ii. It could produce any value whatsoever.
5. (15 points)
Consider a typed λ-calculus with sum types, pair types, recursive types, unit, and int.

(a) Define a type $t_1$ for a binary tree of integers where:
- Each interior node has one integer and two children.
- Each leaf node has no data.
- Your type definition should have the form $\mu \alpha \cdot \cdots$.

(b) Give a type $t_2$ for a binary tree of integers where:
- Each node has one integer and two optional children (meaning each child may or may not be another binary tree).
- Your type definition should have the form $\mu \alpha \cdot \cdots$.

(c) Explain in English how there is exactly one value of type $t_1$ that cannot be translated to an equivalent value of type $t_2$.

Solution:

(a) $\mu \alpha. \text{unit} + (\text{int} \times \alpha \times \alpha)$
(b) $\mu \alpha. \text{int} \times (\text{unit} + \alpha) \times (\text{unit} + \alpha)$
(c) The empty tree can be represented with a value of type $t_1$ but not with $t_2$ because every $t_2$ has at least one int.
6. Continuation passing style in OCaml.

(a) (12 points) Assume that the `eqk`, `addk`, `timesk`, `divk` functions are defined as follows.

```ocaml
let eqk a b k = k (a = b);;
let addk a b k = k ( a + b );;
let subk a b k = k (a - b);;
let times a b k = k ( a * b );;
let divk a b k = k ( a / b );;
```

Using only the above functions, implement a CPS function `abcdk` that takes four integer arguments `a`, `b`, `c`, `d`, a regular continuation `k`, and an exception continuation `xk`, to compute the following integer expression: \( a \times (b + c)/d \). If `d` is 0, call the exception continuation `xk` and pass the offending value to it.

```ocaml
# let abcdk a b c d k xk = ...;;
val abcdk : int -> int -> int -> int -> (int -> 'a) -> (int -> 'a) -> 'a = <fun>
```
(b) (20 points) Consider the direct style function that given a list of integers, returns the sum of squares of all values.

```ml
let rec sumsquares l =
  match l with
  | [] -> 0
  | h::tl -> (h*h) + (sumsquares tl)
```

i. What is the type of `sumsquares` above?

ii. For a given call to `sumsquares` above, approximately how deep would the call-stack grow in terms of the function arguments?

iii. Write a version of `sumsquares` called `sumsquaresk` in continuation-passing style (i.e., it should take as arguments a list of integers and a continuation function):

```ml
let rec sumsquaresk l k = ...
```

You can assume that the following CPS functions are defined (you can assume only integer division is supported, e.g., `divk 5 2 (fun x->x)` returns 2).

```ml
open List;;
let eqk arg1 arg2 k = k (arg1 == arg2);;
let timesk arg1 arg2 k = k (arg1 * arg2);;
let divk arg1 arg2 k = k (arg1 / arg2);;
let hdk lst k = k (hd lst);;
let tlk lst k = k (tl lst);;
let plusk arg1 arg2 k = k (arg1 + arg2);;
```

(continues on next page)
iv. What is the type of the `sumsquaresk` function you wrote in part b.iii?

Solution:

(a) \( \% \ a * (b + c) / d \)

\[
\text{let abck} \ a \ b \ c \ d \ k \ xk = \\
\text{eqk} \ d \ 0 \\
(\text{fun} \ ex \rightarrow \text{if} \ ex \text{then} xk \ d \\
\text{else} \ \text{plusk} \ b \ c \\
(\text{fun} \ bc \rightarrow \timesk \ a \ bc \\
(\text{fun} \ abc \rightarrow \divk \ abc \ d \ k));;
\]

(b) Sum the squares of values in list.

   i. \( \text{int list} \rightarrow \text{int} \)
   
   ii. Its depth will be proportional to the length of the list \( l \).
   
   iii. \text{let rec} \ \text{sumsquaresk} \ l \ k = 

   \[
   \text{eqk} \ l \ [] \\
   (\text{fun} \ \text{empty} \rightarrow \text{if} \ \text{empty} \text{then} k \ 0 \\
   \text{else} \ \text{hdk} \ l \\
   (\text{fun} \ h \rightarrow \timesk \ h \ h \\
   (\text{fun} \ h2 \rightarrow \ \text{tlk} \ l \\
   (\text{fun} \ \text{ltail} \rightarrow \text{sumsquaresk} \ \text{ltail} \\
   (\text{fun} \ t \rightarrow \ \text{plusk} \ h2 \ t \ k))))));;
   \]

(* To test: *)

\[
\text{let} \ \text{print_int} \ i = \\
\text{print_string} \ (\text{string_of_int} \ i); \ \text{print_newline}();;
\]

\[
\text{sumsquaresk} \ [1;2;3] \ \text{print_int};;
\]

iv. \text{sumsquaresk} \ has \ type \ \text{int list} \rightarrow (\text{int list} \rightarrow \ 'a) \rightarrow \ 'a
7. (30 points) In this problem, we consider a call-by-value lambda-calculus with very basic support for profiling: In addition to computing a value, it computes how many times an expression of the form \texttt{count} \( e \) is evaluated. Here is the syntax and operational semantics:

\[ e ::= \lambda x. e \mid x \mid e \ e \mid c \mid \text{count} \ e \]

\[ c; e \rightarrow c'; e' \]

\[
\begin{align*}
  &c;(\lambda x. \ e) \ v \rightarrow c; e[v/x] \\
  &c; e_1 \rightarrow c'; e'_1 \\
  &c; e_2 \rightarrow c'; e'_2 \\
  &c; v \ e_2 \rightarrow c'; v \ e'_2 \\
  &c; \text{count} \ v \rightarrow c + 1; v \\
  &c; \text{count} \ e \rightarrow c'; \text{count} \ e' \\
\end{align*}
\]

Given a source program \( e \), our initial state is 0; \( e \) (i.e., the global count starts at 0). A program state \( c; e \) type-checks if \( e \) type-checks (i.e., the count can be any number).

(a) (5 points) Give a typing rule for \texttt{count} \( e \) that is sound and not unnecessarily restrictive.

\[
\text{T-Count} \quad \frac{}{\Gamma \vdash \text{count} \ e : \tau}
\]

(b) (10 points) State an appropriate Preservation Lemma for this language. Prove just the case(s) directly involving \texttt{count} \( e \) expressions.

(c) (10 points) State an appropriate Progress Lemma for this language. Prove just the case(s) directly involving \texttt{count} \( e \) expressions.

(d) (5 points) Give an example program that terminates in our language and would terminate if we changed function application to be call-by-name but under call-by-name it would produce a different resulting count\(^1\).

Solution:

(a)

\[
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{count} \ e : \tau}
\]

\(^1\)Recall that in the call-by-value parameter passing mechanism the expression argument to a function is evaluated before the function is applied, while in call-by-name, the expression argument to a function is substituted for all the occurrences of the formal parameter and the resulting expression is then evaluated normally.
(b) If $\cdot \vdash e : \tau$ and $c; e \rightarrow c'; e'$, then $\cdot \vdash e' : \tau$. We can prove this by induction on the derivation of $\cdot \vdash e : \tau$. In the case we're asked to prove, the bottom of the derivation looks like:

$$
\begin{array}{c}
\vdash e_0 : \tau \\
\vdash \text{count } e_0 : \tau
\end{array}
$$

There are two possible ways $c; \text{count } e_0$ can step to some $e'$. If $e_0$ is a value, then $e' = e_0$ and the assumed derivation's hypothesis $\cdot \vdash e_0 : \tau$ suffices. If $e_0$ is not a value, then $e' = \text{count } e'_0$ where $c; e_0 \rightarrow c'; e'_0$. So using $\cdot \vdash e_0 : \tau$ and induction, $\cdot \vdash e'_0 : \tau$, so we can derive $\cdot \vdash \text{count } e'_0 : \tau$.

(c) If $\cdot \vdash e : \tau$, then $e$ is a value or there exists an $e'$ and $c'$ such that $c; e \rightarrow c; e'$. In the case we're asked to prove the bottom of the derivation looks like:

$$
\begin{array}{c}
\vdash e_0 : \tau \\
\vdash \text{count } e_0 : \tau
\end{array}
$$

So using $\cdot \vdash e_0 : \tau$, by induction either $e_0$ is a value or $c; e_0 \rightarrow c'; e'_0$ for some $c'$ and $e'_0$. If $e_0$ is a value, then $c; \text{count } e_0 \rightarrow c + 1; e_0$. If $c; e_0 \rightarrow c'; e'_0$, then we can derive $c; \text{count } e_0 \rightarrow c'; \text{count } e'_0$.

(d) One of an infinite number of examples is $(\lambda x. 0)(\text{count } 0)$.