CIS 427/527 Assignment #4 - Due Thursday, February 14, 2008

Another Proof of the Completeness Theorem

The first three exercises provide a proof of completeness. We have broken this proof up into three parts to aid understanding and reduce complexity.

1. Prove the following Lemma A:

   Lemma A:
   $\phi_1, \phi_2, \ldots \phi_n \models \psi \quad \Rightarrow \quad \models \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi))\ldots))))$

2. Prove Lemma B:

   Lemma B:
   Consider a propositional formula ϕ with propositional variables $p_1, p_2, \ldots p_n$. Let T_ϕ be the truth table of ϕ, and let L be the line number of a row of this truth table.
   For all $1 \leq i \leq n$, define $\hat{p_i} = p_i$ if the entry in line L for p_i is 1, and $\hat{p_i} = \neg p_i$ if the entry in line L for p_i is 0.
   Then the following holds:
   $\hat{p_1}, \hat{p_2}, \ldots \hat{p_n} \vdash \phi$ if the truth value (final column) of line L is 1.
   $\hat{p_1}, \hat{p_2}, \ldots \hat{p_n} \vdash \neg\phi$ if the truth value (final column) of line L is 0.

3. Prove the Completeness Theorem for propositional logic:

   Completeness Theorem:
   $\Gamma \models \phi \quad \Rightarrow \quad \Gamma \vdash \phi$

First Order Logic

4. Consider the sentence

   $$\phi = \forall x \exists y \exists z (P(x,y) \land P(z,y) \land P(x,z) \to P(z,x)).$$

   Which of the following models satisfies ϕ?

   a The model $\mathcal{M}$ consists of the set of natural numbers with $P^{\mathcal{M}} = \{(m,n) \mid m < n\}$.
   b The model $\mathcal{M}'$ consists of the set of natural numbers with $P^{\mathcal{M}'} = \{(m, 2{*}m) \mid m \text{ is a natural number}\}$.
   c The model $\mathcal{M}''$ consists of the set of natural numbers with $P^{\mathcal{M}''} = \{(m,n) \mid m < (n+1)\}$.

5. Let ϕ be the formula

   $$\forall x \forall y \exists z (R(x,y) \to R(y,z))$$

   where R is a predicate symbol of two arguments.

   (a) let $A = \{a,b,c\}$ and $R^{\mathcal{M}} = \{(b,c),(b,b),(b,a)\}$. Do we have $\mathcal{M} \models \phi$?
   (b) let $A = \{a,b,c\}$ and $R^{\mathcal{M}'} = \{(b,c),(a,b),(c,b)\}$. Do we have $\mathcal{M}' \models \phi$?