Modelling and reasoning about references

A language with dynamic allocation

$$\tau ::= \mathrm{unit} \mid \mathrm{int} \mid \sigma\,\mathrm{ref} \mid \tau \times \tau \mid \tau + \tau \mid \tau \to \mathbf{T}\tau$$
$$\sigma ::= \mathrm{int} \mid \sigma\,\mathrm{ref}$$
$$\gamma ::= \tau \mid \mathbf{T}\tau$$

$$\begin{aligned} V &::= x \mid \underline{n} \mid \underline{\ell} \mid () \mid (V, V') \mid \mathrm{in}_i^{\tau} V \mid \mathrm{rec}\, f(x : \tau) : \tau' = M \\ M &::= V\, V' \mid \mathrm{let}\, x \Leftarrow M \,\mathrm{in}\, M' \mid \mathrm{val}\, V \mid \pi_i V \mid \mathrm{ref}\, V \mid !V \mid V := V' \\ &\quad \mid \mathrm{case}\, V \,\mathrm{of}\, \mathrm{in}_1 x \Rightarrow M \;;\; \mathrm{in}_2 x \Rightarrow M' \\ &\quad \mid V = V' \mid V + V' \mid \mathrm{iszero}\, V \end{aligned}$$

Store types Δ map locations $\ell \in \mathbb{L}$ to storable types σ

$$(rec)\frac{\Delta; \Gamma, x:\tau, f:\tau \to \mathbf{T}(\tau') \vdash M : \mathbf{T}(\tau')}{\Delta; \Gamma \vdash (\mathrm{rec}\ f(x{:}\tau){:}\tau' = M) : \tau \to \mathbf{T}(\tau')} \qquad (loc)\frac{\ell : \sigma \in \Delta}{\Delta; \Gamma \vdash \underline{\ell} : \sigma\ \mathrm{ref}}$$

$$(app)\frac{\Delta; \Gamma \vdash V_1 : \tau \to \mathbf{T}\tau' \quad \Delta; \Gamma \vdash V_2 : \tau}{\Delta; \Gamma \vdash V_1\ V_2 : \mathbf{T}\tau'}$$

$$(let)\frac{\Delta; \Gamma \vdash M_1 : \mathbf{T}(\tau_1) \quad \Delta; \Gamma, x:\tau_1 \vdash M_2 : \mathbf{T}(\tau_2)}{\Delta; \Gamma \vdash \mathrm{let}\ x \Leftarrow M_1\ \mathrm{in}\ M_2 : \mathbf{T}(\tau_2)} \qquad (val)\frac{\Delta; \Gamma \vdash V : \tau}{\Delta; \Gamma \vdash \mathrm{val}\ V : \mathbf{T}(\tau)}$$

$$(eq)\frac{\Delta; \Gamma \vdash V_1 : \sigma\ \mathrm{ref} \quad \Delta; \Gamma \vdash V_2 : \sigma\ \mathrm{ref}}{\Delta; \Gamma \vdash V_1 = V_2 : \mathbf{T}(\mathrm{unit} + \mathrm{unit})} \qquad (deref)\frac{\Delta; \Gamma \vdash V : \sigma\ \mathrm{ref}}{\Delta; \Gamma \vdash\ !V : \mathbf{T}\sigma}$$

$$(alloc)\frac{\Delta; \Gamma \vdash V : \sigma}{\Delta; \Gamma \vdash \mathrm{ref}\ V : \mathbf{T}(\sigma\ \mathrm{ref})} \qquad (assign)\frac{\Delta; \Gamma \vdash V_1 : \sigma\ \mathrm{ref} \qquad \Delta; \Gamma \vdash V_2 : \sigma}{\Delta; \Gamma \vdash V_1 := V_2 : \mathbf{T}(\mathrm{unit})}$$

Continuation-based termination relation

$$\Sigma, \mathrm{let}\ x \Leftarrow M \ \mathrm{in}\ K \quad \downarrow$$

$$\frac{}{\Delta; \vdash \mathrm{val}\, x : (x : \tau)^{\top}} \qquad \frac{\Delta; x : \tau \vdash M : \mathbf{T}\tau' \quad \Delta; \vdash K : (y : \tau')^{\top}}{\Delta; \vdash \mathrm{let}\ y \Leftarrow M \ \mathrm{in}\ K : (x : \tau)^{\top}}$$

States Σ map locations to $\mathbb{Z} + \mathbb{L}$

$$\frac{}{\Sigma, \mathrm{let}\ x \Leftarrow \mathrm{val}\ V\ \mathrm{in}\ \mathrm{val}\ x\ \downarrow} \qquad \frac{\Sigma, \mathrm{let}\ y \Leftarrow M[V/x]\ \mathrm{in}\ K\ \downarrow}{\Sigma, \mathrm{let}\ x \Leftarrow \mathrm{val}\ V\ \mathrm{in}\ (\mathrm{let}\ y \Leftarrow M\ \mathrm{in}\ K)\ \downarrow}$$

$$\frac{\Sigma, \mathrm{let}\ x_2 \Leftarrow M_1\ \mathrm{in}\ (\mathrm{let}\ x_1 \Leftarrow M_2\ \mathrm{in}\ K)\ \downarrow}{\Sigma, \mathrm{let}\ x_1 \Leftarrow (\mathrm{let}\ x_2 \Leftarrow M_1\ \mathrm{in}\ M_2)\ \mathrm{in}\ K\ \downarrow}$$

$$\frac{\Sigma, \mathrm{let}\ x_1 \Leftarrow M[V/x_2, (\mathrm{rec}\ f(x_2 : \tau_1) : \tau_2 = M)/f]\ \mathrm{in}\ K\ \downarrow}{\Sigma, \mathrm{let}\ x_1 \Leftarrow (\mathrm{rec}\ f(x_2 : \tau_1) : \tau_2 = M)\ V\ \mathrm{in}\ K\ \downarrow}$$

$$\frac{\Sigma, \mathrm{let}\ x \Leftarrow \mathrm{val}\ \mathit{false}\ \mathrm{in}\ K\ \downarrow}{\Sigma, \mathrm{let}\ x \Leftarrow \underline{\ell} = \underline{\ell'}\ \mathrm{in}\ K\ \downarrow}\ \ell \neq \ell' \qquad \frac{\Sigma[\ell \mapsto in_{\mathbb{L}}\ell'], \mathrm{let}\ x \Leftarrow \mathrm{val}\ ()\ \mathrm{in}\ K\ \downarrow}{\Sigma, \mathrm{let}\ x \Leftarrow \underline{\ell} := \underline{\ell'}\ \mathrm{in}\ K\ \downarrow}$$

$$\frac{\Sigma(\ell) = in_{\mathbb{L}}\ell' \quad \Sigma, \mathrm{let}\ x \Leftarrow \mathrm{val}\ \underline{\ell'}\ \mathrm{in}\ K\ \downarrow}{\Sigma, \mathrm{let}\ x \Leftarrow !\underline{\ell}\ \mathrm{in}\ K\ \downarrow}$$

$$\frac{\Sigma[\ell \mapsto in_{\mathbb{L}}\ell'], \mathrm{let}\ x \Leftarrow \mathrm{val}\ \underline{\ell}\ \mathrm{in}\ K\ \downarrow}{\Sigma, \mathrm{let}\ x \Leftarrow \mathrm{ref}\ \underline{\ell'}\ \mathrm{in}\ K\ \downarrow}\ \ell \notin locs(\Sigma) \cup locs(K) \cup \{\ell'\}$$

How to model such a language?

- Nondeterminism and invariance
- Encapsulation
- Functor categories
- FM cpos

Semantics of types

$$[\![\mathrm{unit}]\!] = 1 \qquad [\![\tau_1 \times \tau_2]\!] = [\![\tau_1]\!] \times [\![\tau_2]\!]$$

$$[\![\mathrm{int}]\!] = \mathbb{Z} \qquad [\![\tau_1 + \tau_2]\!] = [\![\tau_1]\!] + [\![\tau_2]\!]$$

$$[\![\sigma\ \mathrm{ref}]\!] = \mathbb{L} \qquad [\![\tau_1 \rightarrow \mathbf{T}\tau_2]\!] = [\![\tau_1]\!] \Rightarrow \mathbf{T}[\![\tau_2]\!]$$

$$\mathbf{T}D \ = \ (\mathbb{S} \Rightarrow D \Rightarrow \mathbb{O}) \multimap (\mathbb{S} \Rightarrow \mathbb{O})$$

$$\mathbb{S} = \mathbb{L} \Rightarrow (\mathbb{Z} + \mathbb{L})$$

$$[\![\Delta;\Gamma \vdash \underline{\ell} : \sigma\ \mathrm{ref}]\!]\ \rho \;=\; \ell$$

$$[\![\Delta;\Gamma \vdash \mathrm{let}\ x \Leftarrow M_1\ \mathrm{in}\ M_2 : \mathbf{T}\tau_2]\!]\ \rho\ k\ S \;=\;$$
$$[\![\Delta;\Gamma \vdash M_1 : \mathbf{T}\tau_1]\!]\ \rho\ (\lambda S' : \mathbb{S}.\lambda d : [\![\tau_1]\!].[\![\Delta;\Gamma, x : \tau_1 \vdash M_2 : \mathbf{T}\tau_2]\!]\ \rho[x \mapsto d]\ k\ S')\ S$$

$$[\![\Delta;\Gamma \vdash \mathrm{val}\ V : \mathbf{T}\tau]\!]\ \rho\ k\ S \;=\; k\ S\ ([\![\Delta;\Gamma \vdash V : \tau]\!]\ \rho)$$

$$[\![\Delta;\Gamma \vdash\ !V : \mathbf{T}\sigma]\!]\ \rho\ k\ S \;=\; \begin{cases} k\ S\ v\ \text{if}\ S([\![\Delta;\Gamma \vdash V : \sigma\ \mathrm{ref}]\!]\ \rho) = in_{[\![\sigma]\!]}v \\ \bot \qquad \text{otherwise} \end{cases}$$

$$[\![\Delta;\Gamma \vdash V_1 := V_2 : \mathbf{T}\mathrm{unit}]\!]\ \rho\ k\ S \;=\;$$
$$k\ S[([\![\Delta;\Gamma \vdash V_1 : \sigma\ \mathrm{ref}]\!]\ \rho) \;\mapsto\; in_{[\![\sigma]\!]}([\![\Delta;\Gamma \vdash V_2 : \sigma]\!]\ \rho)]\ *$$

$$[\![\Delta;\Gamma \vdash \mathrm{ref}\ V : \mathbf{T}\sigma\ \mathrm{ref}]\!]\ \rho\ k\ S \;=\; k\ S[\ell \mapsto in_{[\![\sigma]\!]}([\![\Delta;\Gamma \vdash V : \sigma]\!]\rho)]\ \ell$$
$$\text{for some/any}\ \ell \notin supp(\lambda\ell'.k\ S[\ell' \mapsto in_\sigma([\![\Delta;\Gamma \vdash V : \sigma]\!]\rho)]\ \ell').$$

$$[\![\Delta;\Gamma \vdash (\mathrm{rec}\ f\ x = M) : \tau \to \mathbf{T}\tau']\!]\ \rho \;=\;$$
$$fix(\lambda f' : [\![\tau \to \mathbf{T}\tau']\!].\lambda x' : [\![\tau]\!].[\![\Delta;\Gamma, f : \tau \to \mathbf{T}\tau', x : \tau \vdash M : \mathbf{T}\tau']\!]\ \rho[f \mapsto f', x \mapsto x'])$$

Soundness and adequacy

If $\Delta; \vdash M : \mathbf{T}\tau$, $\Delta; \vdash K : (x : \tau)^\tau$, $\Sigma : \Delta$ and $S \in [\![\Sigma]\!]$ then

$$\Sigma, \mathrm{let}\, x \Leftarrow M \;\mathrm{in}\; K \downarrow \iff [\![\Delta; \vdash M : \mathbf{T}\tau]\!]\{\}[\![\Delta; \vdash K : (x : \tau)^\top]\!]^{\mathcal{K}}\, S = \top.$$

as a corollary

$$[\![\Delta; \Gamma \vdash G_1 : \gamma]\!] = [\![\Delta; \Gamma \vdash G_2 : \gamma]\!] \text{ implies } \Delta; \Gamma \vdash G_1 =_{\mathrm{ctx}} G_2 : \gamma$$

(in)equivalences

$$\frac{\Delta;\Gamma \vdash V_1 : \sigma_1 \quad \Delta;\Gamma \vdash V_2 : \sigma_2 \quad \Delta;\Gamma, x : \sigma_1\,\mathrm{ref}, y : \sigma_2\,\mathrm{ref} \vdash N : \mathbf{T}\tau}{\Delta;\Gamma \vdash \begin{array}{l} \quad\ \mathrm{let}\ x \Leftarrow \mathrm{ref}\ V_1\ \mathrm{in}\ (\mathrm{let}\ y \Leftarrow \mathrm{ref}\ V_2\ \mathrm{in}\ N) \\ =_{\mathrm{ctx}} \mathrm{let}\ y \Leftarrow \mathrm{ref}\ V_2\ \mathrm{in}\ (\mathrm{let}\ x \Leftarrow \mathrm{ref}\ V_1\ \mathrm{in}\ N) : \mathbf{T}\tau \end{array}}$$

$$\frac{\Delta;\Gamma \vdash V : \sigma \quad \Delta;\Gamma \vdash N : \mathbf{T}\tau}{\Delta;\Gamma \vdash \mathrm{let}\ x \Leftarrow \mathrm{ref}\ V\ \mathrm{in}\ N \ =_{\mathrm{ctx}}\ N : \mathbf{T}\tau}\ x \notin fvN$$

A Parametric Logical Relation

- Partially ordered set of parameters *p*
- Parameter-indexed relations

$$\forall p.\, \mathcal{R}_{\mathbb{S}}(p) \subseteq \mathbb{S} \times \mathbb{S}$$

$$\forall p.\, \forall \gamma.\, \mathcal{R}_{\gamma}(p) \subseteq [\![\gamma]\!] \times [\![\gamma]\!]$$

- Show denotation of each term related to itself
- Corollary: terms with related denotations are contextually equivalent

Accessibility maps

- Support turns out not to help in defining "the part of the store about which a relation depends":

$$\{(S_1, S_2) \mid \exists \ell, S_1 \ell = 0 = S_2 \ell\}$$

An accessibility map A is a function from $\mathbb{S}$ to finite subsets of $\mathbb{L}$, such that:

$$\forall S, S' \in \mathbb{S}, (\forall \ell \in AS, S\ell = S'\ell) \implies A(S) = A(S')$$

The subtyping ordering $<:$ is defined as:

$$A <: A' \iff \forall S, A(S) \supseteq A'(S)$$

Accessibility maps from state types

If Δ is a state type, then $\mathsf{Acc}_\Delta : \mathbb{S} \rightarrow \mathbb{P}_{fin}(\mathbb{L})$ is defined by $\mathsf{Acc}_\Delta(S) = \bigcup_{(\ell:\sigma)\in\Delta} \mathsf{Acc}(\ell, \sigma, S)$

where $\mathsf{Acc}(\ell, \mathsf{int}, S) \stackrel{def}{=} \{\ell\}$ and

$$\mathsf{Acc}(\ell, \sigma\mathsf{ref}, S) \stackrel{def}{=} \{\ell\} \cup \begin{cases} \mathsf{Acc}(\ell', \sigma, S) & \text{if } S\,\ell = \mathsf{in}_{\mathbb{L}}\ell' \\ \emptyset & \text{otherwise} \end{cases}$$

If A is an accessibility map, we define $S \sim S' : A$ to mean $\forall \ell \in A(S), S\ell = S'\ell$.

Finitary state relations

A finitary state relation r is a pair $\langle |r|, A_r \rangle$ where $|r| \subseteq \mathbb{S} \times \mathbb{S}$ and A_r is an accessibility map, subject to the following saturation condition: if $S_1 \sim S_1' : A_r$ and $S_2 \sim S_2' : A_r$ then $(S_1, S_2) \in |r| \iff (S_1', S_2') \in |r|$.

Given two finitary state relations, $r_1 = \langle |r^1|, A^1 \rangle$ and $r_2 = \langle |r^2|, A^2 \rangle$, define

$$r^1 \otimes r^2 \stackrel{def}{=} \langle |r^1 \otimes r^2|, A^1 \wedge A^2 \rangle$$

where

$$(S_1, S_2) \in |r^1 \otimes r^2| \iff \begin{cases} (S_1, S_2) \in |r^1| \cap |r^2| \\ \forall i \in \{1, 2\}, A^1(S_i) \cap A^2(S_i) = \emptyset \end{cases}$$

Parameters

A parameter is a pair (Δ, r), where Δ is a state type and r is a finitary relation; we will abbreviate this to Δr. If Δr is a parameter, we define the binary relation on states $\mathcal{R}_{\mathbb{S}}(\Delta r) \stackrel{def}{=} |id_{\Delta} \otimes r|$ and define the partial order $\triangleright$ on parameters by

$$\Delta r \triangleright \Delta' r' \iff (\Delta \supseteq \Delta') \wedge (\exists r'', r = r' \otimes r'')$$

Logical Relation

$$\begin{array}{rcl}
\mathcal{R}_{\mathsf{unit}}(\Delta r) & = & \{(*,*)\} \\
\mathcal{R}_{\mathsf{int}}(\Delta r) & = & \{(n,n) \mid n \in N\} \\
\mathcal{R}_{\sigma\ \mathsf{ref}}(\Delta r) & = & \{(\ell,\ell) \mid (\ell : \sigma) \in \Delta\} \\
\mathcal{R}_{\tau \to \mathbf{T}\tau'}(\Delta r) & = &
\end{array}$$

$$\{(f_1, f_2) \mid \forall \Delta' r' \rhd \Delta r, (v_1, v_2) \in \mathcal{R}_\tau(\Delta' r'), (f_1 v_1, f_2, v_2) \in \mathcal{R}_{\mathbf{T}\tau'}(\Delta' r')\}$$

For continuations, we define $\mathcal{R}_{\tau^\top}(\Delta r)$ to be

$$\{(k_1, k_2) \mid \forall \Delta' r' \rhd \Delta r, (v_1, v_2) \in \mathcal{R}_\tau(\Delta' r'), (S_1, S_2) \in \mathcal{R}_{\mathbb{S}}(\Delta' r'), \\ k_1 S_1 v_1 = k_2 S_2 v_2\}$$

and for computations, $\mathcal{R}_{\mathbf{T}\tau}(\Delta r)$ is defined as

$$\{(f_1, f_2) \mid \forall \Delta' r' \rhd \Delta r, (k_1, k_2) \in \mathcal{R}_{\tau^\top}(\Delta' r'), (S_1, S_2) \in \mathcal{R}_{\mathbb{S}}(\Delta' r'), \\ f_1 k_1 S_1 = f_2 k_2 S_2\}$$

Why?

- Fundamental Lemma:

If $\Delta; \Gamma \vdash G : \gamma$, then

$$\forall r. \, (\llbracket \Delta; \Gamma \vdash G : \gamma \rrbracket, \llbracket \Delta; \Gamma \vdash G : \gamma \rrbracket) \in \mathcal{R}_{\Gamma \vdash \gamma}(\Delta r).$$

- Soundness of relational reasoning:

If $\Delta; \Gamma \vdash G_i : \gamma$ for $i = 1, 2$ and

$$(\llbracket \Delta; \Gamma \vdash G_1 : \gamma \rrbracket, \llbracket \Delta; \Gamma \vdash G_2 : \gamma \rrbracket) \in \mathcal{R}_{\Gamma \vdash \mathbf{T}\tau}(\Delta \top)$$

then $\Delta; \Gamma \vdash G_1 =_{\mathsf{ctx}} G_2 : \gamma$.

Examples

- The garbage collection rule from earlier
- All the Meyer-Sieber examples, e.g

$$
\begin{array}{l}
\mathsf{let}\ x \Leftarrow \mathsf{ref}\ \underline{0}\ \mathsf{in} \\
\qquad \mathsf{let}\ almost\_add2 \Leftarrow \lambda z.\mathsf{if}\ z = x \\
\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{then}\ x := 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{else\ let}\ y \Leftarrow !x\ \mathsf{in\ let}\ y' \Leftarrow y + 2\ \mathsf{in}\ x := y' \mathsf{in} \\
\qquad\qquad p(almost\_add2); \\
\qquad\qquad \mathsf{let}\ y \Leftarrow !x\ \mathsf{in} \\
\qquad\qquad\qquad \mathsf{if}\ !x\ \mathsf{mod}\ 2 = 0\ \mathsf{then\ diverge}_{\mathsf{unit}}\ \mathsf{else\ val}\ ()
\end{array}
$$

Examples

- Pointers between hidden and visible parts:

$$\begin{aligned}&\mathsf{let}\ x \Leftarrow \mathsf{ref}\ 0\ \mathsf{in}\\ &\mathsf{let}\ y \Leftarrow \mathsf{ref}\ x\ \mathsf{in}\\ &p\ \ x;\\ &\mathsf{let}\ z \Leftarrow !y\ \mathsf{in}\\ &\mathsf{if}\ z = x\ \mathsf{then}\ \mathsf{diverge}_{\mathsf{unit}}\ \mathsf{else}\ \mathsf{val}\ ()\end{aligned}$$

- Some very artificial encodings of crypto a la Sumii and Pierce

Non-examples ☹

$$
\begin{aligned}
M = \quad & \mathsf{let}\, x \Leftarrow \mathsf{ref}\, 0\, \mathsf{in} \\
& \quad p(\lambda\_\,.\, x := 1;\ 0); \\
& \quad \mathsf{let}\, y \Leftarrow !x\, \mathsf{in} \\
& \quad \mathsf{if\ iszero}\, y\ \mathsf{then\ val}\,()\ \mathsf{else\ diverge}_{\mathsf{unit}}
\end{aligned}
$$

$$N = \quad p\ (\lambda\_\,.\, \mathsf{diverge}_{\mathsf{int}})$$

$$\mathsf{snapback}\ f\ k\ S \ = \ f\ *\ (\lambda S'.\lambda n.k\ S\ n)\ S$$

Garbage Collection If x is not free in M, and $\Delta;\Gamma \vdash M : \mathbf{T}\tau$, then

$$\Gamma \vdash \text{let } x \Leftarrow \text{ref } V \text{ in } M =_{\text{ctx}} M : \mathbf{T}\tau$$

We prove that $[\![\text{let } x \Leftarrow \text{ref } V \text{ in } M]\!]$ and $[\![M]\!]$ are related by $\mathcal{R}_{\Gamma\vdash\mathbf{T}\tau}(\Delta\top)$, and we conclude using Theorem 17. Let $\Delta' r' \rhd \Delta\top$ be a parameter and $(\rho_1, \rho_2) \in \mathcal{R}_{\Gamma}(\Delta' r')$. We need to prove that $([\![\text{let } x \Leftarrow \text{ref } V \text{ in } M]\!]\rho_1, [\![M]\!]\rho_2) \in \mathcal{R}_{\mathbf{T}\tau}(\Delta' r')$. Let $\Delta'' r'' \rhd \Delta' r'$, $(k_1, k_2) \in \mathcal{R}_{\tau^\top}(\Delta'' r'')$ and $(S_1, S_2) \in \mathcal{R}_{\mathbb{S}}(\Delta'' r'')$. We have to prove that

$$[\![\text{let } x \Leftarrow \text{ref } V \text{ in } M]\!]\rho_1 k_1 S_1 = [\![M]\!]\rho_2 k_2 S_2$$

For $\ell \notin supp(\lambda\ell'.k_1 S_1[\ell' \to [\![V]\!]\rho]\ell')$

$$[\![\text{let } x \Leftarrow \text{ref } V \text{ in } M]\!]\rho_1 k_1 S_1 = [\![M]\!]\rho_1 k_1 S_1[\ell \to [\![V]\!]\rho_1]$$

because x is not free in M. Since we can pick *any* such ℓ, we actually choose one also out of $\text{Acc}_{\Delta''}(S_i) \cup A_{r''}(S_i)$ for $i = 1, 2$. By the fundamental lemma, $[\![M]\!]$ is related to itself by $\mathcal{R}_{\Gamma\vdash\mathbf{T}\tau}(\Delta\top)$, so if we prove that $(S_1[\ell \to [\![V]\!]\rho_1], S_2) \in \mathcal{R}_{\mathbb{S}}(\Delta'' r'')$ we are done.

First, since $\ell \notin \text{Acc}_{\Delta''}(S_i)$, $(S_1[\ell \to [\![V]\!]\rho_1], S_2) \in id_{\Delta''}$, and since $\ell \notin A_{r''}(S_i)$, $(S_1[\ell \to [\![V]\!]\rho_1], S_2) \in r''$. By definition of accessibility maps, $\text{Acc}_{\Delta''}$ and $A_{r''}$ are unchanged, so they still do not overlap, which concludes the proof.