Definition 4. A set S *truth-functionally implies* a formula X, or X is *truth-functionally implied* by S, or is a *truth-functional consequence* of S if X is true in every Boolean valuation which satisfies S. We also say that Y is truth-functionally implied by X if Y is truth functionally implied by the unit set $\{X\}$ ... i. e. if Y is true in every Boolean valuation in which X is true.

Definition 5. Two formulas X, Y are called *truth functionally equivalent* iff X, Y are true in the same Boolean valuations. [The reader should note that X truth-functionally implies Y iff $X \supset Y$ is a tautology, and that X is truth-functionally equivalent to Y iff the formula $X \leftrightarrow Y$ is a tautology].

Truth Sets. Let v be a Boolean valuation, and let S be the set of all formulas which are *true* under v. It is immediate from the definition of a Boolean valuation that the set S obeys the following conditions (for every X, Y):

S_1: Exactly one of the pair $(X, \sim X)$ belongs to S. Stated otherwise $(\sim X) \in S$ iff $X \notin S$.

S_2: $(X \wedge Y)$ is in S iff X, Y are both in S.

S_3: $(X \vee Y)$ is in S iff $X \in S$ or $Y \in S$.

S_4: $(X \supset Y)$ is in S iff $X \notin S$ or $Y \in S$.

A set S obeying the above conditions will be called *saturated* or will be said to be a *truth set*. Thus for any Boolean valuation, the set of all sentences true under the valuation is saturated. Indeed, if v is an arbitrary valuation, and if S is the set of all sentences which are true under v, then the following 2 conditions are *equivalent:*

(1) v is a Boolean valuation,

(2) S is *saturated.*

Now suppose that we start with a set S, and we define v_s to be that valuation which assigns t to every member of S, and f to every formula outside S. [The function v_s is sometimes referred to as the *characteristic* function of the set S.] It is again obvious that S is *saturated* iff v_s is a Boolean valuation.

Now the set of all sentences true under v_s is obviously S itself. Thus a set is saturated iff it is the set of all sentences true under some Boolean valuation. Thus a formula X is a *tautology* iff it is an element of every truth set; stated otherwise, the set of tautologies is the *intersection* of all truth sets and a formula X is *satisfiable* iff it is an element of some truth set. Stated otherwise, the set of satisfiable sentences is the *union* of all truth sets. Likewise a set S truth-functionally implies X iff X belongs to every truth set which includes S.

We thus see that we really do not need to "import" these "foreign" elements t, f in order to define our basic semantic notions. In some

Subformulas. The notion of *immediate subformula* is given explicitly by the conditions:

I_0: Propositional variables have no immediate subformulas.

I_1: $\sim X$ has X as an immediate subformula and no others.

$I_2 - I_4$: The formulas $X \wedge Y$, $X \vee Y$, $X \supset Y$ have X, Y as immediate subformulas and no others.

We shall sometimes refer to X, Y respectively as the *left immediate subformula, right immediate subformula* of $X \wedge Y$, $X \vee Y$, $X \supset Y$.

The notion of *subformula* is implicitly defined by the rules:

S_1: If X is an immediate subformula of Y, or if X is identical with Y, then X is a subformula of Y.

S_2: If X is a subformula of Y and Y is a subformula of Z, then X is a subformula of Z.

The above implicit definition can be made explicit as follows: Y is a subformula of Z iff (i.e. if and only if) there exists a finite sequence starting with Z and ending with Y such that each term of the sequence except the first is an immediate subformula of the preceding term.

The only formulas having no immediate subformulas are propositional variables. These are sometimes called *atomic* formulas. Other formulas are called *compound* formulas. We say that a variable p *occurs* in a formula X, or that p is one of the variables of X, if p is a subformula of X.

Degrees; Induction Principles. To facilitate proofs and definitions by induction, we define the *degree* of a formula as the number of occurrences of logical connectives. Thus:

D_0: A variable is of degree 0.

D_1: If X is of degree n, then $\sim X$ is of degree $n+1$.

$D_2 - D_4$: If X, Y are of degrees n_1, n_2, then $X \wedge Y$, $X \vee Y$, $X \supset Y$ are each of degree $n_1 + n_2 + 1$.

Example.

$p \wedge (q \vee \sim r)$ is of degree 3.

$p \wedge (q \vee r)$ is of degree 2.

We shall use the principle of *mathematical induction* (or of *finite descent*) in the following form. Let S be a set of formulas (S may be finite or infinite) and let P be a certain property of formulas which we wish to show holds for every element of S. To do this it suffices to show the following two conditions:

(1) Every element of S of degree 0 has the property P.

(2) If some element of S of degree >0 fails to have the property P, then some element of S of lower degree also fails to have property P.

Of course, we can also use (2) in the equivalent form:

(2)′ For every element X of S of positive degree, if all elements of S of degree less than that of X have property P, then X also has property P.

§ 2. Boolean Valuations and Truth Sets

Now we consider, in addition to the formulas of propositional logic, a set $\{t, f\}$ of two *distinct* elements, t, f. We refer to t, f as *truth-values*. For any set S of formulas, by a *valuation* of S, we mean a function v from S into the set $\{t, f\}$—i.e. a mapping which assigns to every element X of S one of the two values t, f. The value $v(X)$ of X under v is called the *truth value* of X under v. We say that X is *true under* v if $v(X)=t$, and *false under* v if $v(X)=f$.

Now we wish to consider valuations of the set E of all formulas of propositional logic. We are not really interested in *all* valuations of E, but only in those which are "faithful" to the usual "truth-table" rules for the logical connectives. This idea we make precise in the following definition.

Definition 1. A valuation v of E is called a *Boolean* valuation if for every X, Y in E, the following conditions hold:

B_1: The formula $\sim X$ receives the value t if X receives the value f and f if X receives the value t.

B_2: The formula $X \wedge Y$ receives the value t if X, Y both receive the value t, otherwise $X \wedge Y$ receives the value f.

B_3: The formula $X \vee Y$ receives the value t if at least one of X, Y receives the value t, otherwise $X \vee Y$ receives the value f.

B_4: The formula $X \supset Y$ receives the value f if X, Y receive the respective values t, f, otherwise $X \supset Y$ receives the value t.

This concludes our definition of a Boolean valuation. We say that two valuations *agree* on a formula X if X is either true in both valuations or false in both valuations. And we say that 2 valuations agree on a *set* S of formulas if they agree on every element of the set S.

If S_1 is a subset of S_2 and if v_1, v_2 are respective valuations of S_1, S_2, then we say that v_2 is an *extension* of v_1 if v_2, v_1 agree on the smaller set S_1.

It is obvious that if 2 Boolean valuations agree on X then they agree on $\sim X$ (why?), and if they agree on both X, Y they must also agree on each of $X \wedge Y$, $X \vee Y$, $X \supset Y$ (why?). By mathematical induction it follows that if 2 Boolean valuations of E agree on the set of all atomic elements of E (i.e., on all propositional variables) then they agree on all of E. Stated otherwise, a valuation v_0 of the set of all *atomic* elements of E can be extended to *at most* one Boolean valuation of E.

By an *interpretation* of a formula X is meant an assignment of truth values to all of the variables which occur in X. More generally, by an interpretation of a set W (of formulas) is meant an assignment of truth values to all the variables which occur in any of the elements of W. We can thus rephrase the last statement of the preceding paragraph by saying that any interpretation v_0 of E can be extended to at most one Boolean valuation of E. That v_0 can be extended to at *least* one Boolean valuation of E will be clear from the following considerations.

Consider a single formula X and an interpretation v_0 of X—or for that matter any assignment v_0 of truth values to a set of propositional variables which include at least all variables of X (and possibly others). It is easily verified by induction on the degree of X that there exists one

and only one way of assigning truth values to all *subformulas* of X such that the *atomic* subformulas of X (which are propositional variables) are assigned the same truth values as under v_0, and such that the truth value of each *compound* subformula Y of X is determined from the truth values of the immediate subformulas of Y by the truth-table rules $B_1 - B_4$. [We might think of the situation as first constructing a formation tree for X, then assigning truth values to the end points in accordance with the interpretation v_0, and then working our way up the tree, successively assigning truth values to the junction and simple points, in terms of truth values already assigned to their successors, in accordance with the truth-table rules]. In particular, X being a subformula of itself receives a truth value under this assignment; if this value is t then we say that X is true *under the interpretation* v_0, otherwise *false* under v_0. Thus we have now defined what it means for a formula X to be true under an *interpretation*.

Now consider an interpretation, v_0, for the entire set E. Each element, X, of E has a definite truth value under v_0 (in the manner we have just indicated); we let v be that valuation which assigns to each element of E its truth value under the interpretation v_0. The valuation v is on the entire set E, and it is easily verified that v is a Boolean valuation, and of course, v is an extension of v_0. Thus it is indeed the case that every interpretation of E can be extended to one (and only one) Boolean valuation of E.

Tautologies. The notion of *tautology* is perhaps the fundamental notion of propositional logic.

Definition 2. X is a *tautology* iff X is true in all Boolean valuations of E.

Equivalently, X is a *tautology* iff X is true under every *interpretation* of E. Now it is obvious that the truth value of X under an interpretation of E depends only on the truth values assigned to the variables which occur in X. Therefore, X is a tautology if and only if X is true under every interpretation of X. Letting n be the number of variables which occur in X, there are exactly 2^n distinct interpretations of X. Thus the task of determining whether X is or is not a tautology is purely a finite and mechanical one—just evaluate its truth value under each of its 2^n interpretations (which is tantamount to the familiar truth-table analysis).

Definition 3. A formula X is called (truth-functionally) *satisfiable* iff X is true in at least one Boolean valuation. A set S of formulas is said to be (simultaneously) truth-functionally *satisfiable* iff there exists at least one Boolean valuation in which every element of S is true. Such a valuation is said to *satisfy* S.