Minimum spanning trees

algorithms

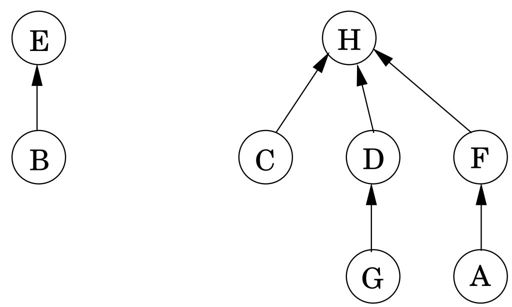
Kruskal's Method

```
1) A = ∅
2) for each v ∈V
3)       makeSet(v)
4) sort E by weight
5) for each (u,v) ∈ E
6)      if findSet(u) ≠ findSet(v)
7)           then  A = A ∪{(u,v)}
8)                 union(u, v)
9) return A
```

timing:
lines 2-3: O(V)
line 4: O(E lg E)   -- faster if small edge weights (counting sort)?
lines 5-8: E calls to 3 union-find operations, each O(lg*V) amortized
lines 5-8: total O(E lg* V)
overall total: O(E lgE)

aside: disjoint sets

Figure 5.5 A directed-tree representation of two sets $\{B, E\}$ and $\{A, C, D, F, G, H\}$.



from Dasgupta-Papadimitriou-Vazirani

union-find by rank with path compression

<u>procedure makeset</u>(x)
$\pi(x) = x$
rank$(x) = 0$

~~<u>function find</u>(x)~~
~~while $x \neq \pi(x)$: $x = \pi(x)$~~
~~return x~~

<u>function find</u>(x)
if $x \neq \pi(x)$: $\pi(x) = \texttt{find}(\pi(x))$
return $\pi(x)$

<u>procedure union</u>(x, y)
$r_x = \texttt{find}(x)$
$r_y = \texttt{find}(y)$
if $r_x = r_y$: return
if rank$(r_x) >$ rank(r_y):
    $\pi(r_y) = r_x$
else:
    $\pi(r_x) = r_y$
    if rank$(r_x) =$ rank(r_y) : rank$(r_y) =$ rank$(r_y) + 1$

Any sequence of m operations, n of which are makeset, takes time O(m lg*n)

- lg*n is minimum k such that lg lg lg lg n ≤ 1 (k iterations)
- actually better -- O(mα(n)) -- α(n) is inverse Ackermann function
- both lg*n and α(n) are very very slow growing, essentially constant

Prim's method

```
for each u ∈ V
    u.key = ∞
    u.prev = nil
r.key = 0                    -- start point

priority queue Q ← V     -- insert all of V into Q

while Q not empty
    u = Q.extractMin
    for each v ∈ adj[u]
        if v∈Q and W[u,v] < v.key
            then
                v.prev = u
                v.key = W[u,v]     -- use heap decreaseKey operation
```

time for Prim

- there is one buildHeap
- V extractMin operations
- E decreaseKey operations
- time using binary heap

$$O((V+E) \lg V)$$

- time using Fibonacci heap

$$O(V \lg V + E)$$

generic MST proof with loop invariant!

```
A = Ø
while A not yet spanning tree
      choose a safe edge (u,v) for A
      add (u,v) to A
```

Definition: Suppose A is a subset of a MST of the graph G. A **safe edge** for A is an edge (u,v) such that AU{(u,v)} is also a subset of a MST of G.

- so our algorithm is trivially correct (think about initialization, maintenance, and termination)
- still need to fill it out

safe edges and cuts

- Prim and Kruskal choose safe edges by means of cuts
- let G=(V,E) be the (weighted) graph, and let $A \subseteq E$ be a set of edges
- the idea is that A is a subset of a MST
- a cut that respects A is a proper subset of vertices $S \subset V$,…,so (S,V-S) partitions the vertices
- … and no edge of A is allowed to cross (S,V-S)

light edge

- a light edge for a cut (S,V-S) is a minimum weight edge crossing the cut
- main theorem: for any cut (S,V-S) respecting A, a light edge for the cut is safe for A
- both Prim and Kruskal pick light edges for some cut
- therefore, they are both correct

the dual to a cut is a cycle

```
input: graph G=(V,E), with weights

T=E
while T has a cycle
     pick a cycle C in T
     find a max weight edge (u,v) in T
     remove edge (u,v) from T
```

- does this work?
- can it be proved correct loop invariantly?
- efficiency?

the greedy algorithm

- red rule
  - Let C be a cycle with no red edges.
  - Select an uncolored edge of C of max cost and color it red
- blue rule
  - Let D be a cutset with no blue edges.
  - Select an uncolored edge in D of min cost and color it blue.
- greedy algorithm
  - Apply the red and blue rules (nondeterministically!) until all edges are colored. The blue edges form a MST.
  - Note: can stop once n – 1 edges colored blue.