Art of Invariant Generation
applied to
Symbolic Bound Computation

Part 2

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Oregon Summer School
July 2009

Art of Invariant Generation

1. Program Transformations

   - Reduce need for sophisticated invariant generation.
   - E.g., control-flow refinement, loop-flattening/peeling, non-standard cut-points, quantitative attributes instrumentation.

2. Colorful Logic

   - Language of Invariants
   - E.g., arithmetic, uninterpreted fns, lists/arrays

3. Fixpoint Brush

   - Automatic generation of invariants in some shade of logic, e.g., conjunctive/k-disjunctive/predicate abstraction.
   - E.g., Iterative, Constraint-based, Proof Rules

Logic

- To validate correctness of loop-free programs, or programs annotated with loop invariants, decision procedures are enough.
  - Provided the logic is closed under WP.

    $\langle Pre, S, Post \rangle$ is valid

    iff $Pre \Rightarrow WP(S, Post)$ is valid

    iff $Pre \wedge \neg WP(S, Post)$ is unsatisfiable

- To validate correctness of programs with loops, we need to automatically discover loop invariants using fixpoint computation techniques.
  - This requires algorithms that are more sophisticated than decision procedures.

Fixpoint Brush

We will briefly study fixpoint techniques for discovering loop invariants in conjunctive fragments of various logics.

- Iterative
  - Forward
  - Backward
- Constraint-based
- Proof-rules

Iterative Forward: Examples

Start with Precondition and propagate facts forward using Existential elimination operator until fixpoint.

- Data-flow analysis
  - Join at merge points
  - Finite abstract domains
- Abstract Interpretation
  - Join at merge points
  - Widening for infinite height abstract domains.
- Model Checking
  - Analyze all paths precisely without join at merge points.
  - Finite abstractions (e.g., boolean abstraction) required
    - Counterexample guided abstraction refinement
    - BDD based data-structures for efficiency

Key Operators needed by Iterative Forward

- Fixpoint checking: Decision Procedure.

$Decide_T(\phi)$ = Yes, iff ϕ is satisfiable

- Transfer Function for Assignment Node: Existential Quantifier Elimination

Eliminate(ϕ, V) = strongest ϕ' such that $V \in Vars(\phi')$

- Transfer Function for Merge Node: Join

$Join(\phi_1, \phi_2)$ = strongest ϕ s.t. $\phi_1 \Rightarrow \phi$ and $\phi_2 \Rightarrow \phi$

Eliminate and Join are usually harder than Decide.

Difference Constraints

- Abstract element:
  - conjunction of $x_i - x_j \leq c_{ij}$
  - can be represented using matrix M, where $M[i][j]=c_{ij}$

- Decide(M):
  1. M' := Saturate(M);
  2. Declare unsat iff $\exists i$: $M'[i][i] < 0$

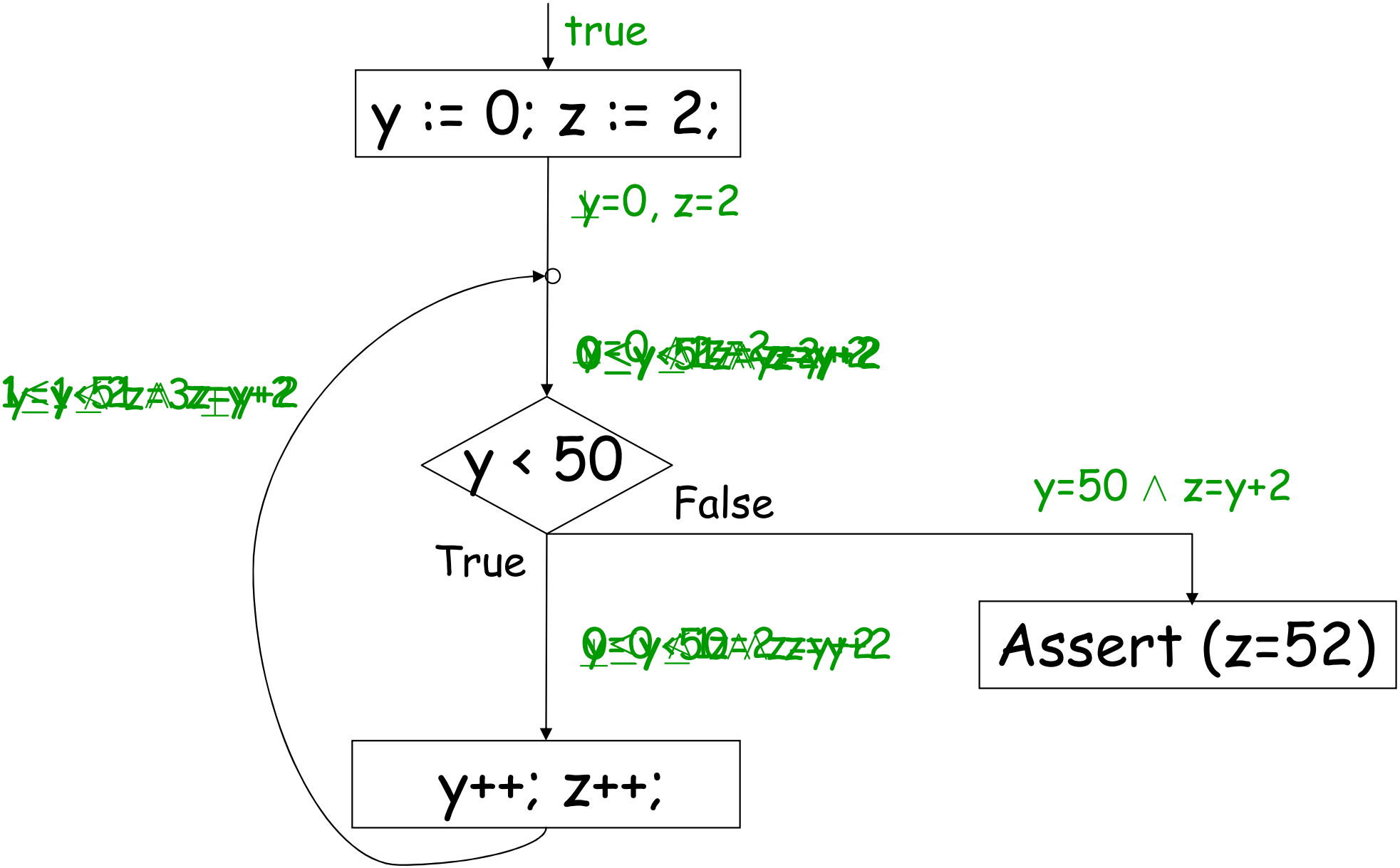
- Join(M_1, M_2):
  1. M'_1 := Saturate(M_1); M'_2 := Saturate(M_2);
  2. Let M_3 be s.t. $M_3[i][j]$ = Max { $M'_1[i][j]$, $M'_2[i][j]$ }
  3. return M_3

Difference Constraints

- Eliminate(M, x_i):
  1. M′ := Saturate(M);
  2. Let M_1 be s.t. $M_1[j][k] = \infty$ (if j=i or k=i)
     $= M'[j][k]$ otherwise
  3. return M_1

- Widen(M_1, M_2):
  1. M'_1 := Saturate(M_1); M'_2 := Saturate(M_2);
  2. Let M_3 be s.t. $M_3[i][j] = M_1[i][j]$ (if $M_1[i][j] = M_2[i][j]$))
     $= \infty$ (otherwise)
  3. return M_3

Example: Abstract Interpretation using Difference Constraints



Uninterpreted Functions

- Abstract element:
  - conjunction of $e_1=e_2$, where e := y | $F(e_1,e_2)$
  - can be represented using EDAGs

- Decide(G):
  1. G' := Saturate(G);
  2. Declare unsat iff G contains $e_1 \neq e_2$ and G' has e_1, e_2 in the same congruence class.

- Eliminate(G, y):
  1. G' := Saturate(G);
  2. Erase y; (might need to delete some dangling expressions)
  3. return G'

Uninterpreted Functions

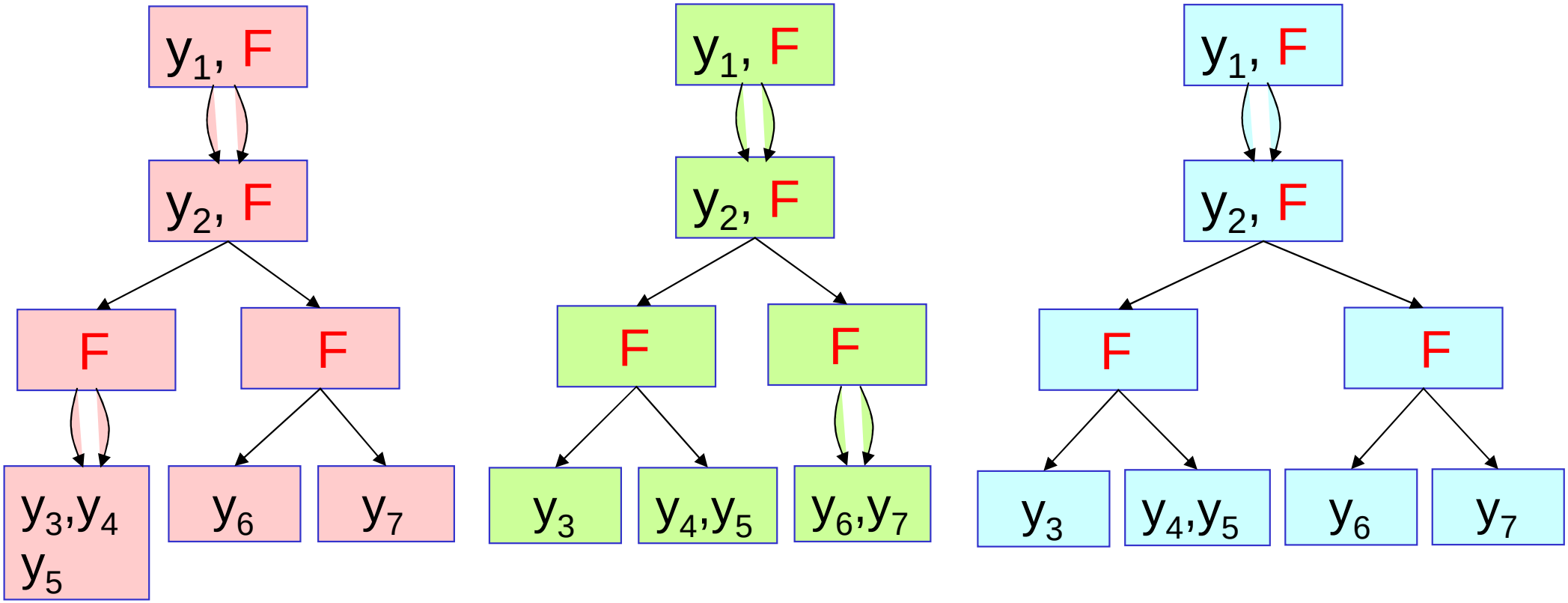
- Join(G_1, G_2):
  1. G'_1 := Saturate(G_1); G'_2 := Saturate(G_2);
  2. G := Intersect(G'_1, G'_2);
  3. return G;

For each node n = <U, $F(n_i,n'_i)$> in G'_1

and node m = <V, $F(m_j,m'_j)$> in G'_2,

G contains a node [n,m] = <$U \cap V$, $F([n_i, m_j],[n'_i,m'_j])$>

Uninterpreted Functions: Example of Join



G_1

G_2

$G = \text{Join}(G_1, G_2)$

Recap: Combination of Decision Procedures

- Decide(E_{12}):
  1. $\langle E_1, E_2 \rangle$:= Purify&Saturate(E_{12});
  2. Return $\text{Decide}_{T_1}(E_1) \wedge \text{Decide}_{T_2}(E_2)$;

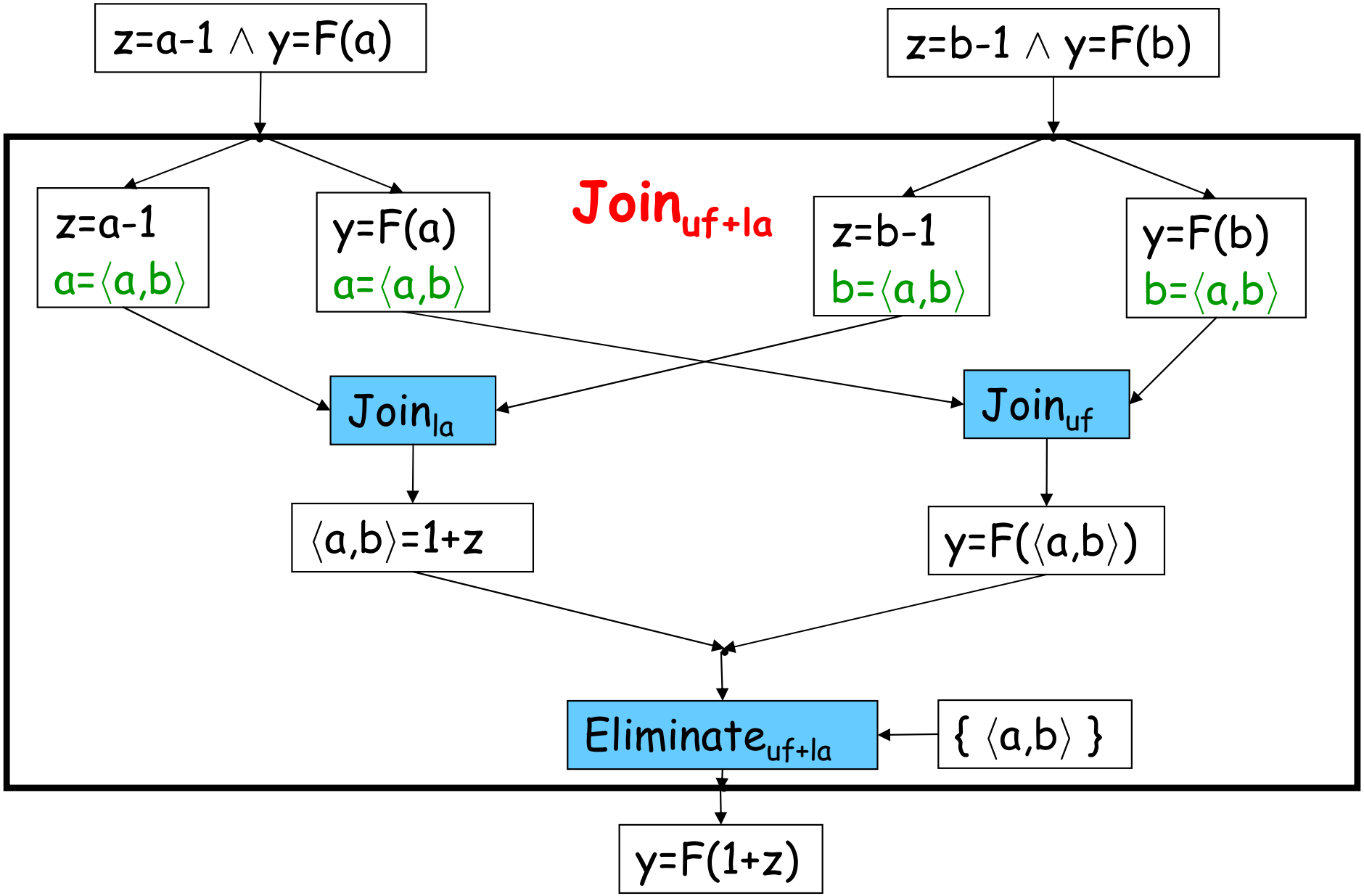
Combination: Join Algorithm (1st attempt)

- $Join_{T_{12}}(L_{12}, R_{12})$:
  1. $\langle L_1, L_2 \rangle$:= Purify&Saturate(L_{12});
     $\langle R_1, R_2 \rangle$:= Purify&Saturate(R_{12});
  2. $A_1 := Join_{T_1}(L_1, R_1)$;
     $A_2 := Join_{T_2}(L_2, R_2)$;
  3. Return $A_1 \wedge A_2$;

Combination: Join Algorithm

- $Join_{T_{12}}(L_{12}, R_{12})$:
  1. $\langle L_1, L_2\rangle$:= Purify&Saturate(L_{12});
     $\langle R_1, R_2\rangle$:= Purify&Saturate(R_{12});
  2. $D_L := \bigwedge \{v_i=\langle v_i,v_j\rangle \mid v_i \in Vars(L_1 \wedge L_2),\ v_j \in Vars(R_1 \wedge R_2)\}$;
     $D_R := \bigwedge \{v_j=\langle v_i,v_j\rangle \mid v_i \in Vars(L_1 \wedge L_2),\ v_j \in Vars(R_1 \wedge R_2)\}$;
  3. $L'_1 := L_1 \wedge D_L$; $R'_1 := R_1 \wedge D_R$;
     $L'_2 := L_2 \wedge D_L$; $R'_2 := R_2 \wedge D_R$;
  4. $A_1 := Join_{T_1}(L'_1, R'_1)$;
     $A_2 := Join_{T_2}(L'_2, R'_2)$;
  5. $V := Vars(A_1 \wedge A_2)$ – Program Variables;
     $A_{12} := Eliminate_{T_{12}}(A_1 \wedge A_2, V)$;
  6. Return A_{12};

Combination: Example of Join Algorithm

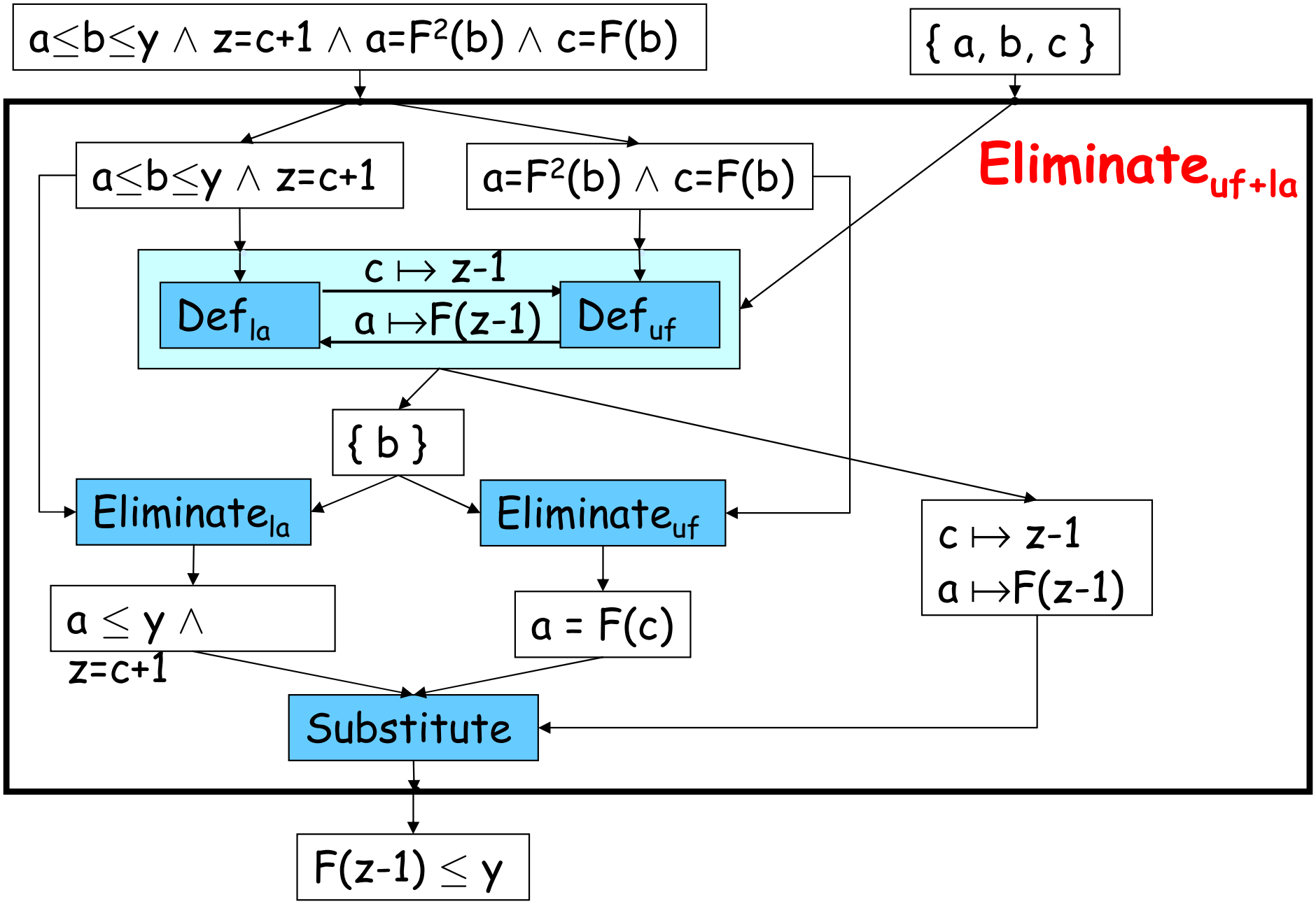


Combination: Existential Quantifier Elimination

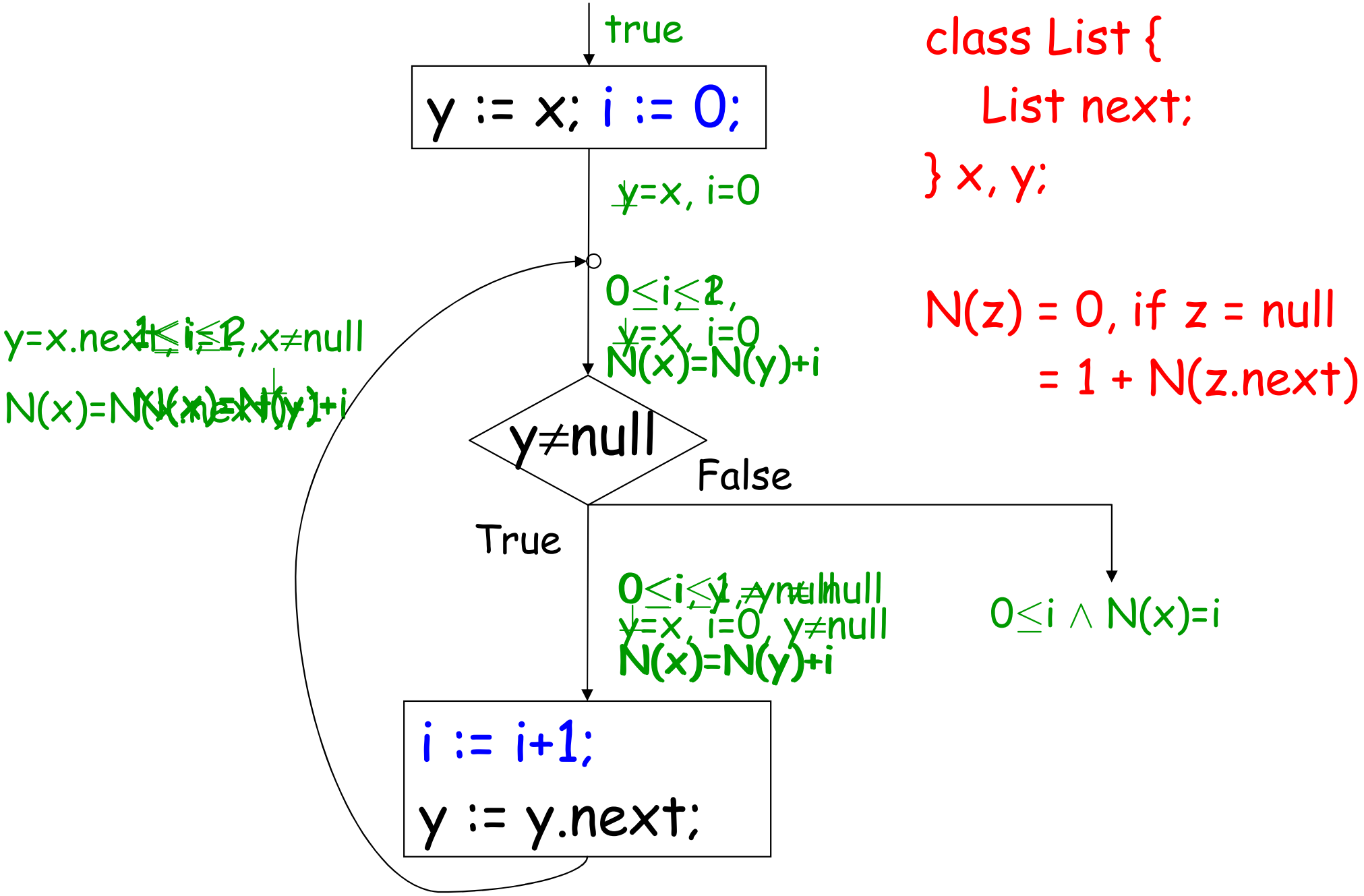
- $\text{Elimintate}_{T_{12}}(E_{12}, V)$:
  1. $\langle E_1, E_2 \rangle$:= Purify&Saturate(E_{12});
  2. $\langle D, Defs \rangle$:= DefSaturate(E_1, E_2, $V \cup$ Temp Variables);
  3. V' := $V \cup$ Temp Variables – D;
     E'_1 := $\text{Eliminate}_{T_1}(E_1, V')$;
     E'_2 := $\text{Eliminate}_{T_2}(E_2, V')$;
  4. $E := (E'_1 \wedge E'_2)$ [Defs(y)/y];
  5. Return E;

DefSaturate(E_1, E_2, U) returns the set of all variables D that have definitions Defs in terms of variables not in U as implied by $E_1 \wedge E_2$.

Combination: Example of Existential Elimination



Example: Abstract Interpretation over Combined Domain




```
class List {
    List next;
} x, y;
```

N(z) = 0, if z = null
= 1 + N(z.next)

Iterative Forward: References

- Uninterpreted Functions
  - A Polynomial-Time Algorithm for Global Value Numbering; Gulwani, Necula; SAS '04
- Linear Arithmetic + Uninterpreted Functions
  - Combining Abstract Interpreters; Gulwani, Tiwari; PLDI '06
- Theory of Arrays/Lists
  - Quantified Abstract Domains
    - Lifting Abstract Interpreters to Quantified Logical Domains; Gulwani, McCloskey, Tiwari; POPL '08
    - Discovering Properties about Arrays in Simple Programs; Halbwachs, Péron; PLDI '08
  - Shape Analysis
    - Parametric Shape Analysis via 3-Valued Logic; Sagiv, Reps, Wilhelm; POPL '99, TOPLAS '02

Iterative Forward: References

- Theory of Arrays/Lists
  - Combination of Shape Analysis + Arithmetic
    - A Combination Framework for tracking partition sizes; Gulwani, Lev-Ami, Sagiv; POPL '09
- Non-linear Arithmetic
  - User-defined axioms + Expression Abstraction
    - A Numerical Abstract Domain Based on Expression Abstraction and Max Operator with Application in Timing Analysis; Gulavani, Gulwani; CAV '08
  - Polynomial Equalities
    - An Abstract Interpretation Approach for Automatic Generation of Polynomial Invariants; Rodriguez-Carbonell, Kapur; SAS '04

Fixpoint Brush

- Iterative
  - Forward
  - ➢ Backward
- Constraint-based
- Proof-rules

Iterative Backward

- Comparison with Iterative Forward
  - Positives: Can compute preconditions, Goal-directed
  - Negatives: Requires assertions or template assertions.
- Transfer Function for Assignment Node is easier.
  - Substitution takes role of existential elimination.
- Transfer Function for Conditional Node is challenging.
  - Requires abductive reasoning/under-approximations.
    - Abduct(ϕ,g) = weakest ϕ' s.t. $\phi' \wedge g \Rightarrow \phi$
    - Case-split reasoning as opposed to Saturation based, and hence typically not closed under conjunctions.
    - Optimally weak solutions for negative unknowns as opposed to optimally strong solutions for positive unknowns.

Iterative Backward: References

- Program Verification using Templates over Predicate Abstraction;
  Srivastava, Gulwani; PLDI '09
- Assertion Checking Unified;
  Gulwani, Tiwari; VMCAI '07
- Computing Procedure Summaries for Interprocedural Analysis;
  Gulwani, Tiwari; ESOP '07

Fixpoint Brush

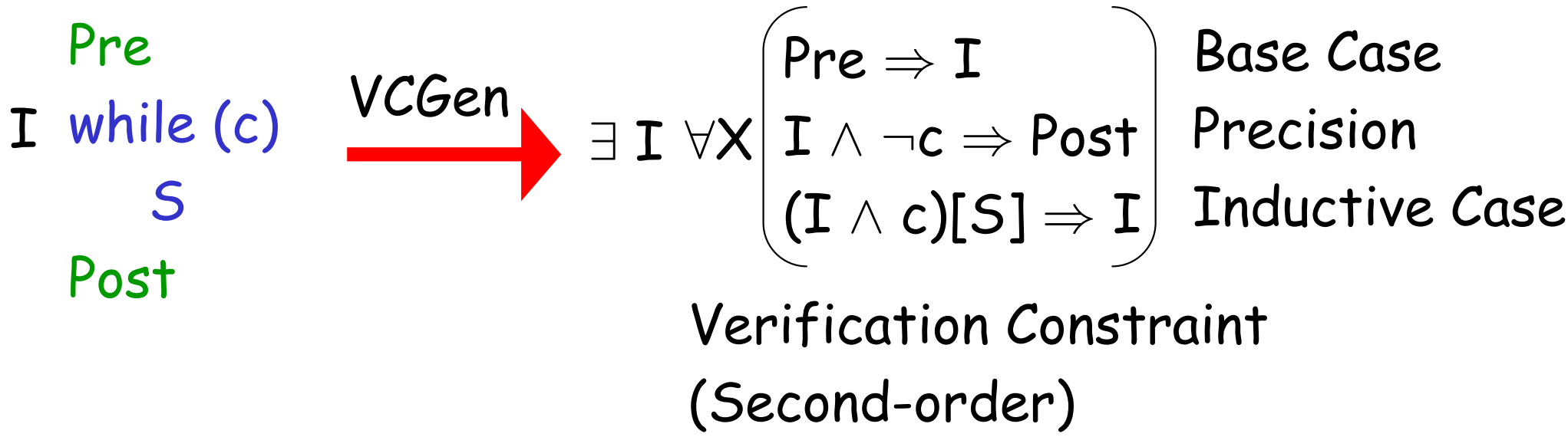
- Iterative
  - Forward
  - Backward

➢ Constraint-based

- Proof-rules

Constraint-based Invariant Generation

- Goal-directed invariant generation for verification of a Hoare triple (Pre, Program, Post)



- Key Idea: Reduce the second-order verification constraint to a first-order satisfiability constraint that can be solved using off-the-shelf SAT/SMT solvers
  - Choose a template for I (specific color/shade in some logic).
  - Convert ∀ into ∃.

Key Idea in reducing $\forall$ to $\exists$ for various Domains

Trick for converting $\forall$ to $\exists$ is known for following domains:

- Linear Arithmetic
  - Farkas Lemma
- Linear Arithmetic + Uninterpreted Fns.
  - Farkas Lemma + Ackerman's Reduction
- Non-linear Arithmetic
  - Grobner Basis
- Predicate Abstraction
  - Boolean indicator variables + Cover Algorithm (Abduction)
- Quantified Predicate Abstraction
  - Boolean indicator variables + More general Abduction

Constraint-based Invariant Generation: References

- Linear Arithmetic
  - Constraint-based Linear-relations analysis; Sankaranarayanan, Sipma, Manna; SAS '04
  - Program analysis as constraint solving; Gulwani, Srivastava, Venkatesan; PLDI '08
- Linear Arithmetic + Uninterpreted Fns.
  - Invariant synthesis for combined theories; Beyer, Henzinger, Majumdar, Rybalchenko; VMCAI '07
- Non-linear Arithmetic
  - Non-linear loop invariant generation using Gröbner bases; Sankaranarayanan, Sipma, Manna; POPL '04
- Predicate Abstraction
  - Constraint-based invariant inference over predicate abstraction; Gulwani, Srivastava, Venkatesan; VMCAI '09
- Quantified Predicate Abstraction
  - Program verification using templates over predicate abstraction; Srivastava, Gulwani; PLDI '09

Constraint-based Invariant Generation

- Linear Arithmetic
- Linear Arithmetic + Uninterpreted Fns.
- Non-linear Arithmetic
- Predicate Abstraction
- Quantified Predicate Abstraction

Farkas Lemma

$$\forall X \wedge_k (e_k \geq 0) \Rightarrow e \geq 0$$
$$\text{iff}$$
$$\exists \lambda_k \geq 0 \; \forall X \; (e \equiv \lambda + \sum_k \lambda_k e_k)$$

Example

- Let's find Farkas witness $\lambda, \lambda_1, \lambda_2$ for the implication
$$x \geq 2 \wedge y \geq 3 \Rightarrow 2x+y \geq 6$$
- $\exists \lambda, \lambda_1, \lambda_2 \geq 0$ s.t. $\forall x,y \; [2x+y-6 \equiv \lambda + \lambda_1(x-2) + \lambda_2(y-3)]$
- Equating coefficients of x,y and constant terms, we get:
$$2=\lambda_1 \wedge 1=\lambda_2 \wedge -6=\lambda-2\lambda_1-3\lambda_2$$
which implies $\lambda=1$, $\lambda_1=2$, $\lambda_2=1$.

Solving 2nd order constraints using Farkas Lemma

$\exists I\ \forall X\ \phi_1(I,X)$

- Second-order to First-order
  - Assume I has some form, e.g., $\sum_j a_j x_j \geq 0$
  - $\exists I\ \forall X\ \phi_1(I,X)$ translates to $\exists a_j\ \forall X\ \phi_2(a_j,X)$
- First-order to "only existentially quantified"
  - Farkas Lemma helps translate $\forall$ to $\exists$
  - $\forall X\ (\wedge_k(e_k \geq 0) \Rightarrow e \geq 0)$ iff $\exists \lambda_k \geq 0\ \forall X\ (e \equiv \lambda + \sum_k \lambda_k e_k)$
    - Eliminate X from polynomial equality by equating coefficients.
  - $\exists a_j\ \forall X\ \phi_2(a_j,X)$ translates to $\exists a_j\ \exists \lambda_k\ \phi_3(a_j,\lambda_k)$
- "only existentially quantified" to SAT
  - Bit-vector modeling for integer variables

Example

$[n=1 \wedge m=1]$
$x := 0; y := 0;$
I $\quad$ while $(x < 100)$
$\quad\quad x := x+n;$
$\quad\quad y := y+m;$
$[y \geq 100]$

VCGen →

$n=m=1 \wedge x=y=0 \Rightarrow I$

$I \wedge x \geq 100 \Rightarrow y \geq 100$

$I \wedge x<100 \Rightarrow I[x \leftarrow x+n, y \leftarrow y+m]$

Invariant Template	Satisfying Solution	Loop Invariant
$a_0 + a_1x + a_2y + a_3n + a_4m \geq 0$ $b_0 + b_1x + b_2y + b_3n + b_4m \geq 0$ $c_0 + c_1x + c_2y + c_3n + c_4m \geq 0$	$a_2=b_0=c_4=1, a_1=b_3=c_0=-1$ →	$y \geq x$ $m \geq 1$ $n \leq 1$
$a_0 + a_1x + a_2y + a_3n + a_4m \geq 0$ $b_0 + b_1x + b_2y + b_3n + b_4m \geq 0$	$a_2=b_2=1, a_1=b_1=-1$ →	$y \geq x$ $m \geq n$
$a_0 + a_1x + a_2y + a_3n + a_4m \geq 0$	UNSAT →	Invalid triple or Imprecise Template

Constraint-based Invariant Generation

- Linear Arithmetic
- Linear Arithmetic + Uninterpreted Fns.
- Non-linear Arithmetic
- ➢ Predicate Abstraction
- Quantified Predicate Abstraction

Solving 2nd order constraints using Boolean Indicator Variables + Cover Algorithm

$\exists I \; \forall X \; \phi_1(I,X)$

- Second-order to First-order (Boolean indicator variables)
  - Assume I has the form $P_1 \vee P_2$, where $P_1, P_2 \subseteq P$
    Let $b_{i,j}$ denote presence of $p_j \in P$ in P_i
    Then, I can be written as $(\wedge_j b_{1,j} \Rightarrow p_j) \vee (\wedge_j b_{2,j} \Rightarrow p_j)$
    $\exists I \; \forall X \; \phi_1(I,X)$ translates to $\exists b_{i,j} \; \forall X \; \phi_2(b_{i,j},X)$
  - Can generalize to k disjuncts

- First-order to "only existentially quantified"
  - Cover Algorithm helps translate $\forall$ to $\exists$
  - $\exists b_{i,j} \; \forall X \; \phi_2(b_{i,j},X)$ translates to SAT formula $\exists b_{i,j} \; \phi_3(b_{i,j})$

Example

```
[m > 0]
x := 0; y := 0;
I  while (x < m)
       x := x+1;
       y := y+1;
[y=m]
```

VCGen →

$$m>0 \Rightarrow I[x\leftarrow 0, y\leftarrow 0]$$
$$I \wedge x\geq m \Rightarrow y=m$$
$$I \wedge x<m \Rightarrow I[x\leftarrow x+1, y\leftarrow y+1]$$

Suppose P = $x\leq y,\ x\geq y,\ x<y$; $x\leq m,\ x\geq m,\ x<m$; $y\leq m,\ y\geq m,\ y<m$ and k = 1

Then, I has the form P_1, where $P_1 \subseteq P$

$$m>0 \Rightarrow P_1[x\leftarrow 0, y\leftarrow 0] \quad (1)$$
$$P_1 \wedge x\geq m \Rightarrow y=m \quad (2)$$
$$P_1 \wedge x<m \Rightarrow P_1[x\leftarrow x+1, y\leftarrow y+1] \quad (3)$$

Example

(1) $m>0 \Rightarrow P_1[x \leftarrow 0, y \leftarrow 0]$

$x \leq y,\ x \geq y,\ x<y$	$x \leftarrow 0, y \leftarrow 0$	$0 \leq 0,\ 0 \geq 0,\ 0<0$
$x \leq m,\ x \geq m,\ x<m$	→	$0 \leq m,\ 0 \geq m,\ 0<m$
$y \leq m,\ y \geq m,\ y<m$		$0 \leq m,\ 0 \geq m,\ 0<m$

Hence, (1) $\equiv$ P_1 doesn't contain $x<y,\ x \geq m,\ y \geq m$

$\equiv \neg b_{x \geq m} \wedge \neg b_{x<y} \wedge \neg b_{y \geq m}$

(2) $P_1 \wedge x \geq m \Rightarrow y=m$

There are 3 maximally-weak choices for P_1 (computed using Pred)

(i) $y \leq m \wedge y \geq m$

(ii) $x<m$

(iii) $x \leq y \wedge y \leq m$

Hence, (2) $\equiv$ P_1 contains at least one of above combinations

$\equiv (b_{x<m}) \vee (b_{y \leq m} \wedge b_{y \geq m}) \vee (b_{x \leq y} \wedge b_{y \leq m})$

Example

(1) $\neg b_{x\geq m} \wedge \neg b_{x<y} \wedge \neg b_{y\geq m}$

(2) $(b_{x<m}) \vee (b_{y\leq m} \wedge b_{y\geq m}) \vee (b_{x\leq y} \wedge b_{y\leq m})$

(3) $(b_{y\leq m} \Rightarrow (b_{y<m} \vee b_{y\leq x})) \wedge \neg b_{x<m} \wedge \neg b_{y<m}$

Obtained from solving local/small SMT queries

SAT Solver

$b_{y\leq x}$, $b_{y\leq m}$, $b_{x\leq y}$: true

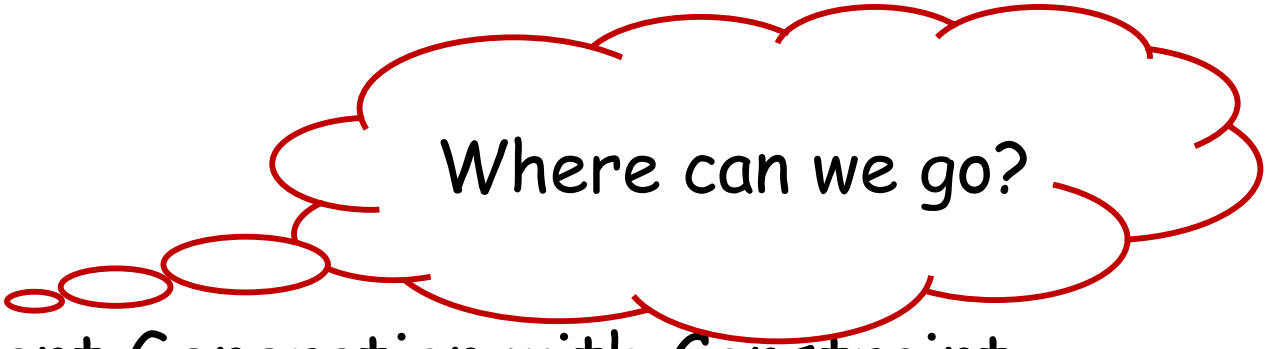
rest: false

I: $(y=x \wedge y\leq m)$

I

```
[m > 0]
x := 0; y := 0;
while (x < m)
    x := x+1;
    y := y+1;
[y=m]
```

Bonus Material



Going beyond Invariant Generation with Constraint-based techniques…

Example: Bresenham's Line Drawing Algorithm

```
[0<Y≤X]
v1:=2Y-X; y:=0; x:=0;
while (x ≤ X)
    out[x] := y;
    if (v1<0) v1:=v1+2Y;
    else v1:=v1+2(Y-X); y++;
return out;
[∀k (0≤k≤X ⇒ |out[k]-(Y/X)k| ≤ ½)]
```

Postcondition: The best fit line shouldn't deviate more than half a pixel from the real line, i.e., $|y - (Y/X)x| \leq 1/2$

Transition System Representation

$[0<Y\leq X]$
$v_1:=2Y-X;\ y:=0;\ x:=0;$
while $(x\leq X)$
    $v_1<0$: $out'=Update(out,x,y) \wedge v'_1=v_1+2Y \wedge y'=y \wedge x'=x+1$
    $v_1\geq 0$: $out'=Update(out,x,y) \wedge v_1=v_1+2(Y-X) \wedge y'=y+1 \wedge x'=x+1$
$[\forall k\ (0\leq k\leq X \Rightarrow |out[k]-(Y/X)k| \leq \frac{1}{2})]$

Or, equivalently,

[Pre]
s_{entry};
while (g_{loop})
    g_{body1}: s_{body1};
    g_{body2}: s_{body2};
[Post]

Where,
g_{body1}: $v_1<0$
g_{body2}: $v_1\geq 0$
g_{loop}: $x\leq X$
s_{entry}: $v_1'=2Y-X \wedge y'=0 \wedge x'=0$
s_{body1}: $out'=Update(out,x,y) \wedge v'_1=v_1+2Y \wedge x'=x+1 \wedge y'=y$
s_{body2}: $out'=Update(out,x,y) \wedge v'_1=v_1+2(Y-X) \wedge x'=x+1 \wedge y'=y+1$

Verification Constraint Generation & Solution

Verification Constraint:

$$
\begin{aligned}
&Pre \wedge s_{entry} \Rightarrow I' \\
&I \wedge g_{loop} \wedge g_{body1} \wedge s_{body1} \Rightarrow I' \\
&I \wedge g_{loop} \wedge g_{body2} \wedge s_{body2} \Rightarrow I' \\
&I \wedge \neg g_{loop} \Rightarrow Post
\end{aligned}
$$

Given Pre, Post, g_{loop}, g_{body1}, g_{body2}, s_{body1}, s_{body2}, we can find solution for I using constraint-based techniques.

I: $0<Y\leq X \wedge v_1=2(x+1)Y-(2y+1)X \wedge 2(Y-X)\leq v_1\leq 2Y \wedge \forall k(0\leq k\leq x \Rightarrow |out[k]-(Y/X)k| \leq \frac{1}{2})$

The Surprise!

Verification Constraint:

$$
\begin{aligned}
&Pre \wedge s_{entry} \Rightarrow I' \\
&I \wedge g_{loop} \wedge g_{body1} \wedge s_{body1} \Rightarrow I' \\
&I \wedge g_{loop} \wedge g_{body2} \wedge s_{body2} \Rightarrow I' \\
&I \wedge \neg g_{loop} \Rightarrow Post
\end{aligned}
$$

- What if we treat each g and s as unknowns like I?
- We get a solution that has $g_{body1} = g_{body2} = false$.
  - This doesn't correspond to a valid transition system.
  - We can fix this by encoding $g_{body1} \vee g_{body2} = true$.
- We now get a solution that has $g_{loop} = true$.
  - This corresponds to a non-terminating loop.
  - We can fix this by encoding existence of a ranking function.
- We now discover each g and s along with I.
  - We have gone from Invariant Synthesis to Program Synthesis.

Fixpoint Brush

- Iterative
  - Forward
  - Backward
- Constraint-based

➢ Proof-rules

Proof Rules

while (cond(X))

π: X' := F(X);

Bounding Loop Iterations

If $(cond(X) \wedge X'=F(X)) \Rightarrow (e>0 \wedge e[X'/X] \leq e-1)$,

Then $Bound(\pi) \leq e$

- Candidate expressions for e: Look inside Cond(X).

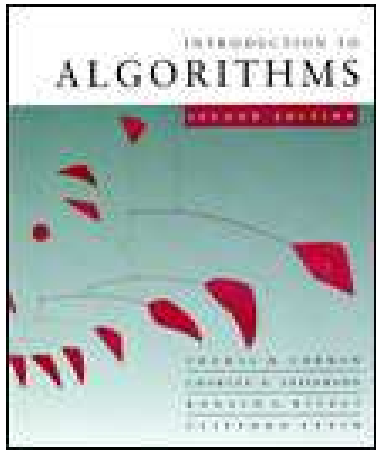
Bounding values of monotonically increasing variables

If $(cond(X) \wedge X'=F(X)) \Rightarrow y' \leq y+c)$,

Then $y^{out} \leq y^{in} + c \times Bound(\pi)$

- Candidate constants for c: 1, 2

Recurrence Solving Techniques vs. Our Fixpoint Brush



Undergraduate Textbook on Algorithms by Cormen, Leiserson, Rivest, Stein describes 3 fundamental methods for recurrence solving:

Example of a recurrence: $T(n)=T(n-1)+2n \wedge T(0)=0$

- Iteration Method
  - Expands/unfolds the recurrence relation
  - Similar to Iterative approach
- Substitution Method
  - Assumes a template for a closed-form
  - Similar to Constraint-based approach
- Masters Theorem
  - Provides a cook-book solution for $T(n)=aT(n/b)+f(n)$
  - Similar to Proof-Rules approach

Fixpoint Brush

- Iterative
  - Forward
  - Backward
- Constraint-based
- Proof-rules
- Learning
  - Iterative, but w/o monotonic increase or decrease.
  - Distance to fixed-point decreases in each iteration.

Learning: References

- Program Verification as Probabilistic Inference;
  Gulwani, Jojic; POPL '07
- Learning Regular Sets from Queries and Counterexamples;
  Angluin; Information and Computing '87
  - Learning Synthesis of interface specifications for Java classes;
    Alur, Madhusudan, Nam; POPL '05.
  - Learning meets verification;
    Leucker; FMCO '06
- May/Must Analyses?
- Game-theoretic Analyses?

Fixpoint Brush: Summary

- Iterative
  - Forward
    - Join, Existential Elimination
  - Backward
    - Abduct
- Constraint-based
  - Exotic; Works well for small code-fragments
  - $\forall$ to $\exists$
- Proof-rules
  - Scalable
  - Requires understanding design patterns
- Learning