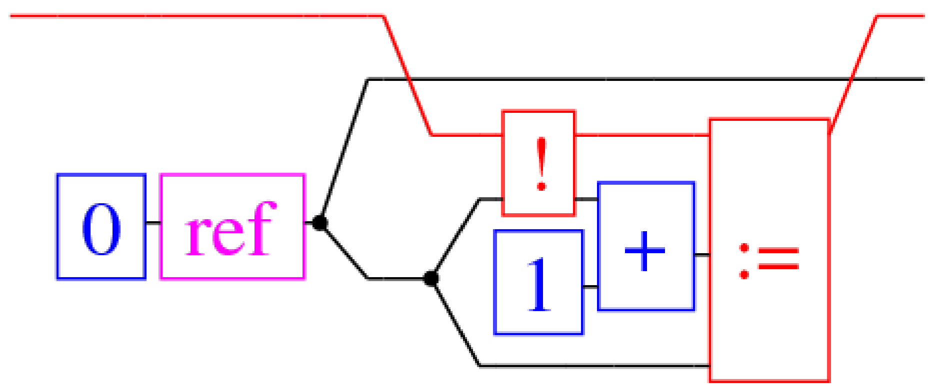
Reasoning about effectful programs: the state of the art

Perspectives

- Abstract structure of effectful programs
- Concrete models for languages with effects
- Logics and reasoning principles
- Proofs of particular programs
- Mechanization
- Automation and tool-building
- Language design and programming patterns

Arrows, premonoidal and Freyd categories

- Power, Robinson, Hughes, Atkey
- Rather than requiring a monad, just ask for identity on objects product-preserving functor from monoidal base category (values) to premonoidal category of computations
- Roughly corresponds to arrow abstraction in Haskell
  - Used for, for example, functional reactive programming
- Nice syntax due to Patterson, Wadler et al



Parameterized monads

- Atkey
- $T : A^{op} \times A \times Set \to Set$
- $\eta_{aX} : X \to T(a,a,X)$
- $\mu_{abcXY} : T(a,b,X) \times (X \to T(b,c,Y)) \to T(a,c,Y)$
- Subject to laws…
- Examples
  - Monads $T(A,B,X)=MX$
  - State transformers $T(S1,S_2,X)=S_1 \to S_2 \times X$
    - Monoidal structure on parameters allows separated states too
  - Composable continuations $T(R1,R_2,X)=(X \to R_1) \to R_2$
- Further applications to, for example, permissions and session types

Algebraic effects

- Plotkin, Power, Hyland, Pretnar, Staton,…
- Focus on operations and equations
- Generate monads from them
- Composite monads from combinations of algebraic theories (sum and tensor (commutation))
- Add generic handlers (destructors) for effects to account for e.g. catching exceptions
- Pretnar and Bauer building a language on these ideas (eff)

$$
\begin{array}{l}
\mathrm{a} : \mathbb{A} \vdash \mathsf{let}\ \mathrm{v} \Leftarrow !\mathrm{a}\ \mathsf{in}\ \mathrm{a} := \mathrm{v}\ \approx\ ()\ : 1 \\
\mathrm{a} : \mathbb{A} \vdash \mathsf{let}\ \mathrm{v} \Leftarrow !\mathrm{a}\ \mathsf{in}\ \mathsf{let}\ \mathrm{w} \Leftarrow !\mathrm{a}\ \mathsf{in}\ (\mathrm{v}, \mathrm{w})\ \approx\ \mathsf{let}\ \mathrm{v} \Leftarrow !\mathrm{a}\ \mathsf{in}\ (\mathrm{v}, \mathrm{v})\ : \mathbb{V} \times \mathbb{V} \\
\mathrm{a} : \mathbb{A}, \mathrm{v}, \mathrm{w} : \mathbb{V} \vdash \mathrm{a} := \mathrm{v};\ \mathrm{a} := \mathrm{w}\ \approx\ \mathrm{a} := \mathrm{w}\ : 1 \\
\mathrm{a} : \mathbb{A}, \mathrm{v} : \mathbb{V} \vdash \mathrm{a} := \mathrm{v};\ \mathsf{let}\ \mathrm{w} \Leftarrow !\mathrm{a}\ \mathsf{in}\ \mathrm{w}\ \approx\ \mathrm{a} := \mathrm{v};\ \mathrm{v}\ : \mathbb{V} \\
\mathrm{a}, \mathrm{b} : \mathbb{A} \vdash \mathsf{let}\ \mathrm{v} \Leftarrow !\mathrm{a}\ \mathsf{in}\ \mathsf{let}\ \mathrm{w} \Leftarrow !\mathrm{b}\ \mathsf{in}\ (\mathrm{v}, \mathrm{w}) \\
\qquad\qquad\qquad\qquad \approx\ \ \mathsf{let}\ \mathrm{w} \Leftarrow !\mathrm{b}\ \mathsf{in}\ \mathsf{let}\ \mathrm{v} \Leftarrow !\mathrm{a}\ \mathsf{in}\ (\mathrm{v}, \mathrm{w})\ : \mathbb{V} \times \mathbb{V} \\
(\mathrm{a}, \mathrm{b}) : \mathbb{A} \otimes \mathbb{A}, \mathrm{v}, \mathrm{w} : \mathbb{V} \vdash \mathrm{a} := \mathrm{v};\ \mathrm{b} := \mathrm{w}\ \approx\ \mathrm{b} := \mathrm{w};\ \mathrm{a} := \mathrm{v}\ : 1 \\
(\mathrm{a}, \mathrm{b}) : \mathbb{A} \otimes \mathbb{A}, \mathrm{v} : \mathbb{V} \vdash \mathrm{a} := \mathrm{v};\ !\mathrm{b}\ \approx\ \mathsf{let}\ \mathrm{w} \Leftarrow !\mathrm{b}\ \mathsf{in}\ \mathrm{a} := \mathrm{v};\ \mathrm{w}\ : \mathbb{V}
\end{array}
$$

Biorthogonality and TT-lifting

- Krivine, Pitts, Katsumata
- Given configurations (p,e) and observation $O \subseteq P \times E$
  - If $P \subseteq$ Progs, $P^{\top} \subseteq$ Envs = $\{e \mid \forall p \in P, (p,e) \in O\}$
  - If $E \subseteq$ Envs, $E^{\top} \subseteq$ Progs = $\{p \mid \forall e \in E, (p,e) \in O$
  - $(.)^{\top\top}$ is a closure operator on 2^{Progs} that "contextualizes" properties
- Gives general approach to lifting logical relations to monadic types (via continuation monad transformer)
- Saw this twice already (logical relation for imperative language, QPER closure in relation for effect system)
- Also used in relations for HO store, compiler correctness, realizability

Enriched effect calculus

- Simson, Egger, Mogelberg
- Generalizes Levy's CBPV and the relationship between the monadic calculus and linear logic to the non-commutative case
- Results on linearly used continuations, parametricity, type isomorphisms etc.
- Promising metalanguage for work on effects

Value types:

$$\mathsf{A}, \mathsf{B}, \ldots \ ::= \ \alpha \mid \underline{\alpha} \mid 1 \mid \mathsf{A} \times \mathsf{B} \mid \mathsf{A} \to \mathsf{B} \mid !\mathsf{A}$$
$$\mid \underline{\mathsf{A}} \multimap \underline{\mathsf{B}} \mid !\mathsf{A} \otimes \underline{\mathsf{B}} \mid \underline{0} \mid \underline{\mathsf{A}} \oplus \underline{\mathsf{B}}$$

Computation types:

$$\underline{\mathsf{A}}, \underline{\mathsf{B}}, \ldots \ ::= \ \underline{\alpha} \mid 1 \mid \underline{\mathsf{A}} \times \underline{\mathsf{B}} \mid \mathsf{A} \to \underline{\mathsf{B}} \mid !\mathsf{A}$$
$$\mid !\mathsf{A} \otimes \underline{\mathsf{B}} \mid \underline{0} \mid \underline{\mathsf{A}} \oplus \underline{\mathsf{B}} \ .$$

Separation logic

- Reynolds, O'Hearn, Bornat, Ishtiaq, Yang, Parkinson, and cast of thousands
- Extends Hoare logic to allow local reasoning about programs that manipulate data structures in the heap
  - Separating conjunction
  - Frame rule
- Revitalised theory of program verification, huge influence on theory and practice
- Tools: Smallfoot, jStar, SpaceInvader, SLAyer,…
  - Memory safety proofs for large bits of industrial code
- Extends beautifully to concurrency
  - Resources, permissions
  - Rely-guarantee reasoning

Higher-order extensions of separation logic

- Higher order functions, flat store (Birkedal, Torp-Smith, Yang)
  - Higher-order frame rules, FM-cpos, relational parametricity and TT-lifting
- Higher order functions and higher-order store (Schwinghammer, Birkedal, Reus, Yang)
  - Foundationally challenging because of, e.g. recursion through the store
  - Nested triples, Kripke semantics, worlds defined recursively using ultrametric spaces

Refining the state monad with dependent types

- Nanevski, Morrisett, Birkedal,…
- Hoare Type Theory {P}x:A{Q} where P,Q are predicates on the state, x binds return value in postcondition
- Embedded in Coq, full dependent types
- Can include separation logic assertions
- Generates VCS, powerful automation
- Cool examples: tricky imperative datastructures, web services, database management system

Relational Hoare Logic

- B, Yang. Extend Hoare logic to reason about multiple runs of programs
- Allows proofs of contextual program transformations enabled by compiler analyses
- Expresses various dependency analyses: information flow, program slicing
- CertiCrypt (Zanella, Barthe,…) probabilistic variant in Coq used to verify digital signature schemes by program transformation
- Relational HTT (Nanevski, Banerjee, Garg) expresses and verifies information flow and access control policies for programs with higher-order functions, dynamically allocated references

Monadic reflection

- Filinski
- Layering of one effect on top of another via monad trasnformers amounts to translation of language with complex effect into one with simpler effect
- Monadic reflection allows one to move between effects as behaviour and effects as data

$$\frac{\Gamma \vdash V : T\alpha}{\Gamma \vdash \mu(V) : \alpha} \qquad and \qquad \frac{\Gamma \vdash E : \alpha}{\Gamma \vdash [E] : T\alpha}$$

- Delimited control is a universal effect
- And can be implemented via call/cc and a reference cell

```
- fun apptwice f = (f 1; f 2; "done");
val apptwice : (int->unit)->string

- val tapptwice = translate ((int-->unit)-->string) apptwice;
val tapptwice : (int->unit t)->string t
```

Taming circularity

- Traditional domain theory (Scott,…)
- Step-indexing (Appel, McAllester, Ahmed)
- Ultrametric spaces and the topos of trees $\mathrm{Sets}^{\omega^{\mathrm{op}}}$ (Birkedal et al)
  - Modalities for well-founded recursion

Models of languages with higher-typed store (and other hard features)

- Domain-theoretic (Bohr, Birkedal)
- Step-indexed Kripke logical relations over operational semantics (Ahmed, Dreyer, Rossberg, Neis, Birkedal)
  - Possible worlds become state transition systems
  - Associated modal logics
  - Fine analysis of the effect on reasoning of flat/ho references, control operators
  - Biorthogonals and much more besides
  - Fully abstract and practically applicable to tricky examples of imperative ADTs

Game semantics

- Abramsky, Ghica, Honda, McCusker, Tzevelekos
- Types as games, terms as strategies
- Has provided fully abstract models for many different kinds of programming languages
- Applied to nu-calc, pi-calc, references Recently extended with nominal structure to provide fully abstract models of ML-like languages
- Game-like ideas apparent in structure of possible worlds in recent logical relations models

Compiler Correctness

- Compcert (Leroy)
- FPCC (Appel,…)
  - Translate high-level types into foundational logic
  - Memory safety
- Compositional compiler correctness (B, Hur, Tabareau, Dreyer)
  - Semantic type soundness
  - Full functional correctness
  - Biorthogonals, separation, logical relations, step indices,…
  - Hur & Dreyer can even verify self-modifying code

```
Fixpoint semantics_of_types (t:ExpType) (Ra:stateRel) ptr ptr' struct t :=
  match t with
  | Int P ⇒ lift (P ptr ∧ (ptr = ptr'))
  | Bool P ⇒ lift (P (n2b ptr) ∧ (n2b ptr = n2b ptr'))
  | a ∗ b  ⇒ Ex value, Ex value2, Ex value', Ex value2',
          (ptr,ptr'↦value,value') ×
           (ptr+1,ptr'+1↦value2,value2') × ⟦b⟧ Ra value value' × ⟦a⟧ Ra value2 value2')
  | a ⟶ b  ⇒ Ex Rprivate,
          (ptr,ptr' ↦ Later ( Perp (Pre_arrow Rprivate ptr ptr' Ra (⟦a⟧) (⟦b⟧))) × Rprivate)
  end
  where "'⟦' t '⟧'" := (semantics_of_types t ).
```

```
Definition Post_arrow b (Ra Rc: stateRel) Rc_cloud (n n' stack_ptr stack_ptr': nat):=
  Ex ptr_result, Ex ptr_result',
    (stack_ptr,stack_ptr' ↦ ptr_result,ptr_result') ⊗ (stack_ptr+1,stack_ptr'+1↦-) ⊗
    ((b Ra ptr_result ptr_result') × Rc_cloud) ⊗ Ra ⊗ Rc ⊗ (spreg↦ stack_ptr,stack_ptr') ⊗
    (envreg↦ n,n') ⊗ unused_space.
```

```
Definition Pre_arrow R_private ptr_function ptr_function' Ra a b:=
  Ex Rc, Ex Rc_cloud, Ex n, Ex n', Ex ptr_arg, Ex ptr_arg', Ex stack_ptr, Ex stack_ptr',
    (stack_ptr,stack_ptr'↦ ptr_arg,ptr_arg') ⊗
    (stack_ptr+1,stack_ptr'+1↦ ptr_function,ptr_function')
    ⊗ (R_private × a Ra ptr_arg ptr_arg' × Rc_cloud) ⊗
    ((n+4,n'+4 ↦ Later (Perp (Post_arrow b Ra Rc Rc_cloud n n' stack_ptr stack_ptr'))) × Rc) ⊗
    Ra ⊗ (spreg↦ stack_ptr+1,stack_ptr'+1) ⊗ (envreg↦ n,n') ⊗ unused_space.
```

Themes

- Monads, algebras, monoidal structure
- Biorthogonals
- Separation
- Logical relations and parameters
- Recursion and approximation
- Invariants and beyond
- Types versus logics
- Mechanization