CS 420/520 Automata Theory
Fall 2023

TAKE-HOME FINAL EXAM

due by 11:59pm, Thursday, December 7, 2023

INSTRUCTIONS Undergraduates should do 4 of the following questions, and graduate students should do 5.

1. Consider the PDA of figure 1 for $\{a^i b^i c^j | i, j \geq 1\}$. Use the construction of lemma 2.27 to construct a CFG directly from it.

2. **Pumping lemma:**

   (a) Show that $A = \{ \ w \in \{a, b, c\}^* |$ w has more a's than b's $\}$ is not regular.

   (b) Show that $B = \{ \ w \ | w \in \{a, b, c\}^*, n_a(w)/n_b(w) = n_c(w)\}$ is not context-free. (Here, $n_a(w)$ means the number of a's in w, similarly for $n_b(w)$ and $n_c(w)$.)

   (c) Show that $C = \{ \ a^{i^2} | i \geq 0\}$ is not context free

3. Convert the NFA of figure 2 to a DFA. The start state is q_0, the accepting set is $F = \{q_3\}$, and "epsilon" means ϵ.

4. Build some context free items:

   (a) Construct a PDA M (show diagram) such that

   $$L(M) = \{ \ ax^n b y^m c z^{2m} d x^n e \mid m, n \geq 0 \ \}.$$

   (b) Show a CFG G such that

   $$L(G) = \{ \ x^n \# y^m \mid 0 \leq 2m \leq n \leq 4m \ \}.$$

5. Consider the grammar G given by $S \rightarrow aSb | bSa | SS | \epsilon$. We want to show *very carefully* that $L(G) = A$ where $A = \{ \ w \in \{a, b\}^* \ | w$ contains an equal number of a's and b's$\}$.

   (a) Prove the following: **(Claim1)** if $w \in A$ and $w = axb$ or $w = bxa$, then $x \in A$.

   (b) Prove the following: **(Claim 2)** if $w \in A$ and $w = axa$ or $w = bxb$, then there are strings $y, z \in A$ such that $w = yz$.

   (c) Prove by induction on the length of w that if $w \in A$ then there is a derivation $S \overset{*}{\Rightarrow} w$.

   (d) Argue that if there is a derivation $S \overset{*}{\Rightarrow} w$, then w has an equal number of a's and b's.

6. Show that any infinite subset of MIN_{TM} is not recognizable. (MIN_{TM} is defined in chapter 6).

7. Show how to compute the descriptive complexity of strings $K(x)$ with an oracle for A_{TM}. Use that to give a function f that is computable with an oracle for A_{TM}, where for each n, $f(n)$ is an incompressible string of length n.

8. Give a Σ_k or Π_k characterization of the following problems

   (a) *(example)* $TOT_{TM} = \{ \langle M \rangle \mid L(M) = \Sigma^* \}$ is Π_2 since we can write $TOT_{TM} = \{ \langle M \rangle \mid \forall w \exists t \ \langle M, w, t \rangle \in B \}$, where $B = \{ \langle M, w, t \rangle \mid M \text{ accepts } w \text{ within } t \}$ is decidable.

   (b) $INF_{TM} = \{ \langle M \rangle \mid L(M) \text{ has an infinite number of strings} \}$

   (c) $COF_{TM} = \{ \langle M \rangle \mid$ the complement of *L(M)* has a finite number of strings $\}$

   (d) $E_{TM} = \{ \langle M \rangle \mid L(M) = \emptyset \}$

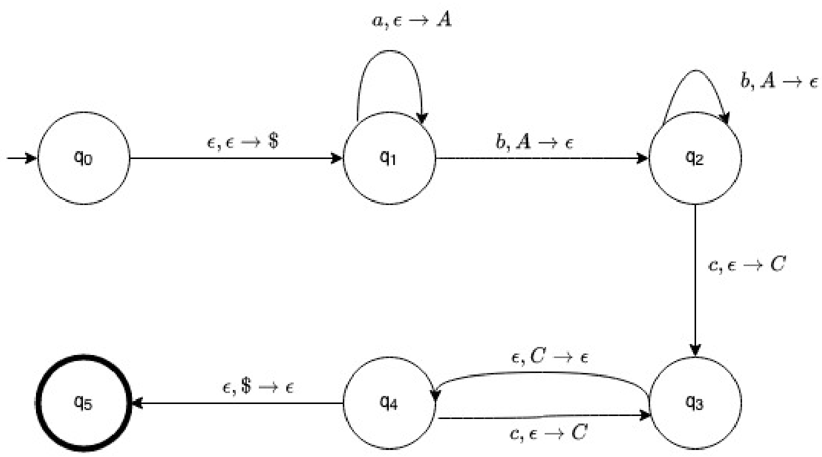

Figure 1: PDA for problem 1

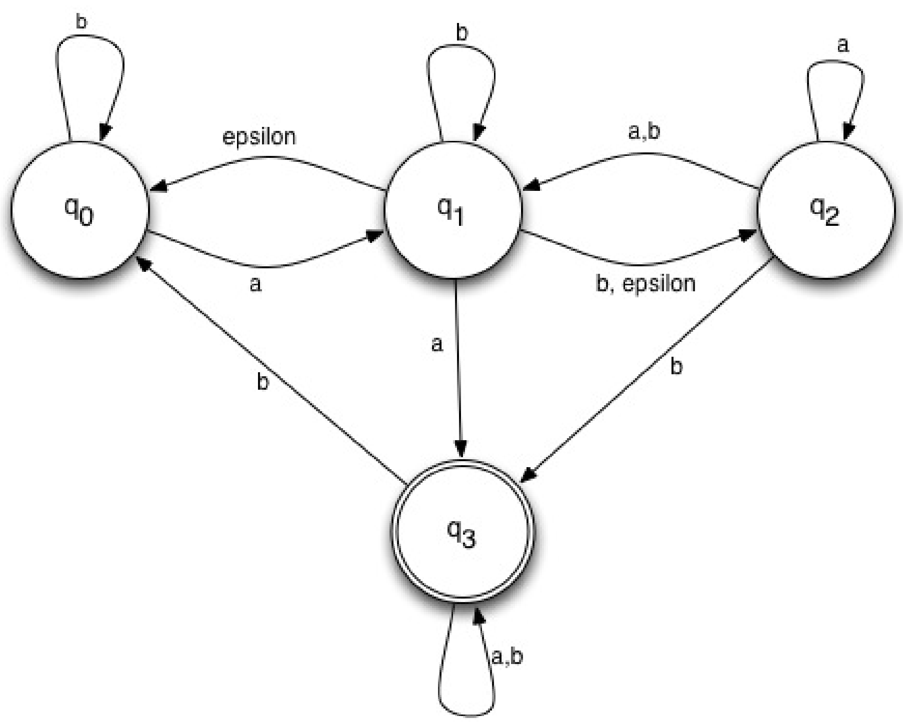

Figure 2: NFA for problem 3