Gödel's T

$$M ::= x \mid z \mid s(M) \mid \lambda x{:}A.M \mid MN \mid \underline{\mathrm{rec}}(M; M_0; x,y.M_1) \mid tt \mid ff$$

$$A ::= \omega \mid A_1 \to A_2 \mid 2$$

Statics $\underline{\Gamma \vdash M : A}$.

$$\frac{\Gamma \vdash M : \omega \qquad \Gamma \vdash M_0 : A \qquad \Gamma, x{:}\omega, y{:}A \vdash M_1 : A}{\Gamma \vdash \underline{\mathrm{rec}}(M; M_0; x,y.M_1) : A}$$

etc. $\left[\dfrac{}{\Gamma \vdash tt : 2} \quad \dfrac{}{\Gamma \vdash ff : 2} \right]$

Dynamics

$$(\lambda x{:}A.M)\,N \longmapsto [N/x]M. \qquad (\text{cbn})$$

$$\underline{\mathrm{rec}}(z; M_0; x,y.M_1) \longmapsto M_0$$

$$\underline{\mathrm{rec}}(s(M); M_0; x,y.M_1) \longmapsto [M, \underline{\mathrm{rec}}(M; M_0; x,y.M_1)/x,y]M_1$$

$$\frac{M_1 \longmapsto M_1'}{M_1 M_2 \longmapsto M_1' M_2} \quad (\text{cbn}) \qquad \left[\frac{M_1 \text{ value} \quad M_2 \longmapsto M_2'}{M_1 M_2 \longmapsto M_1 M_2'} \; (\text{cbv}) \right]$$

$$\frac{M \longmapsto M'}{\underline{\mathrm{rec}}(M; -) \longmapsto \underline{\mathrm{rec}}(M'; -)}$$

$$\frac{M \longmapsto M'}{s(M) \longmapsto s(M')} \qquad (\text{eager}) \quad (\text{vs lazy})$$

If we use lazy evaluation for successor, we add 2 as an <u>observable</u> <u>outcome</u>.

With an eager evaluation for successor, we can use ω as an <u>observable</u> (or else observe only outermost form).

Can do cbv as well, but we'll use cbn. for simplicity.

(eager) <u>Thm</u> If $M : \omega$ then $\exists n$ st $M \mapsto^* \overline{n}$

(lazy) If $M : 2$, then $M \mapsto^* \begin{cases} tt \\ ff \end{cases}$

Tait's Method: how to prove this?

Could consolidate and just say

<u>Thm</u> If $M : A$ then $\exists N$ st N value
$M \mapsto^* N$

where $\dfrac{}{\lambda x{:}A.M \text{ value}}$ $\dfrac{}{z \text{ value}}$ $\dfrac{(M \text{ value})}{s(M) \text{ value}}$.

$\dfrac{}{tt \text{ value}}$ $\dfrac{}{ff \text{ value}}$.

Reducibility

+ $Red_\omega(M)$ iff $M \mapsto^* z$ or
$M \mapsto^* s(M')$ and $Red_\omega(M')$

− $Red_{A_1 \to A_2}(M)$ iff $Red_{A_1}(M_1)$ implies $Red_{A_2}(MM_1)$

1. Red_ω is the strongest P closed under the given conditions.

   $Red_\omega(M)$ iff $\forall P$ closed $P(M)$.

   where P closed means

   a) $M \mapsto^* z$ implies $P(M)$

   b) $M \mapsto^* s(M')$ and $P(M')$ implies $P(M)$

   So: TS $Red_\omega(M)$ implies $P(M)$

   STS P closed.

2. $Red_{A_1 \to A_2} = Red_{A_1} \to Red_{A_2}$

   where $\underbrace{(P \to Q)}_{\text{"relational action" of } \to}(M)$ iff $P(N)$ implies $Q(MN)$

Lemma

1) $Red_\omega(M)$ implies $M\Downarrow$.

2) $Red_A(N)$ $M \mapsto N$ imply $Red_A(M)$

Pf 1) immediate

2) induction on A.

$\boxtimes$

Thm If $\Gamma \vdash M : A$ and $Red_\Gamma(\eta)$

then $Red_A(\hat{\eta}(M))$.

Pf induction on typing.

Notation: $\hat{M}$ for $\hat{\eta}(M)$ when η is clear.

1) $$\frac{}{\Gamma, x:A \vdash x:A}$$

amounts to $Red_A(M)$ implies $Red_A(M)$.

2) $$\frac{}{\Gamma \vdash z : \omega}$$

$Red_\omega(z)$ ✓

3) $$\frac{\Gamma \vdash M : \omega}{\Gamma \vdash s(M) : \omega}$$ IH: $Red_\omega(\hat{M})$

TS: $Red_\omega(s(\hat{M}))$

immediate from defn

3) $$\frac{\Gamma \vdash M : \omega \qquad \Gamma \vdash M_0 : A \qquad \Gamma, x:\omega, y:A \vdash M_1 : A}{\Gamma \vdash \underline{rec}(M; M_0; x,y.M_1) : A}$$

IH_1: $Red_\omega(\hat{M})$

IH_2: $Red_A(\hat{M}_0)$

IH_3: $Red_\omega(N)$ and $Red_A(P)$ imply
$Red_A([N,P/x,y]\hat{M}_1)$.

TS: $Red_A(\underline{rec}(\hat{M}; \hat{M}_0; x,y.\hat{M}_1))$.

Let $P(-) := Red_A(\underline{rec}(-; \hat{M}_0; x,y.\hat{M}_1))$ and $Red_\omega(-)$

STS $P(-)$ closed. (for then $P(\hat{M})$ as req'd)

a) spse $N \mapsto^* z$.

$\underline{red}(N; \hat{M}_0; x,y.\hat{M}_1) \mapsto^* \hat{M}_0$.

Head expansion.

b) spse $N \mapsto^* s(N')$ w/ $P(N')$.

$\underline{red}(N; \hat{M}_0; x,y.\hat{M}_1) \mapsto^* [N', \underline{red}(N'; -)/x,y]\hat{M}_1$

know $Red_\omega(N')$ $Red_A(\underline{rec}(N'; —))$

By IH_3 $Red_A(\underline{rec}(N; \hat{M}_0; x,y.\hat{M}_1))$ ✓.

4) $$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

IH$_1$ $Red_{A\to B}(\hat{M})$

IH$_2$ $Red_A(\hat{N})$

$\therefore$ $Red_A(\hat{M}\hat{N})$

$\quad \equiv \widehat{MN}$.

5) $$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \to B}$$

IH: $Red_A(N)$ implies $Red_B([N/x]\hat{M})$

TS: $Red_{A\to B}(\lambda x{:}A.M)$

spse $Red_A(N)$. then $Red_B([N/x]\hat{M})$

by head expansion result follows.

$\boxtimes$

Extend T with a type 2 with values ~~tt~~ and ~~ff~~. tt ff

$\to$ $Red_2(M)$ iff $M \mapsto^* tt$ or $M \mapsto^* ff$.

(Kleene equiv.) $P \simeq Q$ iff $P \mapsto^* tt$ and $Q \mapsto^* tt$
or $P \mapsto^* ff$ and $Q \mapsto^* ff$

contextual equiv. $\Gamma \vdash M \cong N : A$ iff
$\forall C[-]$ st $C[M], C[N] : 2$
$C[M] \simeq C[N]$.

<u>Fact</u> $\Gamma \vdash M \cong N : A$ is the <u>COARSEST</u> <u>CONSISTENT</u> <u>CONGRUENCE</u>.

1) CONSISTENT : ~~immediate~~. ($M \cong N : 2$ implies $M \simeq N$.)

immediate, take $C[-] = [-]$.

2) CONGRUENCE :

$$\frac{\Gamma \vdash M \cong N : A \qquad C[-] : \Delta \rightsquigarrow \Gamma}{\Delta \vdash C[M] \cong C[N] : A.}$$

Follows from closure of contexts under composition.

3) COARSEST :

if $\Gamma \vdash M \equiv N : A$ is a consistent congruence then

$$\Gamma \vdash M \equiv N : A \supset \Gamma \vdash M \cong N : A .$$

$C[M] \equiv C[N] : 2$

$\therefore C[M] \simeq C[N]$.

$\therefore \Gamma \vdash M \cong N : A$.

⊠

coinduction principle.

TS: $\Gamma \vdash M \cong N : A$

STS: $\Gamma \vdash M \equiv N : A$ for some consistent congruence.

Logical Equivalence (= Binary Reducibility)

We will define $M \sim_\tau N$ on closed terms $M, N : \tau$ s.t.

1) if $M : \tau$ then $M \sim_\tau M$.
2) if $M \sim_\tau N$ then $M \cong_\tau N$.

Def By ind on τ:

$M \sim_2 N$ iff $M \simeq N$ $\left(M \mapsto^* \underset{ff}{tt} \text{ iff } M \mapsto^* \underset{ff}{tt} \right)$

$M \sim_\omega N$ iff $(M \mapsto^* z$ and $N \mapsto^* z)$ or $(M \mapsto^* s(M')$ and $N \mapsto^* s(N')$ and $M' \sim_\omega N')$

$M \sim_{\tau_1 \to \tau_2} N$ iff $M_1 \sim_{\tau_1} N_1 \supset M M_1 \sim_{\tau_2} N N_1$

Note $M \sim_\omega N$ is the strongest such relation!

Def $\Gamma \vdash M \sim N : A$ iff, where $\Gamma = x_1 : A_1 \ldots x_k : A_k$,

$M_1 \sim_{A_1} N_1 \cdots M_k \sim_{A_k} N_k$ imply $[M_i/x_i]M \sim_A [N_i/x_i]N$.

WE

We will show (sketch) that *open logical equivalence* is a consistent congruence.

$\therefore\ \Gamma \vdash M \sim N : A$ implies $\Gamma \vdash M \doteq N : A$.

So logical equivalence will be a *necessary* condition for equality.

PFPL

Conversely, if $\Gamma \vdash M \doteq N : A$ then $\Gamma \vdash M \sim N : A$.

This amounts to showing that log equiv can be thought of as building a pgm context.

Fact open logical equiv is consistent
$\longrightarrow$ immediate!