CIS 420/520 Automata Theory

Lecture Notes for Reductions and Completeness

Alternate Terms

- *Turing-decidable*: decidable, recursive, rec
- *Turing-recognizable*: recognizable, semi-decidable, recursively enumerable, RE, re
- $\leq_T$: Turing-reducible, reducible
- $\leq_m$: mapping-reducible, many-one-reducible

Definitions

1. A set B is used as an *oracle* if we allow a computation to ask questions of the form "$y \in B$" as a basic step. Think of it as allowing the computation as having a sub-routine that determines membership in B.
2. A set A is B-recursive if A can be decided using a B-oracle. Similarly, A is B-re if it can be recognized using a B-oracle.
3. We say A *reduces* to B ($A \leq_T B$) if A is B-recursive.
4. A m-reduces to B ($A \leq_m B$) if there is a Turing-computable string function $f : \Sigma^* \to \Sigma^*$ such that for all strings $w \in \Sigma^*$ we have
$$w \in A \Leftrightarrow f(w) \in B$$
5. A set K is *RE-hard* if, for all recognizable sets A, $A \leq_m K$.
6. K is *RE-complete* if K is both recognizable and RE-hard. Alternate terms:
    - m-complete for RE
    - complete for the recognizable sets under $\leq_m$

"Obvious?" Facts

- Both $\leq_T$ and $\leq_m$ are reflexive ($A \leq_m A$) and transitive ($A \leq_m B$ and $B \leq_m C$ implies $A \leq_m C$)
- $A \leq_m B$ implies $A \leq_T B$
- $A \leq_T \bar{A}$

- $A \leq_m B$ iff $\bar{A} \leq_m \bar{B}$
- the following are equivalent: $A \leq_T B$, $A \leq_T \bar{B}$, $\bar{A} \leq_T B$, $\bar{A} \leq_T \bar{B}$
- A_{TM} is RE-complete (proof below)
- If
  - K is RE-complete,
  - $K \leq_m K'$, and
  - K' is RE

  then K' is RE-complete (follows from definition of complete and transitivity of $\leq_m$).
- If both A and $\bar{A}$ are RE, then A is recursive. (Theorem 4.22 of text says it better.)

More Facts, Definitions

A_{TM} is RE-complete
(proof) First, A_{TM} is RE, thanks to the existence of a universal TM $\mathcal{U}$. Now we have to show that any RE set A m-reduces to A_{TM}. Since A is RE, it is recognized by some TM M. By defining $f(w) = \langle M, w \rangle$ we get

$$w \in A \Leftrightarrow M \text{ accepts } w \Leftrightarrow f(w) = \langle M, w \rangle \in A_{TM}$$

$A_{TM} \leq_T HALT_{TM}$
(proof) pretty easy, done in text

$A_{TM} \leq_m HALT_{TM}$, so therefore $HALT_{TM}$ is also RE-complete
(proof) Also sort of easy, but subtle. We show a computable f such that on input $\langle M, w \rangle$, $f(\langle M, w \rangle) = \langle M', w \rangle$ so that

$$M \text{ accepts } w \Leftrightarrow M' \text{ halts on } w$$

What M' will do is simulate M on its input: M' halts if M accepts but it goes into ∞-loop if M rejects. (Obviously, if M loops then so will M'.) Notably, the constructor for f does not run M, instead it wraps the "code" for M with "code" that will simulate it and behave as described.

Closure properties for $\leq_m$ and $\leq_T$

1. The recursive sets are closed under $\leq_m$: if $A \leq_m B$ and B is recursive, then A is recursive.
2. The RE sets are closed under $\leq_m$: if $A \leq_m B$ and B is RE, then A is RE.
3. The recursive sets are closed under $\leq_T$: if $A \leq_T B$ and B is recursive, then A is recursive.
4. The RE sets are **not** closed under $\leq_T$: for example A_{TM} is RE and $\bar{A}_{TM} \leq_T A_{TM}$, but $\bar{A}_{TM}$ is not RE (if it were, then A_{TM} would be recursive).

Arithmetic Hierarchy

- $\Sigma_0 = \Delta_0 = \Delta_1 = \Pi_0 =$ recursive
- $\Sigma_1 =$ recursively-enumerable
- $\Pi_1 =$ co-RE $= \{ A \mid \bar{A} \in \Sigma_1 \}$
- in general $\Pi_k =$ co-$\Sigma_k = \{ A \mid \bar{A} \in \Sigma_k \}$
- $\Delta_{k+1} = \{ A \mid A$ is B-recursive for some $B \in \Sigma_k \} = \{ A \mid A \leq_T B$ for some $B \in \Sigma_k \}$
- $\Sigma_{k+1} = \{ A \mid A$ is B-RE for some $B \in \Sigma_k \}$

Basic AH Facts

- $\Delta_k = \Sigma_k \cap \Pi_k$
- $\Delta_k =$ co-Δ_k
- $\Delta_k \neq \Sigma_k$ (follows from ex 7, hw 7) , so $\Sigma_k \neq \Pi_k$

Post's Theorem

A is Σ_k if it can be characterized as

$$A = \{ x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q_k y_k \ \langle x, y_1, y_2, y_3, \ldots, y_k \rangle \in B \}$$

where B is decidable and quantifier $Q_k = \exists$ if k is odd and $\forall$ if it is even. (Here x and all the y_i's are strings over the same alphabet.) Similarly, A is Π_k if we can write

$$A = \{ x \mid \forall y_1 \exists y_2 \forall y_3 \ldots \overline{Q}_k y_k \ \langle x, y_1, y_2, y_3, \ldots, y_k \rangle \in B \}$$

where B is decidable and quantifier $\overline{Q}_k = \forall$ if k is odd and $\exists$ if it is even.

Friedberg-Muchnik Theorem

There are two RE sets A, B such that $A \not\leq_T B$ and $B \not\leq_T A$.