CIS 313:
Intermediate Data Structure

sixth slide

expected behavior

- if list a is chosen randomly from among all n! permutations
- how long does "`for i=1 to n T.insert(a`$_i$`)`" take?
- worst case: $O(n^2)$
- want to argue: on average O(n *lg* n)

- main fact: expected search time (1+I/n) in BST built from randomly chosen permutation is $2 \cdot \ln(n+1) + O(1) \approx 1.38\ \log_2 n + O(1)$

observations

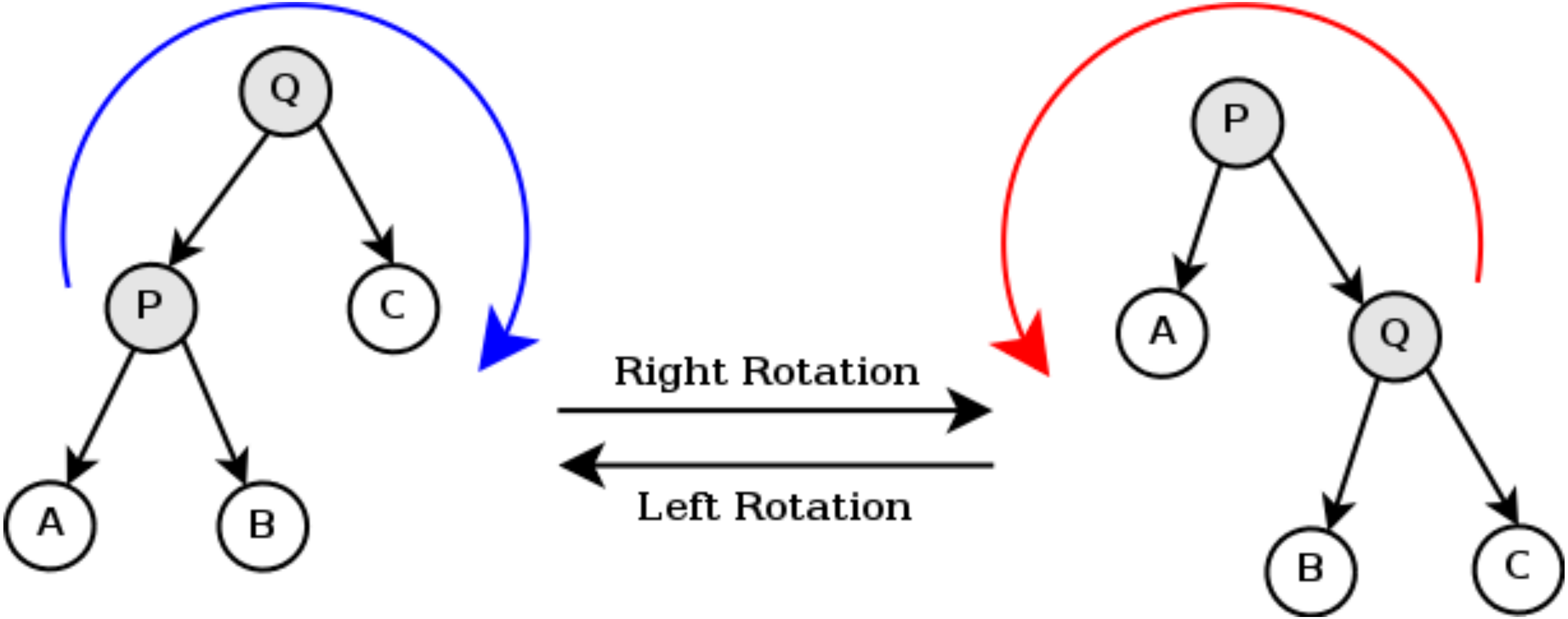
- this does not bound the height of the tree
- exercise 12.4-2, p 303: describe a binary search tree on n nodes such that the average depth of a node in the tree is $\Theta(\lg n)$ but the height of the tree is $\omega(\lg n)$
- stronger result: height of randomly built BST is is $\Theta(\lg n)$

- new goal: maintain BST whose height is is $\Theta(\lg n)$ in the *worst case*
- self balancing search trees: AVL, red-black, B-trees

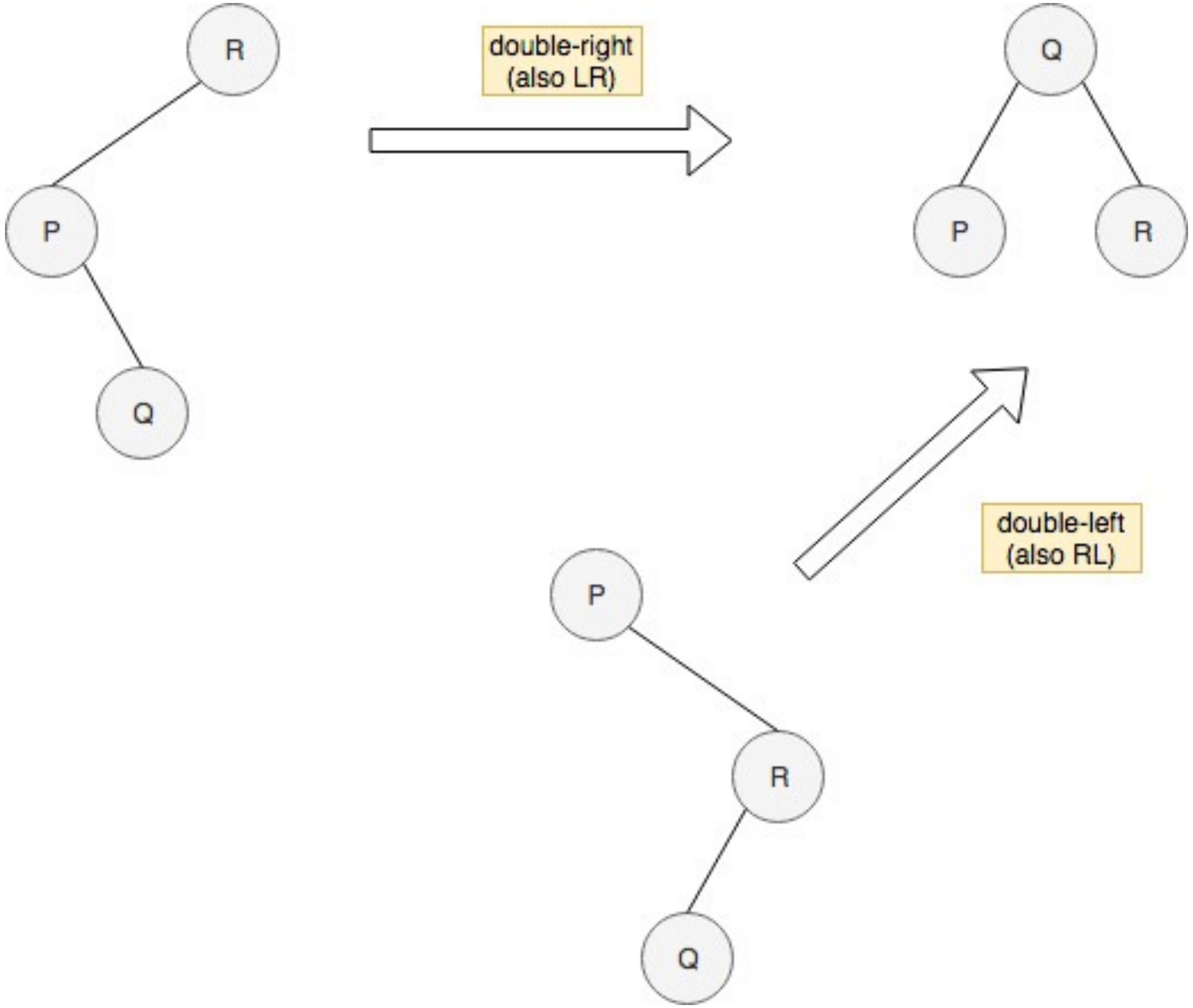
balanced tree

- not realistic to expect perfectly balanced tree
- one attempt (not common): *weight-balance*, where the number of nodes in left and right subtrees of any node must be close to each other
- better: *height-balance*, the height of the left and right subtrees must be close
- AVL: differ by one
- red-black: differ by factor of two
- balance maintained by rotations

rotation: single



rotations: double



Composed from two single rotations.

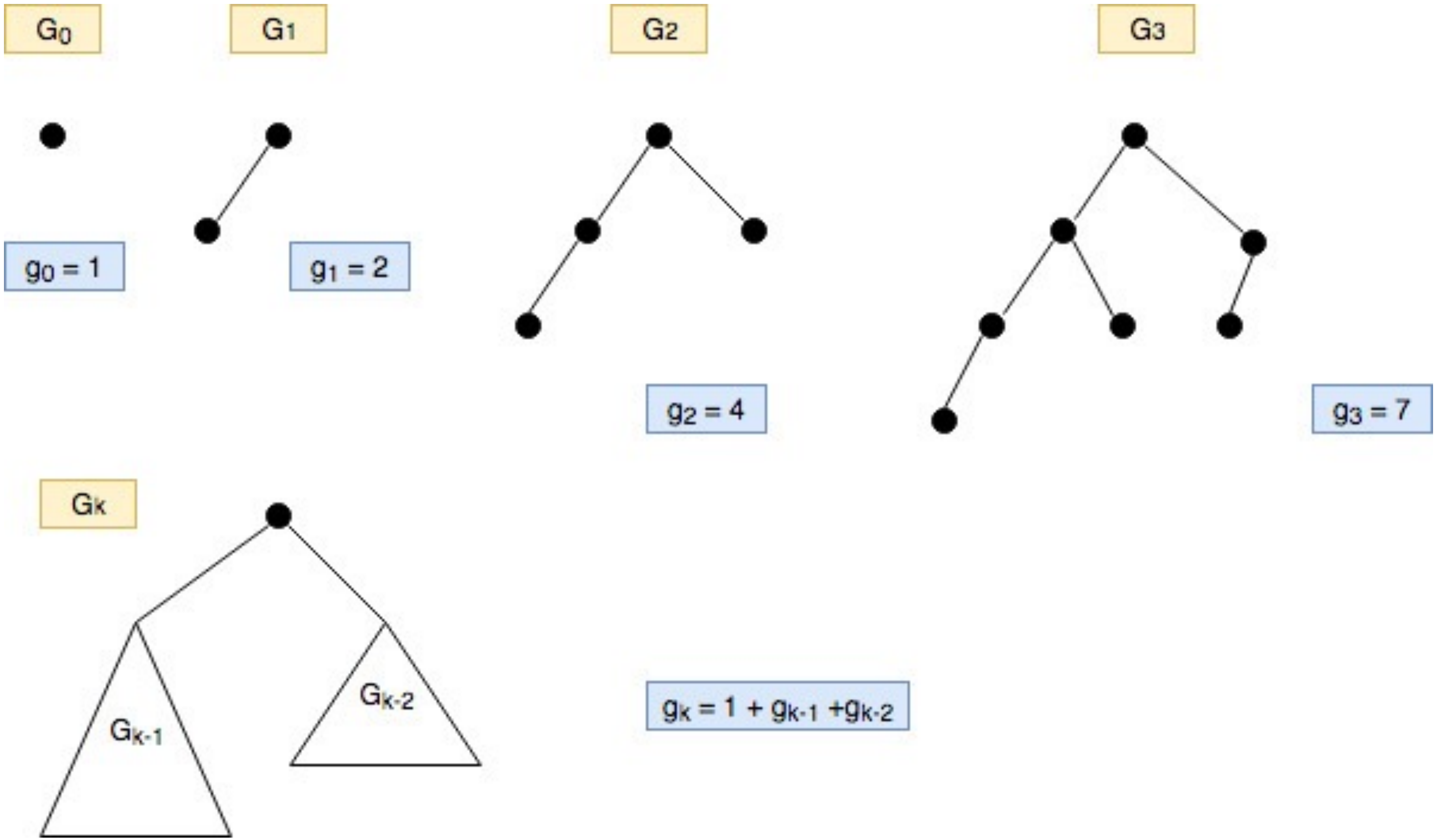
AVL trees

- (not in text)
- named after inventors Adelson-Velskii and Landis
- store at each node the balance factor:
  - bf(p) = height(p.lchild) – height(p.rchild)
  - requirement: for every node p, bf(p) equals -1, 0, or 1
- requires two bits extra storage at each node

AVL height is O(*lg*n)

- let G_k be an AVL tree (shape) of height k with the fewest number of nodes
- G_k can be constructed inductively as a node with a G_{k-1} left child and a G_{k-2} right child
- define g_k to be the number of nodes in a G_k tree
- $g_0 = 1$, $g_1 = 2$, $g_k = 1+g_{k-1}+g_{k-2}$
- sequence: 1, 2, 4, 7, 12, 20
- fact: $g_k = F_{k+3} - 1$ ("easy" to prove with induction)

trees G_k and values g_k



AVL tree height: the punchline

- if n is the number of nodes in an AVL tree of height H then
$$n \geq g_H = F_{H+3} - 1$$
- we know $F_k = \left[\varphi^k/\sqrt{5}\right]$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$
- $\lg F_{H+3} \geq \lg \frac{\varphi^{H+3}}{\sqrt{5}} - 1 = (H+3)\lg\varphi - \lg\sqrt{5} - 1 \geq (H+3)\lg\varphi - 4$
- so $(H+3)\lg\varphi - 4 \leq \lg F_{H+3} \leq \lg(n+1)$ *(take log of both sides of top line)*
- moving terms around: $H \leq \frac{\lg(n+1)+4}{\lg\varphi} - 3 \approx 1.44\lg(n+1) + O(1)$

AVL insertion

- insert node as with a BST (add it to a null pointer)
- update balance factors along path from new node to root
- the balance factors of some nodes may in violation: 2 or -2
- find the *critical node*: the lowest out of balance node
- perform the appropriate rotation

- note: this will affect the balance factors of nodes above it
- total insertion time O(*lg* n)

AVL insertion

Four Possible Cases

bf(x) = +2 and bf(x.left) = 1

rightRotate(x)

bf(x) = +2 and bf(x.left) = -1

leftRotate(x.left)

rightRotate(x)

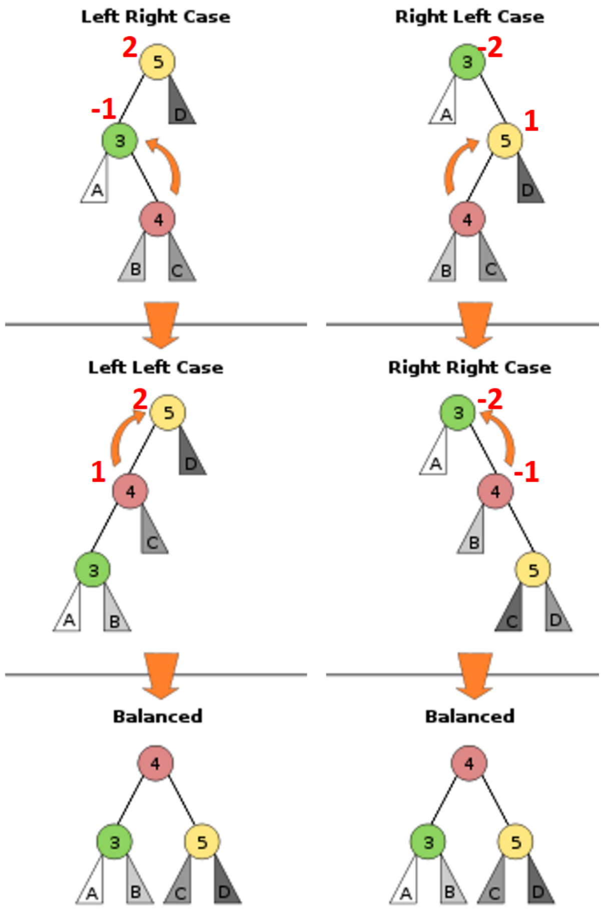
bf(x) = -2 and bf(x.right) = -1

leftRotate(x)

bf(x) = -2 and bf(x.right) = 1

rightRotate(x.right)

leftRotate(x)



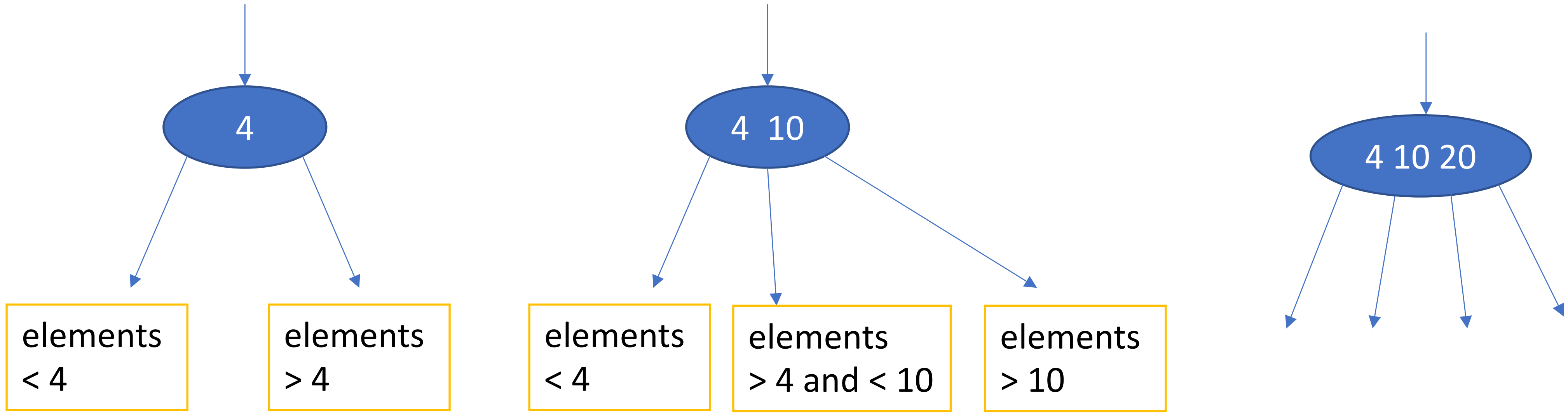
Pictures from Wikipedia

2-3 and 2-3-4 trees

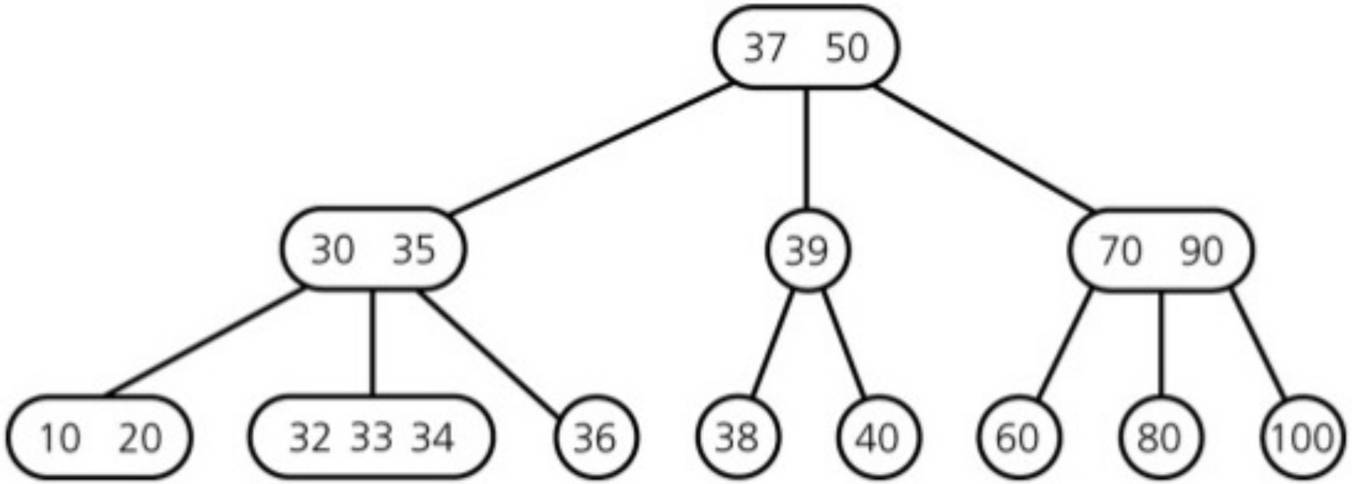
- quick intro here, we will return to them later as B-trees
- a 2-3 tree is a B-tree of order 3 *(see ex 18-2, p 503, of text)*
- these use multi-way search nodes
- must be perfectly balanced: all paths from the root to a null node have the same length
- insertions cause splits rather than rotations

- *important*: red-black trees (our real focus) are a binary implementation of 2-3-4 trees

multiway search nodes



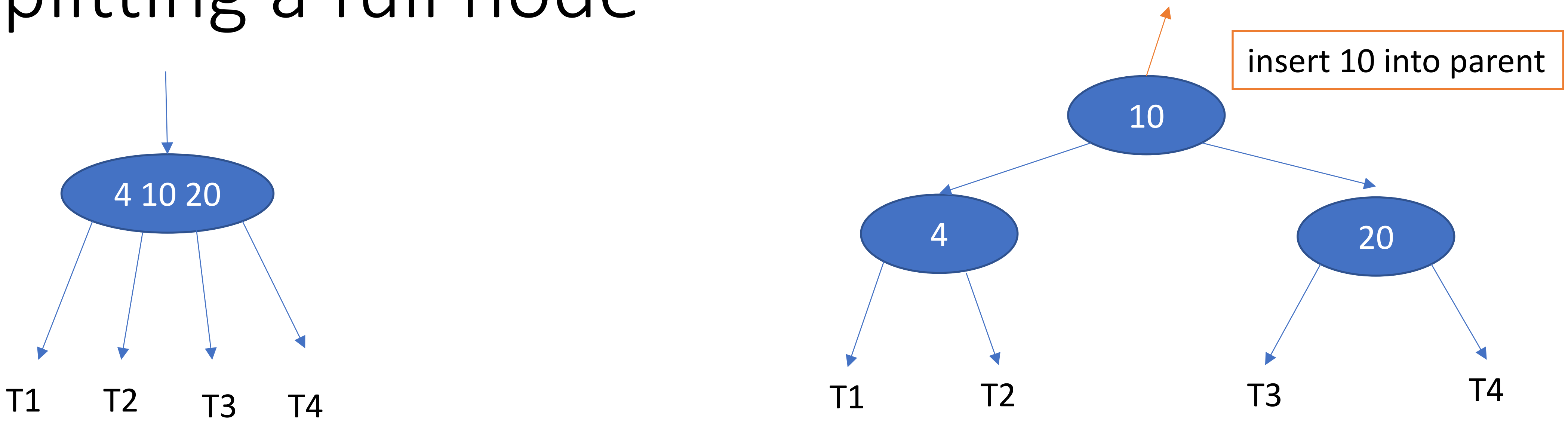
example



insertion: splitting nodes

- can split a node when it is full or has overflowed
- splitting on insertion can be bottom-up
  - put node at bottom of tree, if over-flow, split on the way up
- or top-down
  - when looking for insertion point, if full node seen, split it
- most B-tree implementations use bottom up (less space)

splitting a full node

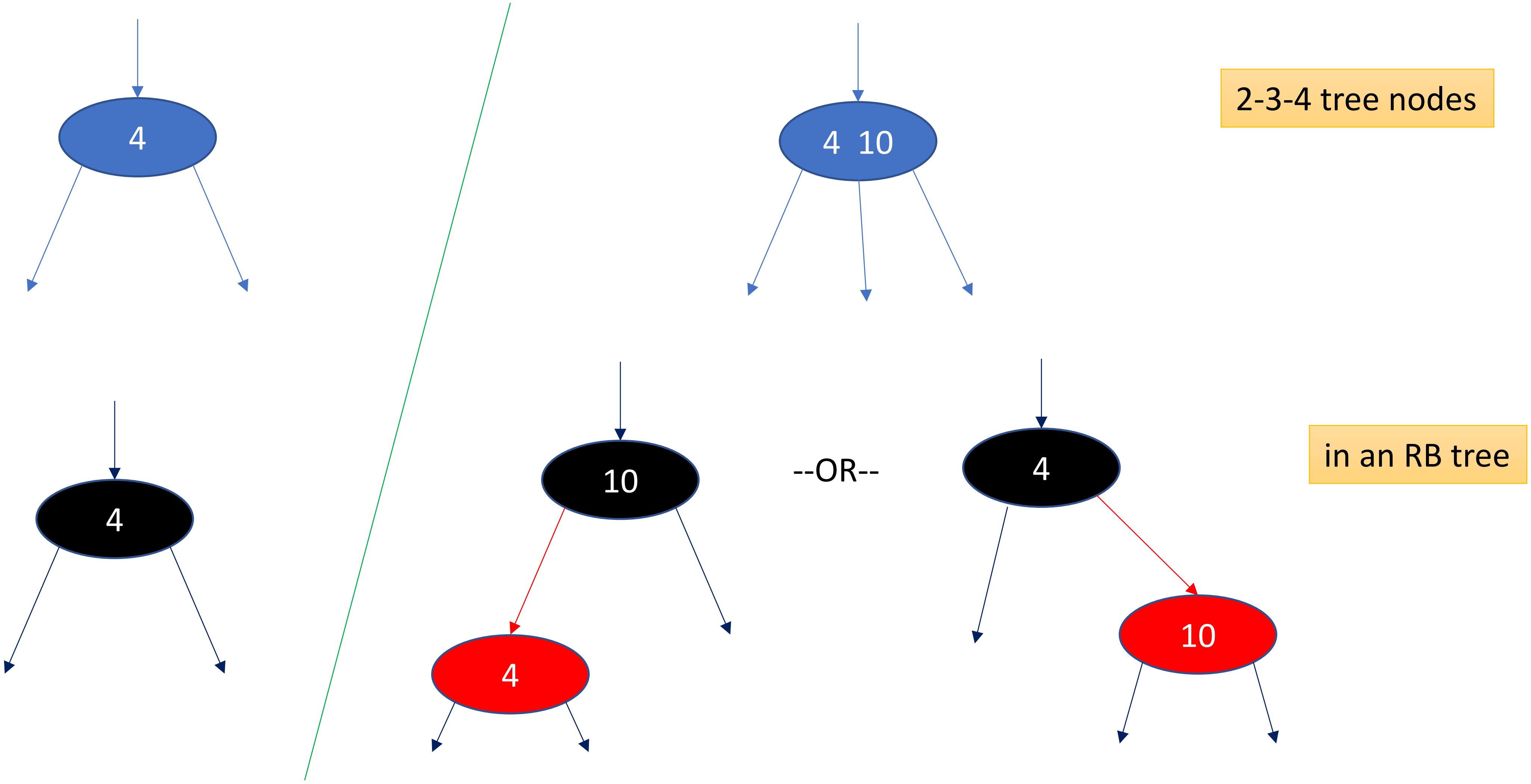


red-black trees and 2-3-4 trees

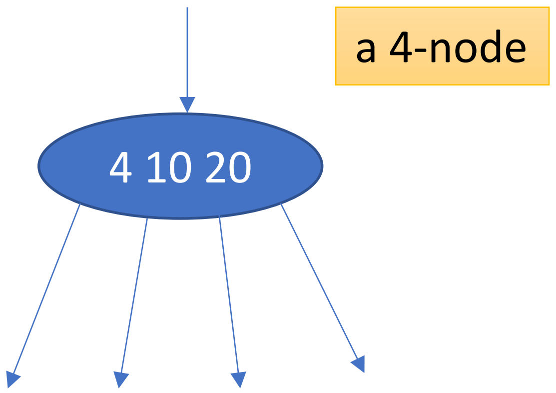
- a 2-3-4 tree node would need up to 4 child pointers
- frequently unused so waste of space
- red-black tree is binary tree implementation of 2-3-4 tree
- uses rotations to handle the splits
- need one bit to indicate color
  - descending the tree, black means ”new node”
  - red means “belong to parent”

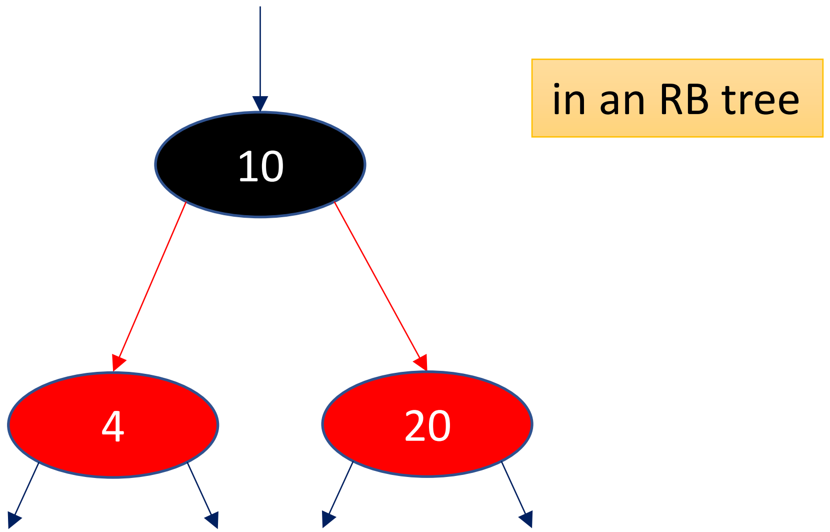
- Java uses RB trees in the TreeMap class (https://docs.oracle.com/javase/7/docs/api/java/util/TreeMap.html)

2-3-4 nodes as RB nodes (2- and 3-nodes)

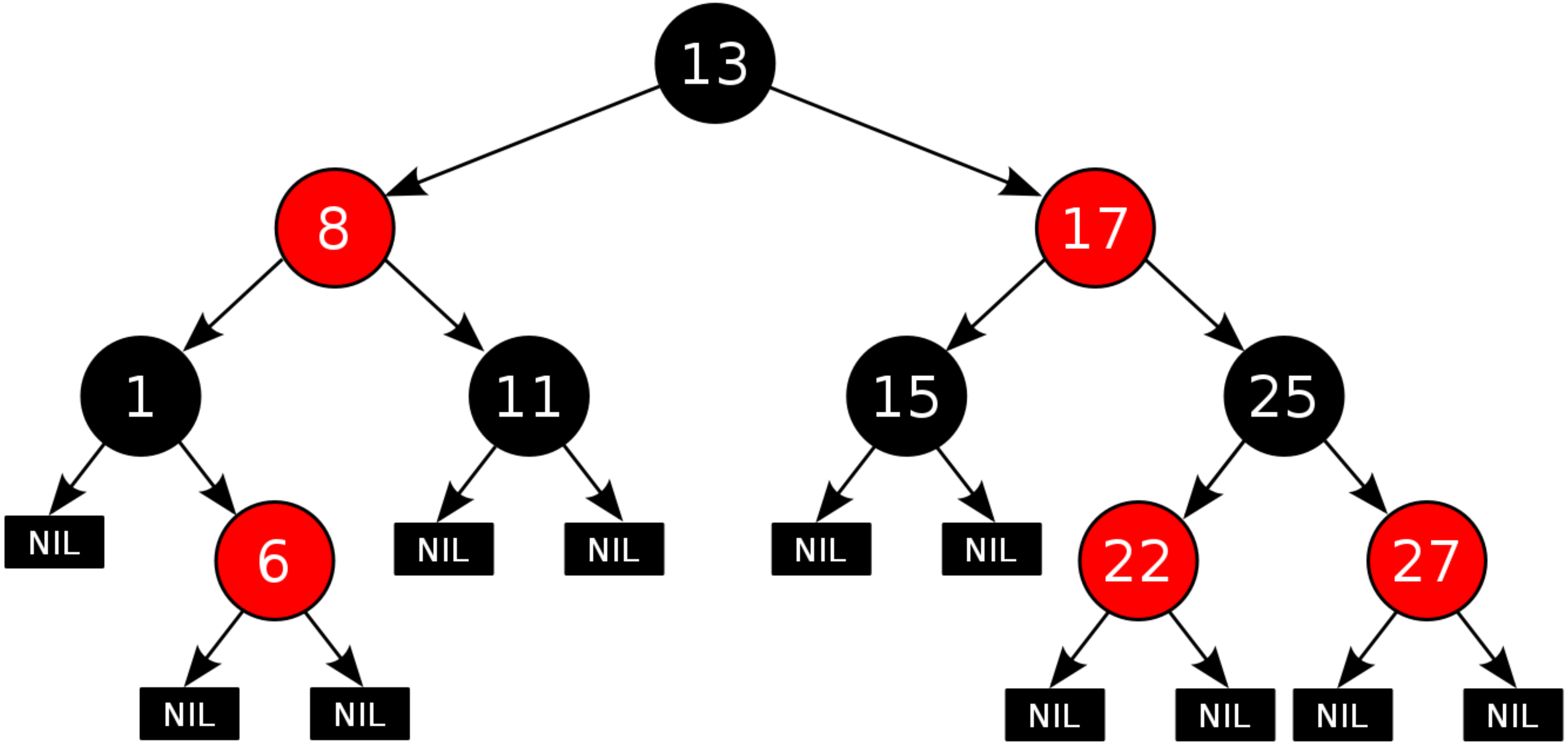


2-3-4 nodes as RB nodes (4-nodes)

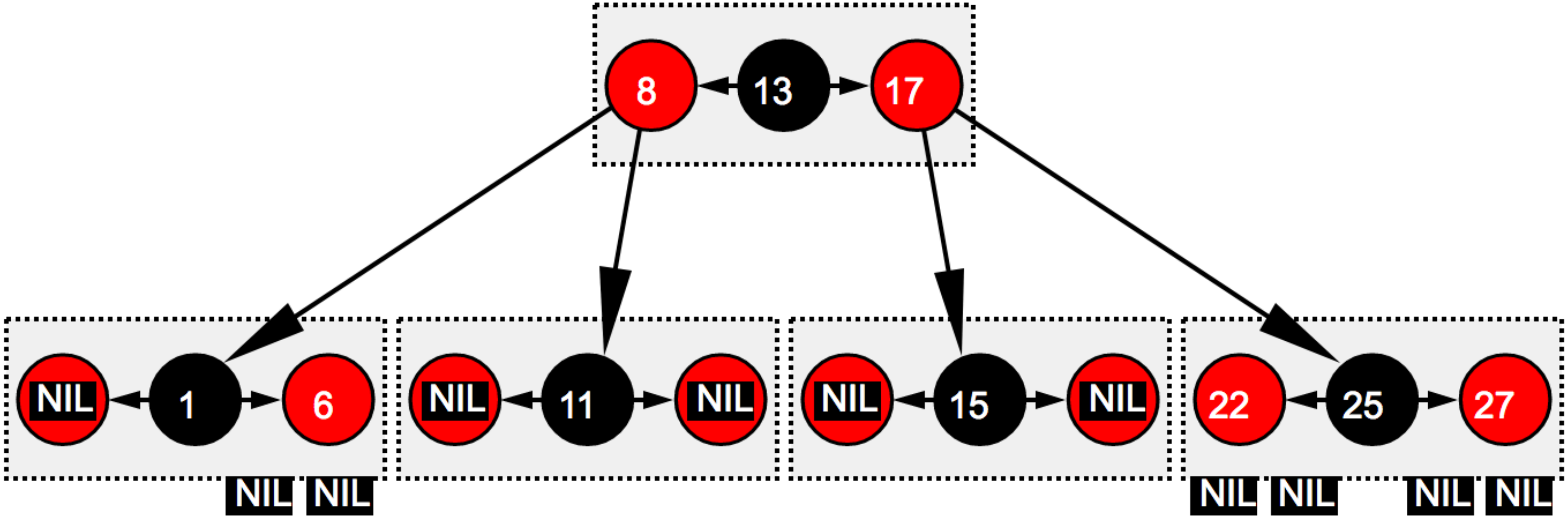




example RB tree



viewed as 2-3-4 tree



red-black tree rules

1. every node is either red or black
2. the root is black
3. every leaf (null) is black
4. if a node is red, both of its children are black
5. for each node, all simple paths from the node to descendant leaves contain the same number of black nodes