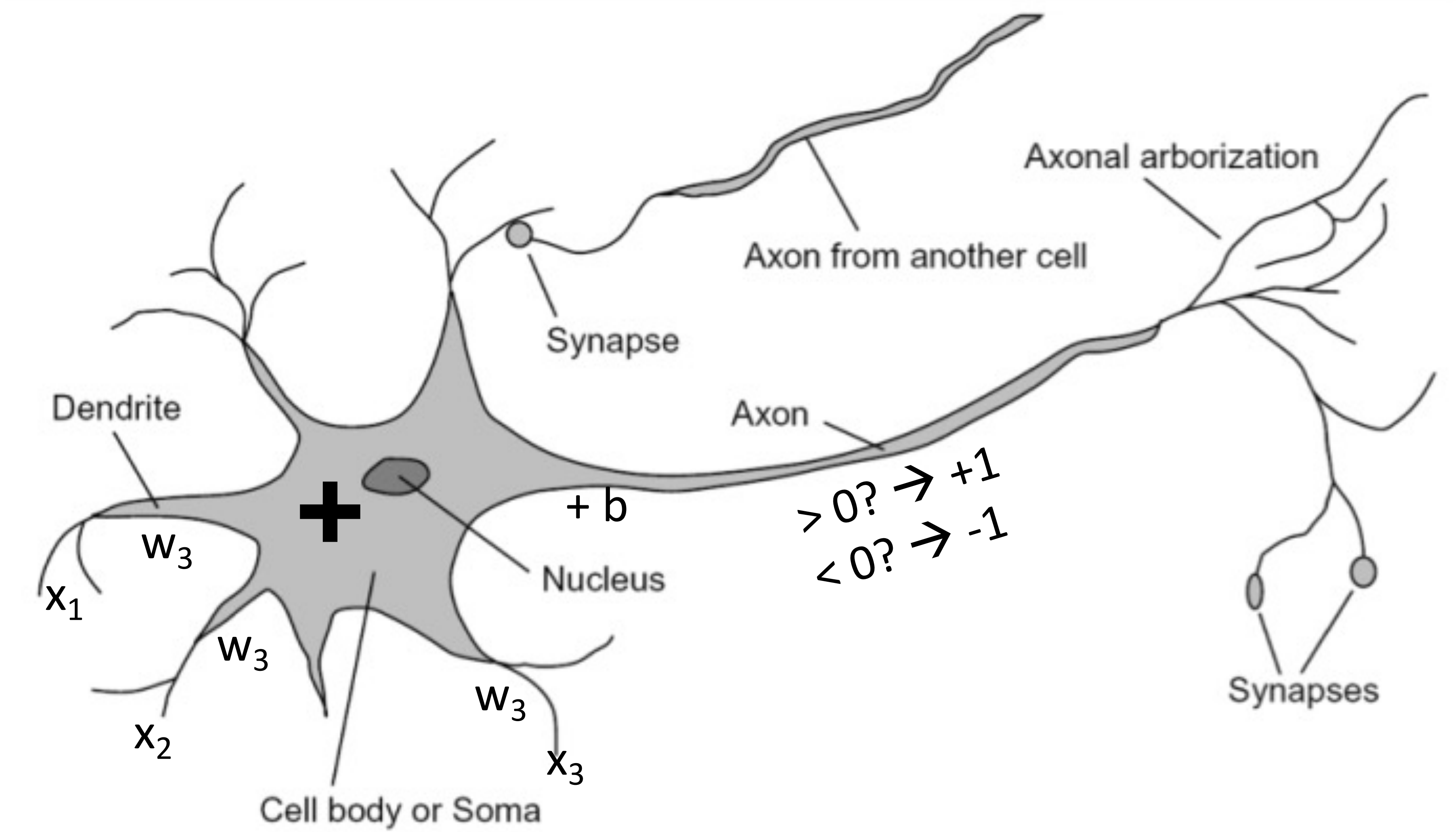
Perceptrons

Based on slides by Daniel Lowd, Vibhav Gogate, Pedro Domingos, Tom Mitchell, Carlos Guestrin, Luke Zettlemoyer and Dan Weld.

Neurons



Axonal arborization
Axon from another cell
Synapse
Dendrite
Axon
+ b
> 0? → +1
< 0? → -1
Nucleus
w3
w3
w3
x1
x2
x3
Synapses
Cell body or Soma

Perception / Linear Models

Example: $x = (x_1, x_2, \dots, x_D), y \in \{-1,1\}$

Perceptron finds a weight for each features and a bias

$w = (w_1, w_2, \dots, w_D), b \longleftarrow$ Model Parameters

Activation: $a = \left[\sum_{d=1}^{D} w_d x_d\right] + b$

Prediction: $sign(a) = \begin{cases} +1 \; if \; a > 0 \\ -1 \; otherwise \end{cases}$

Example: Spam

- Imagine 2 features (spam is "positive" class):
  - free (number of occurrences of "free")
  - money (occurrences of "money")

$$w \cdot f(x)$$

$$\sum_i w_i \cdot f_i(x)$$

x	$f(x)$	w
"free money"	free : 1 money : 1 ...	free : 4 money : 2 ...

b: -3

$$\begin{aligned} &(1)(-3) \quad + \\ &(1)(4) \quad + \\ &(1)(2) \quad + \\ &\ldots \\ &= 3 \end{aligned}$$

w.f(x) > 0 ➔ SPAM!!!

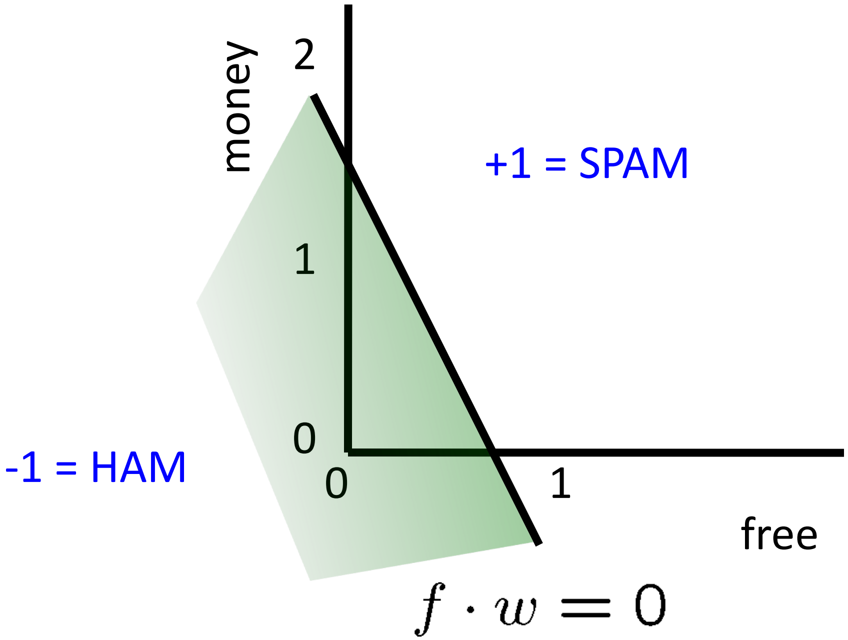
Binary Decision Rule

- In the space of feature vectors
  - Examples are points
  - Any weight vector is a hyperplane
  - One side corresponds to Y=+1
  - Other corresponds to Y=-1

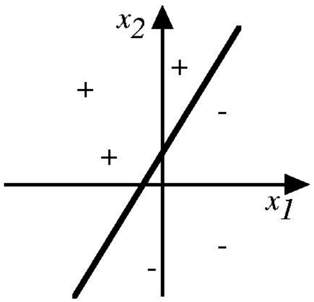
w

```
free  :  4
money :  2
...
```

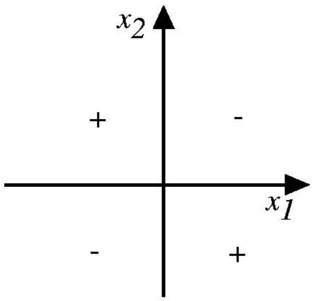
b: -3



Decision Surface of a Perceptron



(a)



(b)

Represents some useful functions

- What weights represent $g(x_1, x_2) = AND(x_1, x_2)$?

But some functions not representable

- All not linearly separable
- Therefore, we'll want networks of these...

How to find the parameters?

Algorithm 5 PERCEPTRONTRAIN($\mathbf{D}$, $MaxIter$)

```
w_d ← 0, for all d = 1 ... D                          // initialize weights
b ← 0                                                 // initialize bias
for iter = 1 ... MaxIter do   ← epoch
    for all (x,y) ∈ D do
        a ← Σ_{d=1}^{D} w_d x_d + b                   // compute activation for this example
        if ya ≤ 0 then
            w_d ← w_d + y x_d, for all d = 1 ... D    // update weights
            b ← b + y                                 // update bias
        end if
    end for
end for
return w_0, w_1, ..., w_D, b
```

Algorithm 6 PERCEPTRONTEST($w_0, w_1, \ldots, w_D, b, \hat{\boldsymbol{x}}$)

```
a ← Σ_{d=1}^{D} w_d x̂_d + b                           // compute activation for the test example
return SIGN(a)
```

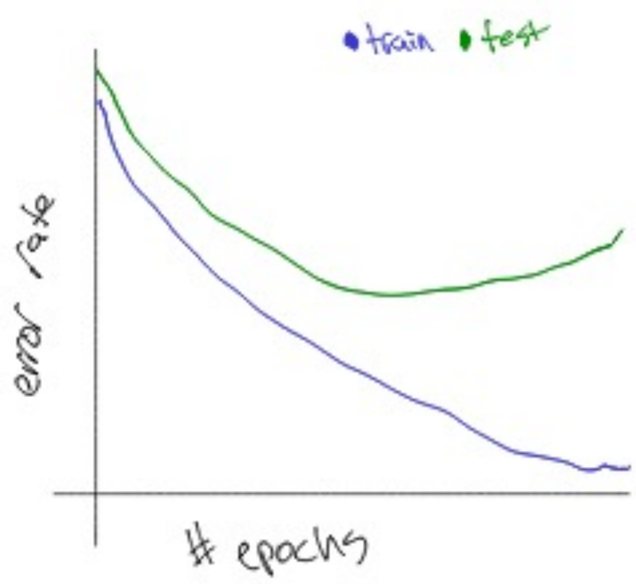
What the updates do?

- Moves the decision boundary in the direction of the training examples
- Intuition: adjust parameters so that they are better for the current example (i.e., if we see an example two times, the parameters should do a better job on the second example than on the first one).
- Example:
  - (x, +1): the current example, $(w_1, w_2, \dots, w_D), b$: the current parameters, the parameters assign incorrect class in this case (i.e., $a < 0$)
  - New parameters: $(w_1', w_2', \dots, w_D'), b'$, new activation: a'

$$
\begin{aligned}
a' &= \sum_{d=1}^{D} w_d' x_d + b' \\
&= \sum_{d=1}^{D} (w_d + x_d) x_d + (b + 1) \\
&= \sum_{d=1}^{D} w_d x_d + b + \sum_{d=1}^{D} x_d x_d + 1 \\
&= a + \sum_{d=1}^{D} x_d^2 + 1 \quad > \quad a
\end{aligned}
$$

How to iterate over data?

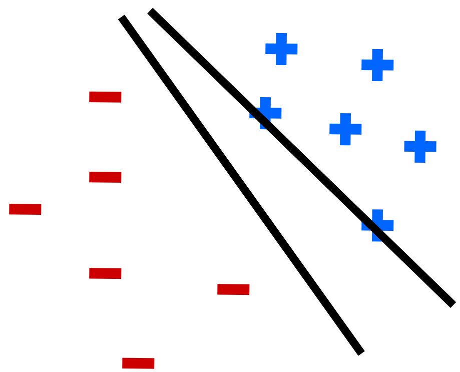
- IMPORTANT: permute the training dataset $\boldsymbol{D}$ at the beginning of each epoch
  - Why? Image having a fixed order of data in the iterations (i.e., positive example come first, followed by negative examples)
- How many epochs should we run (i.e., what's the good value for $MaxIter$)?
  - Many epochs tend to overfit (overtraining) while little epochs would underfit



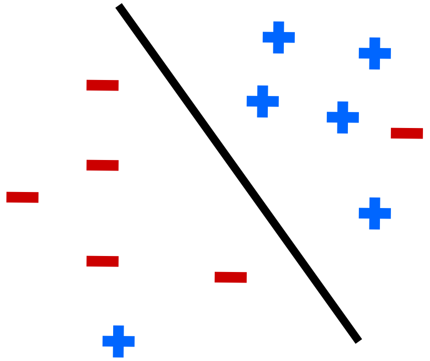
Does the perceptron converge? How long does it take?

- Convergence in perceptron: a classifier is considered as converged if it can correctly classify every training example
- Training data is linearly separable if we can find some hyperplane that puts positive examples on one side and negative examples on the other side (we say the hyperplane/parameter separates the data)

Separable



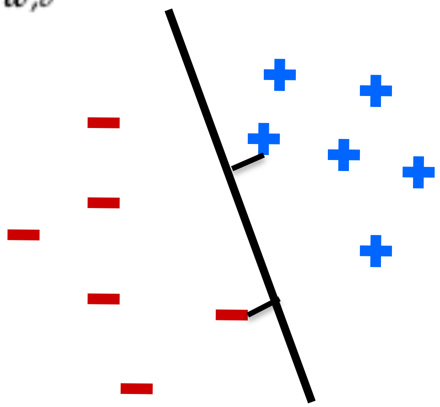
Non-Separable



Margins

$$margin(\mathbf{D}, \boldsymbol{w}, b) = \begin{cases} \min_{(x,y)\in\mathbf{D}} y(\boldsymbol{w}\cdot\boldsymbol{x} + b) & \text{if } \boldsymbol{w} \text{ separates } \mathbf{D} \\ -\infty & \text{otherwise} \end{cases}$$

$$margin(\mathbf{D}) = \sup_{\boldsymbol{w},b} margin(\mathbf{D}, \boldsymbol{w}, b)$$

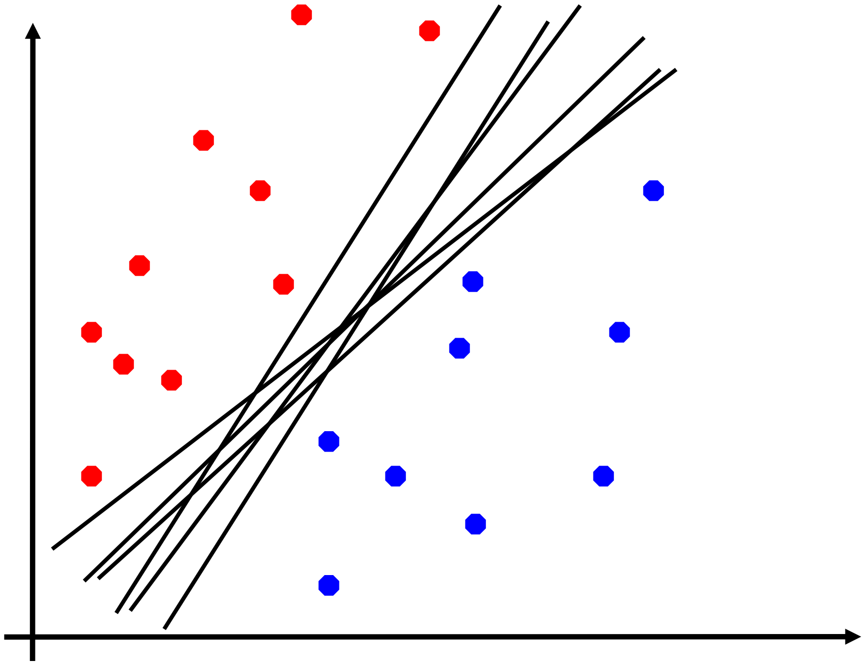


What does the margin tell you about your problem?

Convergence in perceptron

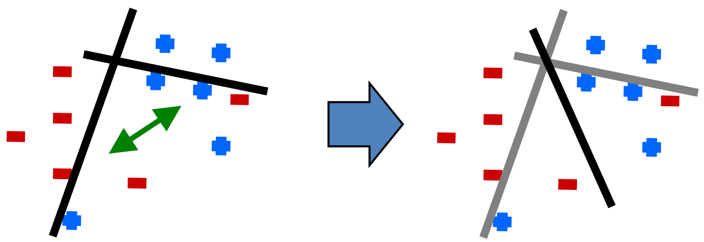
Theorem 2 (Perceptron Convergence Theorem). *Suppose the perceptron algorithm is run on a linearly separable data set* $\mathbf{D}$ *with margin* $\gamma > 0$. *Assume that* $||\boldsymbol{x}|| \leq 1$ *for all* $\boldsymbol{x} \in \mathbf{D}$. *Then the algorithm will converge after at most* $\frac{1}{\gamma^2}$ *updates.*

- What does it tell you?

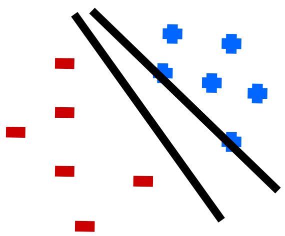


Problems with the Perceptron

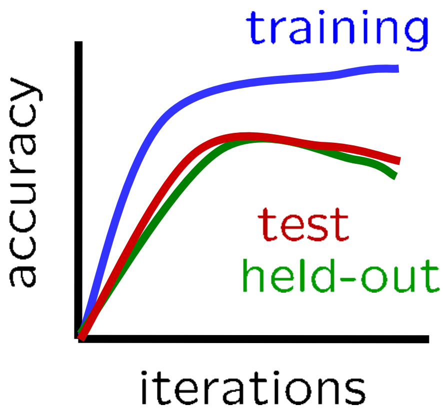
- Noise: if the data isn't separable, weights might thrash
  - Averaging weight vectors over time can help (averaged perceptron)



- Mediocre generalization: finds a "barely" separating solution



- Overtraining: test / validation accuracy usually rises, then falls
  - Overtraining is a kind of overfitting



Improved Generalization

- Problem: The vanilla perceptron counts later points more than it counts earlier points (i.e., the last examples might have significant effect on the parameters)
- Fix: ensure that all the weight vectors encountered during training contribute to the prediction, the weight vectors that survive/appear for longer time should contribute more than those with shorter time.
- Voted perceptron:

$$\hat{y} = \operatorname{sign}\left(\sum_{k=1}^{K} c^{(k)} \operatorname{sign}\left(\boldsymbol{w}^{(k)} \cdot \hat{\boldsymbol{x}} + b^{(k)}\right)\right)$$

  – But: need storage to store many weight vectors, prediction time is also slower
- Averaged perceptron:

$$\hat{y} = \operatorname{sign}\left(\sum_{k=1}^{K} c^{(k)} \left(\boldsymbol{w}^{(k)} \cdot \hat{\boldsymbol{x}} + b^{(k)}\right)\right)$$

$$\hat{y} = \operatorname{sign}\left(\left(\sum_{k=1}^{K} c^{(k)} \boldsymbol{w}^{(k)}\right) \cdot \hat{\boldsymbol{x}} + \sum_{k=1}^{K} c^{(k)} b^{(k)}\right)$$

  – so, same prediction time as vanilla perceptron, good practical performance

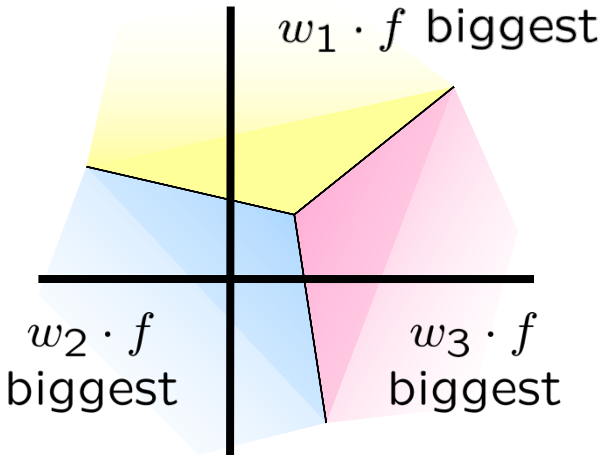
Algorithm 7 AveragedPerceptronTrain($\mathbf{D}$, $MaxIter$)

$\boldsymbol{w} \leftarrow \langle 0, 0, \ldots 0 \rangle$, $b \leftarrow 0$ // initialize weights and bias
$\boldsymbol{u} \leftarrow \langle 0, 0, \ldots 0 \rangle$, $\beta \leftarrow 0$ // initialize cached weights and bias
$c \leftarrow 1$ // initialize example counter to one
for $iter = 1 \ldots MaxIter$ **do**
  for all $(x,y) \in \mathbf{D}$ **do**
    if $y(\boldsymbol{w} \cdot \boldsymbol{x} + b) \leq 0$ **then**
      $\boldsymbol{w} \leftarrow \boldsymbol{w} + y\, \boldsymbol{x}$ // update weights
      $b \leftarrow b + y$ // update bias
      $\boldsymbol{u} \leftarrow \boldsymbol{u} + y\, c\, \boldsymbol{x}$ // update cached weights
      $\beta \leftarrow \beta + y\, c$ // update cached bias
    end if
    $c \leftarrow c + 1$ // increment counter regardless of update
  end for
end for
return $\boldsymbol{w} - \frac{1}{c}\, \boldsymbol{u},\ b - \frac{1}{c}\, \beta$ // return averaged weights and bias

Why?

Multiclass Decision Rule

- If we have more than two classes:
  - Have a weight vector for each class: w_y
  - Calculate an activation for each class



$$\text{activation}_w(x, y) = w_y \cdot f(x)$$

  - Highest activation wins

$$y = \arg\max_y \left(\text{activation}_w(x, y)\right)$$

The Multi-class Perceptron Alg.

- Start with zero weights
- Iterate training examples
  - Classify with current weights

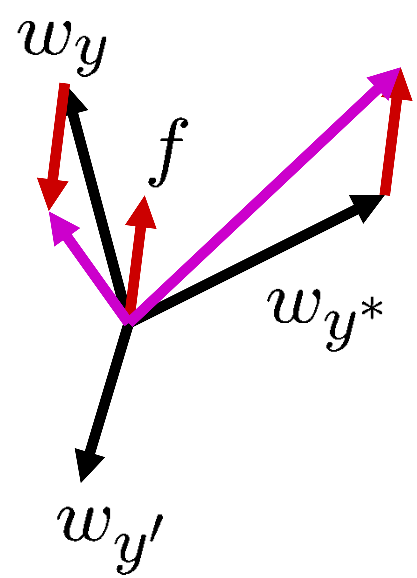
$$y = \arg\max_y \; w_y \cdot f(x)$$

$$= \arg\max_y \; \sum_i w_{y,i} \cdot f_i(x)$$

  - If correct, no change!
  - If wrong: lower score of wrong answer, raise score of right answer

$$w_y = w_y - f(x)$$

$$w_{y^*} = w_{y^*} + f(x)$$



Example

"win the vote"

"win the election"

"win the game"

w_{SPORTS}	$w_{POLITICS}$	w_{TECH}
BIAS :	BIAS :	BIAS :
win :	win :	win :
game :	game :	game :
vote :	vote :	vote :
the :	the :	the :
...	...	...

Example

"win the vote" $\Rightarrow$

```
BIAS  :  1
win   :  1
game  :  0
vote  :  1
the   :  1
...
```

w_{SPORTS}

```
BIAS  : -2
win   :  4
game  :  4
vote  :  0
the   :  0
...
```

$w_{POLITICS}$

```
BIAS  :  1
win   :  2
game  :  0
vote  :  4
the   :  0
...
```

w_{TECH}

```
BIAS  :  2
win   :  0
game  :  2
vote  :  0
the   :  0
...
```

Which type wins?