System L syntax for sequent calculi

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(based on works of or with Guillaume Munch-Maccagnoni, and nourished by an on-going collaboration with Marcelo Fiore)

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Plan

1. Some features of system L syntax
2. linear and classical logic (non focalised), with an analysis of confluence issues
3. Focalised systems in "direct style" (no shifts) : LK_{foc}, LL_{foc} (closely related to systems proposed in Guillaume Munch-Maccagnoni's *master* thesis, cf. his CSL 2009 paper)
4. Systems in "indirect style" (with shifts) : $\mathsf{LL}_{\downarrow}$, Melliès' tensor logic, $\mathsf{LK}_{\downarrow}$, Laurent's LLP, (a sequent calculus version of) CBPV (for $\mathsf{LK}_{\downarrow}$, cf. Curien - Munch-Maccagnoni's IFIP TCS Conference 2010 paper)
5. Double shift $\downarrow\uparrow$ as **a** monad (CBPV) , versus double shift as **continuation** monad (LLP, or TL)

Asides :

A. Type-free versions / general connectives (in the style of ludics), adapted from **Herbelin** (unpublished)

B. System L as an intermediate language / abstract machine

General roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect		$\mathsf{LL}_{\downarrow}$, Tensor Logic

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		$\mathsf{LK}_{\downarrow}$, LLP, CBPV

Oregon's roadmap (July 2012)

Linear :

	Non focalised	Focalised
Direct	LL	
Indirect		

Classical :

	Non focalised	Focalised
Direct		LK$_{foc}$
Indirect		LK$_{\downarrow}$ (monolateral and bilateral)

I) A syntactic tool-box

for sequent calculus proofs

The basic kit

Consider the cut rule, classically presented as :

$$\frac{\Gamma_1 \vdash A, \Delta'_1 \qquad \Gamma'_2, A \vdash \Delta_2}{\Gamma_1, \Gamma'_2 \vdash \Delta'_1, \Delta_2}$$

But $\Delta_1 = A, \Delta'_1$ and $\Gamma_2 = \Gamma'_2, A$ might have several copies of A. One needs to specify which A is *active* in both assumptions.

For term assignments to natural deduction proofs, one associates variables to the formulas in a sequent $\vdash \Gamma$. Here too, contexts are lists of typed variable declarations. In system L notation, we set :

$$\frac{\dfrac{c : (\Gamma_1 \vdash \alpha : A, \Delta'_1)}{\Gamma_1 \vdash \mu\alpha.c : A \,|\, \Delta'_1} \qquad \dfrac{c' : (\Gamma'_2, x : A \vdash \Delta_2)}{\Gamma'_2 \,|\, \tilde{\mu}x.c' : A \vdash \Delta_2}}{\langle \mu\alpha.c \,|\, \tilde{\mu}x.c' \rangle : (\Gamma_1, \Gamma'_2 \vdash \Delta'_1, \Delta_2)}$$

(note that μ, $\tilde{\mu}$ are binding operators)

Different judgements

Therefore, we distinguish different kinds of judgements :

- commands $c : (\Gamma \vdash \Delta)$ with no active formula which under Curry-Howard (and head reduction) will read as machine states

- terms $\Gamma \vdash v : A \,|\, \Delta$ which under Curry-Howard will read as programs of type A

- contexts $\Gamma \,|\, e : A \vdash \Delta$ which under Curry-Howard read as contexts expecting to interact with a program of type A

In *focused* systems, we shall also have *value* and *covalue* judgements in which the active formula is moreover *under focus*.

In monolateral systems, considered first in this talk, the context (and covalue) judgements disappear (replaced with terms or values of the dual type). But they will feature prominently at the end of the talk.

Pattern-matching

Logical connectives are polarised according to the rules used to introduce them, which are irreversible=positive or reversible=negative.

We shall use constructors for denoting the irreversible rules, and **structured** binding operations μ (and $\tilde{\mu}$ on the left of sequents in bilateral systems) for the reversible rules. The dual of an irreversible connective being reversible, this will lead to **"cut-elimination through pattern-matching"** :

Irreversible

$$\frac{\vdash t_1 : A_1 \mid \Delta_1 \quad \vdash t_2 : A_2 \mid \Delta_2}{\vdash (t_1, t_2) : A_1 \otimes A_2 \mid \Delta_1, \Delta_2}$$

Reversible

$$\frac{c : (\vdash x_1 : A_1, x_2 : A_2, \Delta)}{\vdash \mu(x_1, x_2).c : A_1 \parr A_2 \mid \Delta}$$

$$\langle (t_1, t_2) \mid \mu(x_1, x_2).c \rangle \to c[t_1/x_1, t_2/x_2]$$

To make this pregnance of polarities clear, we start from linear logic, where it is explicitly present from the beginning (even if the initial motivating divide was rather *additive* versus *multiplicative*).

What is “system L” ?

Summarising, we use “system L” (“L” for Gentzen’s terminology of sequent calculus systems) for term assignment systems for sequent calculus presentations of various logical systems that share the following features :

- different kinds of judgements, that make explicit the notion of active formula (possibly under focus) and coercions between them. We have seen activation via μ and $\tilde{\mu}$. Deactivation is achieved via “cut with axiom” :

$$\frac{\Gamma \vdash v : A \,|\, \Delta \qquad \overline{\,|\,\alpha : A \vdash \alpha : A}}{\langle v \,|\, \alpha \rangle : (\Gamma, \vdash \alpha : A, \Delta)}$$

This is the only form of cut that will *not* be evaluated in our formalism.

- structured pattern-matching for reversible rules

The first feature was put forward in Curien-Herbelin’s duality of computation paper (ICFP 2000).

II) Linear and classical logic

Roadmap

Linear :

	Non focalised	Focalised
Direct	$\boxed{LL}$	
Indirect		

Classical :

	Non focalised	Focalised
Direct		
Indirect		

Linear logic

$$\frac{}{\vdash A, \overline{A}} \qquad \frac{\vdash P, \Gamma \quad \vdash \overline{P}, \Delta}{\vdash \Gamma, \Delta}$$

$$\frac{\vdash A_1, \Gamma \quad \vdash A_2, \Delta}{\vdash A_1 \otimes A_2, \Gamma, \Delta} \qquad \frac{\vdash A_1, \Gamma}{\vdash A_1 \oplus A_2, \Gamma} \qquad \frac{\vdash A_2, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}$$

$$\frac{\vdash A_1, A_2, \Gamma}{\vdash A_1 \parr A_2, \Gamma} \qquad \frac{\vdash A_1, \Gamma \quad \vdash A_2, \Gamma}{\vdash A_1 \& A_2, \Gamma}$$

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}$$

The connectives of linear logic

$$A ::= X \mid \overline{X} \mid A \otimes A \mid A \parr A \mid A \oplus A \mid A \& A \mid {!A} \quad \mid {?A}$$

$\overline{A}$ (involutive negation) is defined by induction (De Morgan duality).

$\overline{A \otimes B} = \overline{A} \parr \overline{B} \qquad \overline{A \oplus B} = \overline{A} \,\&\, \overline{B} \qquad \overline{!A} = ?\overline{A}$ "and conversely"

Some terminology :

	Multiplicatives	Additives	Exponentials
Irreversible	$\otimes$	$\oplus$	?
Reversible	$\parr$	$\&$	!

Semantic explanation for the terminology (relational model)

– Interpret formulas as sets :

$[\![X]\!] = [\![\overline{X} =$ "your choice"	
$[\![A \otimes B]\!] = [\![A \parr B = [\![A]\!] \times [\![B]\!]$	cartesian product
$[\![A \oplus B]\!] = [\![A \& B]\!] = [\![A]\!] + [\![B]\!]$	disjoint sum
$[\![!A]\!] = [\![?A]\!] = \mathcal{M}_{fin}([\![A]\!])$	finite multisets

– Interpret a proof of $\vdash A_1, \ldots, A_n$ as a subset of $[\![A_1 \parr \cdots \parr A_n]\!]$: very instructive exercise, by induction on the size of the proof !

Syntax for linear logic

Formulas :

$A ::= P \parallel N \quad P ::= X \parallel A \otimes A \parallel A \oplus A \parallel !A \quad N ::= \overline{X} \parallel A \parr A \parallel A \& A \parallel ?A$

We use overlining for De Morgan duality.

There are three kinds of judgements :

Commands	Positive terms	Negative terms
$c : (\vdash \Gamma)$	$\vdash t^+ : P \mid \Gamma$	$\vdash t^- : N \mid \Gamma$

Terms :

$c ::= \langle t^+ \mid t^- \rangle$ which we also write if needed as $\langle t^- \mid t^+ \rangle$

$t ::= t^+ \parallel t^-$

$x ::= x^+ \parallel x^-$

$t^+ ::= x^+ \parallel \mu x^-.c \parallel (t_1, t_2) \parallel inl(t) \parallel inr(t) \parallel \mu x^!.c$

$t^- ::= x^- \parallel \mu x^+.c \parallel \mu(x_1, x_2).c \parallel \mu[inl(x_1).c_1, inr(x_2).c_2] \parallel t^! \parallel \mathtt{w}(c) \parallel \mathtt{c}_{x_1^+, x_2^+}(c)$

Typing rules for LL

Contexts Γ consist of declarations $x^+ : N$ and $x^- : P$:

$$\frac{}{\vdash x : A \mid x : \overline{A}} \qquad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x.c : A \mid \Gamma}$$

$$\frac{\vdash t^+ : P \mid \Gamma \quad \vdash t^- : \overline{P} \mid \Delta}{\langle t^+ \mid t^- \rangle : (\vdash \Gamma, \Delta)} \quad \frac{\vdash t_1 : A_1 \mid \Gamma \quad \vdash t_2 : A_2 \mid \Delta}{\vdash (t_1, t_2) : A_1 \otimes A_2 \mid \Gamma, \Delta} \quad \frac{\vdash t_1 : A_1 \mid \Gamma}{\vdash inl(t_1) : A_1 \oplus A_2 \mid \Gamma}$$

$$\frac{c : (\vdash x_1 : A_1, x_2 : A_2, \Gamma)}{\vdash \mu(x_1, x_2).c : A_1 \invamp A_2 \mid \Gamma} \qquad \frac{c_1 : (\vdash x_1 : A_1, \Gamma) \quad c_2 : (\vdash x_2 : A_2, \Gamma)}{\vdash \mu[inl(x_1).c_1, inr(x_2).c_2] : A_1 \& A_2 \mid \Gamma}$$

$$\frac{c : (\vdash x : A, ?\Gamma)}{\vdash \mu x^!.c : !A \mid ?\Gamma} \quad \frac{\vdash t : A \mid \Gamma}{\vdash t^! : ?A \mid \Gamma} \quad \frac{c : (\vdash \Gamma)}{\vdash \mathtt{w}(c) : ?A \mid \Gamma} \quad \frac{c : (\vdash x_1^+ : ?A, x_2^+ : ?A, \Gamma)}{\vdash \mathtt{c}_{x_1^+, x_2^+}(c) : ?A \mid \Gamma}$$

Illustrating activation and deactivation

The term decoration for

$$\dfrac{\dfrac{\vdash N \oplus P, A, B, \Gamma_1}{\vdash N \oplus P, {\color{blue}A \parr B}, \Gamma_1} \qquad \vdash M, \Gamma_2}{\vdash {\color{red}(N \oplus P) \otimes M}, A \parr B, \Gamma_1, \Gamma_2}$$

is as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{c : (\vdash x : N \oplus P, y_1 : A, y_2 : B, \Gamma_1)}{\vdash \mu(y_1, y_2).c : {\color{blue}A \parr B} \mid x : N \oplus P, \Gamma_1}}{\langle y \mid \mu(y_1, y_2).c\rangle : (\vdash y : A \parr B, x : N \oplus P, \Gamma_1)}}{\vdash \mu x.\langle y \mid \mu(y_1, y_2).c\rangle : N \oplus P \mid y : A \parr B, \Gamma_1} \qquad \vdash t : M \mid \Gamma_2}{\vdash (\mu x.\langle y \mid \mu(y_1, y_2).c\rangle, t) : {\color{red}(N \oplus P) \otimes M} \mid y : A \parr B, \Gamma_1, \Gamma_2}$$

Reduction rules for LL

$$\langle t^+ \mid \mu x^+.c\rangle \to c[t^+/x^+]$$
$$\langle \mu x^-.c \mid t^-\rangle \to c[t^-/x^-]$$
$$\langle (t_1, t_2) \mid \mu(x_1, x_2).c\rangle \to c[t_1/x_1, t_2/x_2]$$
$$\langle \mathit{inl}(t_1) \mid \mu[\mathit{inl}(x_1).c_1, \mathit{inr}(x_2).c_2]\rangle \to c_1[t_1/x_1]$$
$$\langle \mu x^!.c \mid t^!\rangle \to c[t/x]$$
$$\langle t^+ \mid \mathtt{w}(c)\rangle \to \mathtt{W}(c)$$
$$\langle t^+ \mid \mathsf{c}_{x_1^+, x_2^+}(c)\rangle \to \mathtt{C}(c[t^+/x_1^+, t^+/x_2^+])$$

if the free variables of t^+ are in a list $l = y_1, \ldots, y_n$ (with each y_i of type $?B_i$), then

– $\mathtt{W}(c)$ stands for $\mathtt{W}_l(c)$, where $\mathtt{W}_{\mathtt{nil}}(c) = c \quad \mathtt{W}_{y^+ \cdot l} = \langle y^+ \mid \mathtt{w}(\mathtt{W}_l(c))\rangle$

– $\mathtt{C}(c[t^+/x_1^+, t^+/x_2^+])$ stands for $\mathtt{C}_l(c[t^+[l'/l]/x_1^+, t^+[l''/l]/x_2^+])$, where

$$\mathtt{C}_{\mathtt{nil}}(c) = c \quad \mathtt{C}_{y^+ \cdot l} = \langle y^+ \mid \mathsf{c}_{y'^+, y''^+}(\mathtt{C}_l(c))\rangle$$

(by l' we mean $y'^+_1, \ldots, y'^+_n$, and by $t^+[l'/l]$ we mean the simultaneous substitutions of the y_i^+'s by the y'^+_i's).

Substitution accounts for commutative cuts

Lemma : If $c :\vdash x : A, \Gamma$, then the unique occurrence of x in c occurs as a deactivation : $c = C[\langle x \mid t\rangle]$.

The left hand side of the first and second computation rules codify a situation where one of the cut formulas has *not* been just introduced, and the reduction commutes the cut upwards on the right (resp. on the left) to the place where it was introduced, so that eventually a logical cut rule such as the third or the fourth rule can be applied :

$$\begin{array}{c} \langle t_1^+ \mid \mu x^+.c\rangle = \langle t_1^+ \mid \mu x^+.C[\langle x^+ \mid t_2^-\rangle]\rangle \\ \downarrow \\ c[t_1^+/x^+] = C[\langle t_1^+ \mid t_2^-\rangle \end{array}$$

On the confluence of cut elimination in linear logic

The critical pairs are $\langle \mu x_1^-.c_1 | \mu x_2^+.c_2\rangle$, $\langle \mu x_1^-.c_1 | \mathtt{w}(c_2)\rangle$, $\langle \mu x_1^-.c_1 | \mathtt{c}_{x_1^+,x_2^+}(c_2)\rangle$
Exploiting linearity (on each branch) we can set schematically

$$c_1 = C_1[\langle t_1^+ \mid x_1^-\rangle] \quad \text{and} \quad c_2 = C_2[\langle x_2^+ \mid t_2^-\rangle]$$

With these notations, our critical pair sounds less desperate :

$$\begin{array}{rcl}\langle \mu x_1^-.C_1[\langle t_1^+ | x_1^-\rangle] \mid \mu x_2^+.C_2[\langle x_2^+ | t_2^-\rangle]\rangle & \to & C_1[\langle t_1^+ \mid \mu x_2^+.C_2[\langle x_2^+ \mid t_2^-\rangle]\rangle] \\ & \to & C_1[C_2[\langle t_1^+ \mid t_2^-\rangle]]\end{array}$$

while the other branch of the critical pair reduces symmetrically :

$$\langle \mu x_1^-.C_1[\langle t_1^+ \mid x_1^-\rangle] \mid \mu x_2^+.C_2[\langle x_2^+ \mid t_2^-\rangle]\rangle \to^* C_2[C_1[\langle t_1^+ \mid t_2^-\rangle]]$$

One sees that the “space” between $C_1[C_2[\langle t_1^+ | t_2^-\rangle]]$ and $C_2[C_1[\langle t_1^+ | t_2^-\rangle]]$ can be filled by elementary commutations.

The other critical pairs are handled similarly. The system is thus (locally) confluent modulo commutation rules.

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	
Indirect		

Classical :

	Non focalised	Focalised
Direct	“LK”	
Indirect		

Linear logic versus classical logic

A syntax for (a polarised version of) classical logic is obtained from the above by removing the exponential modalities and the term constructions for dereliction and promotion, but keeping explicit contraction and weakening, now expressed as :

$$\frac{c : (\vdash \Gamma)}{\vdash \mathtt{w}(c) : A \,|\, \Gamma} \qquad \frac{c : (\vdash x_1 : A, x_2 : A, \Gamma)}{\vdash \mathtt{c}_{x_1,x_2}(c) : A \,|\, \Gamma}$$

Note that now explicit weakenings and contractions can be either positive or negative terms.

A syntax for a "polarity-aware" version of classical logic

Formulas :

$$A ::= P \parallel N \qquad P ::= X \parallel A \otimes A \parallel A \oplus A \qquad N ::= \overline{X} \parallel A \invamp A \parallel A \& A$$

Terms :

$$\begin{aligned}
&c ::= \langle t^+ \mid t^- \rangle \\
&t ::= t^+ \parallel t^- \\
&x ::= x^+ \parallel x^- \\
&t^+ ::= x^+ \parallel \mu x^-.c \parallel (t_1, t_2) \parallel inl(t) \parallel inr(t) \parallel \mathtt{w}(c) \parallel \mathtt{c}_{x_1^-, x_2^-}(c) \\
&t^- ::= x^- \parallel \mu x^+.c \parallel \mu(x_1, x_2).c \parallel \mu[inl(x_1).c_1, inr(x_2).c_2] \parallel \mathtt{w}(c) \parallel \mathtt{c}_{x_1^+, x_2^+}(c)
\end{aligned}$$

Tentative reduction rules for classical logic

$$
\begin{array}{l}
\langle t^+ \mid \mu x^+.c\rangle \to c[t^+/x^+] \\
\langle \mu x^-.c \mid t^-\rangle \to c[t^-/x^-] \\
\langle (t_1,t_2) \mid \mu(x_1,x_2).c\rangle \to c[t_1/x_1, t_2/x_2] \\
\langle inl(t_1) \mid \mu[inl(x_1).c_1, inr(x_2).c_2]\rangle \to c_1[t_1/x_1] \\
\langle t^+ \mid \mathtt{w}(c)\rangle \to \mathtt{W}(c) \\
\langle \mathtt{w}(c) \mid t^-\rangle \to \mathtt{W}(c) \\
\langle t^+ \mid \mathtt{c}_{x_1^+,x_2^+}(c)\rangle \to \mathtt{C}(c[t^+/x_1^+, t^+/x_2^+]) \\
\langle \mathtt{c}_{x_1^-,x_2^-}(c) \mid t^-\rangle \to \mathtt{C}(c[t^-/x_1^-, t^-/x_2^-])
\end{array}
$$

But now there are more critical pairs (known as Lafont's critical pairs), like the weakening/weakening pair

$$\mathtt{W}(c_1) \quad \leftarrow \quad \langle \mathtt{w}(c_1) \mid \mathtt{w}(c_2)\rangle \quad \to \quad \mathtt{W}(c_2)$$

(for arbitrary proofs c_1, c_2), which collapses all proofs !

Discussion

Under these glasses, and in retrospect, linear logic and focalisation have provided two alternative routes to get out of Lafont's critical pairs :

1. focalised cut elimination (see below) restricts the dynamics in such a way that all the reduction rules are only applicable when they substitute values for positive variables. Then the bad critical pairs (as well as the non harmful ones) disappear, and one gets a confluent system without the need of appealing to commutation rules. This is a constraint on the syntax that still makes sense in an untyped setting.
2. the introduction of the modalities makes the bad critical pairs ill-typed. This is a constraint on types.

We note a third route in between : remove the cases $\mathtt{w}(c)$ and $\mathtt{c}_{x_1^-,x_2^-}(c)$ from the syntax of positive terms. (i.e. allow all contractions and weakenings on negative formulas, and only on them), and keep an unconstrained classical cut-elimination.

III) Focalised systems

1. LK$_{foc}$, where focalisation is badly needed for confluence
2. LL$_{foc}$

Focalisation

A focalised proof

$$\dfrac{\dfrac{\vdash N \mid A \parr B, \Gamma_1}{\vdash N \oplus P \,;\, A \parr B, \Gamma_1} \qquad \vdash M \mid \Gamma_2}{\vdash (N \oplus P) \otimes M \,;\, A \parr B, \Gamma_1, \Gamma_2}$$

A non focalised proof

$$\dfrac{\dfrac{\vdash N \oplus P, A, B, \Gamma_1}{\vdash N \oplus P, A \parr B, \Gamma_1} \qquad \vdash M, \Gamma_2}{\vdash (N \oplus P) \otimes M, A \parr B, \Gamma_1, \Gamma_2}$$

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	
Indirect		

Classical :

	Non focalised	Focalised
Direct		$\boxed{\text{LK}_{foc}}$
Indirect		

Syntax for focalised classical logic LK$_{foc}$

$$A ::= P \parallel N \qquad P ::= X \parallel A \otimes A \parallel A \oplus A \qquad N ::= \overline{X} \parallel A \parr A \parallel A \& A$$

There are now four kinds of judgements :

Commands	Values	Positive terms	Negative terms
$c : (\vdash \Gamma)$	$\vdash V^+ : P \,;\, \Gamma$	$\vdash t^+ : P \mid \Gamma$	$\vdash t^- : N \mid \Gamma$

We set

$$V ::= V^+ \parallel t^-$$

$\vdash V : A \parallel \Gamma$ stands for either $\vdash V^+ : P \,;\, \Gamma$ or $\vdash t^- : N \mid \Gamma$

Terms :

$$c ::= \langle t^+ \mid t^- \rangle$$
$$x ::= x^+ \parallel x^-$$
$$V^+ ::= x^+ \parallel (V_1, V_2) \parallel inl(V) \parallel inr(V)$$
$$t^+ ::= V^+ \parallel \mu x^-.c \parallel \mathtt{w}(c) \parallel \mathtt{c}_{x_1^-,x_2^-}(c)$$
$$t^- ::= x^- \parallel \mu x^+.c \parallel \mu(x_1, x_2).c \parallel \mu[inl(x_1).c_1, inr(x_2).c_2] \parallel \mathtt{w}(c) \parallel \mathtt{c}_{x_1^+,x_2^+}(c)$$

Typing rules for LK_{foc}

$$\frac{}{\vdash x^{+} : P \,;\, x^{+} : \overline{P}} \qquad \frac{}{\vdash x^{-} : N \mid x^{-} : \overline{N}} \qquad \frac{\vdash t^{+} : P \mid \Gamma \quad \vdash t^{-} : \overline{P} \mid \Delta}{\langle t^{+} \mid t^{-} \rangle : (\vdash \Gamma, \Delta)}$$

$$\frac{\vdash V^{+} : P \,;\, \Gamma}{\vdash V^{+} : P \mid \Gamma} \qquad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x.c : A \mid \Gamma}$$

$$\frac{\vdash V_1 : A_1 \parallel \Gamma \qquad \vdash V_2 : A_2 \parallel \Delta}{\vdash (V_1, V_2) : A_1 \otimes A_2 \,;\, \Gamma, \Delta} \qquad \frac{\vdash V_1 : A_1 \parallel \Gamma}{\vdash \mathit{inl}(V_1) : A_1 \oplus A_2 \,;\, \Gamma}$$

$$\frac{c : (\vdash x_1 : A_1, x_2 : A_2, \Gamma)}{\vdash \mu(x_1, x_2).c : A_1 \mathbin{⅋} A_2 \mid \Gamma} \qquad \frac{c_1 : (\vdash x_1 : A_1, \Gamma) \quad c_2 : (\vdash x_2 : A_2, \Gamma)}{\vdash \mu[\mathit{inl}(x_1).c_1, \mathit{inr}(x_2).c_2] : A_1 \& A_2 \mid \Gamma}$$

$$\frac{c : (\vdash \Gamma)}{\vdash \mathtt{w}(c) : A \mid \Gamma} \qquad \frac{\vdash x_1 : A, x_2 : A, \Gamma}{\vdash \mathsf{c}_{x_1, x_2}(c) : A \mid \Gamma}$$

Reduction rules for LK_{foc}

$$
\begin{array}{l}
\langle V^+ \mid \mu x^+.c\rangle \to c[V^+/x^+] \\
\langle \mu x^-.c \mid t^-\rangle \to c[t^-/x^-] \\
\langle (V_1, V_2) \mid \mu(x_1, x_2).c\rangle \to c[V_1/x_1, V_2/x_2] \\
\langle inl(V_1) \mid \mu[inl(x_1).c_1, inr(x_2).c_2]\rangle \to c_1[V_1/x_1] \\
\langle V^+ \mid \mathtt{w}(c)\rangle \to \mathtt{W}(c) \\
\langle \mathtt{w}(c) \mid t^-\rangle \to \mathtt{W}(c) \\
\langle V^+ \mid \mathtt{c}_{x_1^+, x_2^+}(c)\rangle \to \mathtt{C}(c[V^+/x_1^+, V^+/x_2^+]) \\
\langle \mathtt{c}_{x_1^-, x_2^-}(c) \mid t^-\rangle \to \mathtt{C}(c[t^-/x_1^-, t^-/x_2^-])
\end{array}
$$

Note that there is no critical pair anymore. We have regained consistency. The system presented here is a (close) variant of **Girard**'s LC. It is also very close to Liang and Miller's LKF system. One can easily provide precise system L syntax for LC or LKF.

Plotkin meets Andreoli

We have

– a call-by-value regime for positive variables

– a call-by-name regime for negative variables

Plotkin’s values correspond to positive phases in the focalisation discipline.

Removing a bit of bureaucracy

Now that we have carefully discussed the barriers to confluence, we can keep weakening and contraction implicit in the term syntax (both for LL and LK$_{foc}$) by defining $\mathtt{w}(c) = \mu x.c$ (with x fresh) and $\mathtt{c}_{x_1,x_2}(c) = \mu x_1.c[x_2/x_1]$. Then the reductions rules consist only of :

$$
\begin{array}{l}
\langle V^+ \mid \mu x^+.c\rangle \rightarrow c[V^+/x^+] \\
\langle \mu x^-.c \mid t^-\rangle \rightarrow c[t^-/x^-] \\
\langle (V_1, V_2) \mid \mu(x_1, x_2).c\rangle \rightarrow c[V_1/x_1, V_2/x_2] \\
\langle inl(V_1) \mid \mu[inl(x_1).c_1, inr(x_2).c_2])\rangle \rightarrow c_1[V_1/x_1]
\end{array}
$$

since now the dynamics of weakening and contraction is integrated in the dynamics of (implicit) substitution. This is the choice adopted for the rest of this talk.

A short perspective on focalisation

Focalisation appeared in the context of linear logic programming (Andreoli, 1992) : the goal was to *reduce the search space*.

Shortly before, as an independent fore-runner, appeared the notion of *uniform* (intuitionistic) proof by Miller, Nadathur, Pfenning, and Scedrov (1991), in which however polarities were not highlighted (negative fragment !).

The work of Andreoli influenced Girard for the design of LC.

The line of work of **Griffin** is independent, but (negative) polarisation is implicit in **Felleisen**'s CBN $\lambda\mathcal{C}$-calculus , and focalisation is implicit in natural deduction (see below).

Other term syntaxes for classical logic have been given by Urban, and Wadler. System L's distinguishing feature is its design around the capital notion of polarity.

A first type-free aside (monolateral, direct)

In the spirit of **ludics**, we can "define" connectives by syntax and behaviour. In this view, a general connective is entirely defined by its arity, which is of the form

$$\{(n_i) \mid i \in I\}$$

where i ranges over some (finite) set I, and $n_i \in \mathbb{N}$.

$V^+ ::= x^+ \parallel \iota_i(V_1, \ldots, V_{n_i}) \parallel \ \ldots$ (for each connective)
$t^+ ::= V^+ \parallel \mu x^-.c$
$t^- ::= x^- \parallel \mu x^+.c \parallel \mu[\ldots, \iota_i(x^i_1, \ldots, x^i_{n_i}).c_i, \ldots] \parallel \ \ldots$ (for each connective)

(adapted from **Herbelin**, unpublished notes)

Reduction rules for the syntax with general connectives

$$\langle V^+ \mid \mu x^+.c\rangle \to c[V^+/x^+]$$
$$\langle \mu x^-.c \mid t^-\rangle \to c[t^-/x^-]$$
$$\langle \iota_i(V_1,\ldots,V_{n_i}) \mid \mu[\ldots,\iota_i(x_1^i,\ldots,x_{n_i}^i).c_i,\ldots]\rangle \to c_i[V_1/x_1^i,\ldots,,V_{n_i}/x_{n_i}^i]$$

One recovers :

$$\otimes/\parr \qquad I=\{*\}\ ,\ n_*=2\ ,\ \iota_*(\_,\_)=(\_,\_)$$
$$\oplus/\& \qquad I=\{1,2\}\ ,\ n_1=n_2=1\ ,\ \iota_1(\_)=inl(\_)\ ,\ \iota_2(\_)=inr(\_)$$

Another general connective that we shall meet later (the direct style shift operators) :

$$\Downarrow/\Uparrow \qquad I=\{*\}\ ,\ n_*=1\ ,\ \iota_*(\_)=\_^{\Downarrow}$$

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	$\boxed{\text{LL}_{foc}}$
Indirect		

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		

Syntax for LL_{foc}

The formulas are those of linear logic.

The judgements are the same as for LK_{foc}. As above, we use a common notation V for values and negative terms.

The syntax of terms is as for LK_{foc}, with addtional constructs for exponentials :

$$\begin{array}{l} \vdots \\ V^+ ::= \ldots \mid \mu x^!.c \\ \vdots \\ t^- ::= \ldots \mid V^! \end{array}$$

Typing rules for LL_{foc}

$$\frac{c : (\vdash x : A, ?\Gamma)}{\vdash \mu x^{!}.c : !A \,;\, ?\Gamma} \qquad \frac{\vdash V : A \parallel \Gamma}{\vdash V^{!} : ?A \mid \Gamma}$$

$$\frac{c : (\vdash \Gamma)}{c : (\vdash x^{+} : ?A, \Gamma)} \qquad \frac{c : (\vdash x_1^{+} : ?A, x_2^{+} : ?A, \Gamma)}{c[x_2^{+}/x_1^{+}] : (\vdash x_2^{+} : ?A, \Gamma)}$$

Rest of the rules as for LK_{foc}

Reduction rules for LL_{foc}

$$\vdots$$

$$\langle \mu x^{!}.c \mid V^{!} \rangle \to c[V/x]$$

Again, no critical pairs anymore.

Completeness of LL_{foc}

Following a technique in Girard's LC paper (adapted to LL in Laurent's notes on focalisation), we exhibit a translation from LL proofs to LL_{foc} proofs. The translation is the identity on formulas and on judgements. The translation maps variables to themselves, commutes with all μ constructs, and with the command building construct. The remaining cases are :

$$\begin{array}{l}
[\![(t_1,t_2)]\!]_{\mathtt{foc}} = \mu x^-.\langle [\![t_1]\!]_{\mathtt{foc}} \mid \mu x_1.\langle [\![t_2]\!]_{\mathtt{foc}} \mid \mu x_2.\langle (x_1,x_2) \mid x^- \rangle\rangle\rangle \\
[\![inl(t_1)]\!]_{\mathtt{foc}} = \mu x^-.\langle [\![t_1]\!]_{\mathtt{foc}} \mid \mu x_1.\langle inl(x_1) \mid x^- \rangle\rangle \\
[\![t^!]\!]_{\mathtt{foc}} = \mu y^+.\langle [\![t]\!]_{\mathtt{foc}} \mid \mu x.\langle y^+ \mid x^! \rangle\rangle
\end{array}$$

Note the (arbitrary) choice of order of evaluation in the first rule.

The translation introduces cuts, which are then eliminated by cut-elimination. Therefore, every provable sequent of LL (possibly with cuts) admits a cut-free focalised proof (Andreoli). The translation achieves the most important part of the job of CPS translations, which is to **fix an order of evaluation** !

A variation : focalised reduction of non-focalised proofs

In the systems LK$_{foc}$ and LL$_{foc}$ presented here, we restrict both
– the space of proofs, and
– the reduction rules.

One may stay with a non-focalised syntax that does not restrict the space of proofs. This is the choice adopted in Guillaume Munch-Maccagnoni's writings : One then has to add further rules, such as

$$\langle (t_1, t_2) \mid t^- \rangle \rightarrow \langle t_1 \mid \mu x_1.\langle t_2 \mid \mu x_2.\langle (x_1, x_2) \mid t^- \rangle\rangle\rangle$$

that force focalisation "on the fly" (cf. translation in previous slide). We propose to reserve the subscript $_{pol}$ for such systems (not considered here).

IV) Indirect style

1. $\mathsf{LL}_{\downarrow}$
2. Translation into (a subset TL_{foc}) of Melliès' tensor logic
3. $\mathsf{LK}_{\downarrow}$ (monolateral)
4. $\mathsf{LK}_{\downarrow}$ (bilateral, distinguishing lazy programs from contexts of positive type)
5. Levy's CBPV
6. Perspective on the monadic reading of shifts

Focalised syntax in indirect style

We move from polarised formulas to **polarised connectives** : we now force the positive connectives $\otimes$, $\oplus$ to have positive formulas as arguments, and dually for the negative connectives $⅋$, $\&$.

For achieving this, we need two new connectives, which cristallised in ludics and game semantics (Girard, Laurent) : the shifts (for which one may also have under certain circumstances a monadic reading, as we shall see, whence the title of this slide).

We shall call the resulting linear and classical systems $\mathrm{LL}_{\downarrow}$, $\mathrm{LK}_{\downarrow}$.

Illustrating indirect versus direct

Direct

$$\frac{\vdash N \mid \Gamma_1 \quad \vdash P \,;\, \Gamma_2}{\vdash N \otimes P \,;\, \Gamma_1, \Gamma_2}$$

Indirect

$$\frac{\dfrac{\vdash N \mid \Gamma_1}{\vdash {\downarrow}N \,;\, \Gamma_1} \quad \vdash P \,;\, \Gamma_2}{\vdash {\downarrow}N \otimes P \,;\, \Gamma_1, \Gamma_2}$$

Read (bottom-up) $\downarrow$ as marking explicitly the exit from the focalisation phase.

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect		$\boxed{\mathrm{LL}_{\downarrow}}$

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		

Syntax for LL$_\downarrow$

Formulas :

$$P ::= X \mid P \otimes P \mid P \oplus P \mid !N \mid \downarrow N \qquad N ::= \overline{X} \mid N \parr N \mid N \& N \mid ?P \mid \uparrow P$$

We still have the same four kinds of judgements :

Commands	Values	Positive terms	Negative terms
$c : (\vdash \Gamma)$	$\vdash V^+ : P\,;\,\Gamma$	$\vdash t^+ : P \mid \Gamma$	$\vdash t^- : N \mid \Gamma$

Terms :

$$\begin{array}{l} c ::= \langle t^+ \mid t^- \rangle \\ V^+ ::= x^+ \mid (V_1^+, V_2^+) \mid inl(V^+) \mid inr(V^+) \mid \mu(x^+)^!.c \mid (t^-)^\downarrow \\ t^+ ::= V^+ \mid \mu x^-.c \\ t^- ::= x^- \mid \mu x^+.c \mid \mu(x_1^+, x_2^+).c \mid \mu[inl(x_1^+).c_1, inr(x_2^+).c_2] \\ \qquad \mid (V^+)^! \mid \mu(x^-)^\downarrow.c \end{array}$$

Typing rules for $\mathsf{LL}_{\downarrow}$

$$\frac{}{\vdash x^+ : P \,;\, x^+ : \overline{P}} \qquad \frac{}{\vdash x^- : N \mid x^- : \overline{N}} \qquad \frac{\vdash t^+ : P \mid \Gamma \quad \vdash t^- : \overline{P} \mid \Delta}{\langle t^+ \mid t^- \rangle : (\vdash \Gamma, \Delta)}$$

$$\frac{\vdash V^+ : P \,;\, \Gamma}{\vdash V^+ : P \mid \Gamma} \quad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x.c : A \mid \Gamma} \quad \frac{c : (\vdash \Gamma)}{c : (\vdash x^+ : ?A, \Gamma)} \quad \frac{c : (\vdash x_1^+ : ?A, x_2^+ : ?A, \Gamma)}{c[x_2^+/x_1^+] : (\vdash x_2^+ : ?A, \Gamma)}$$

$$\frac{\vdash V_1^+ : P_1 \,;\, \Gamma \qquad \vdash V_2^+ : P_2 \,;\, \Delta}{\vdash (V_1^+, V_2^+) : P_1 \otimes P_2 \,;\, \Gamma, \Delta} \qquad \frac{\vdash V_1^+ : P_1 \,;\, \Gamma}{\vdash \mathit{inl}(V_1^+) : P_1 \oplus P_2 \,;\, \Gamma}$$

$$\frac{c : (\vdash x_1^+ : N_1, x_2^+ : N_2, \Gamma)}{\vdash \mu(x_1^+, x_2^+).c : N_1 \parr N_2 \mid \Gamma} \qquad \frac{c_1 : (\vdash x_1^+ : N_1, \Gamma) \quad c_2 : (\vdash x_2^+ : N_2, \Gamma)}{\vdash \mu[\mathit{inl}(x_1^+).c_1, \mathit{inr}(x_2^+).c_2] : N_1 \& N_2 \mid \Gamma}$$

$$\frac{c : (\vdash x^+ : N, ?\Gamma)}{\vdash \mu(x^+)^!.c : !N \,;\, ?\Gamma} \quad \frac{\vdash V^+ : P \,;\, \Gamma}{\vdash (V^+)^! : ?P \mid \Gamma} \quad \frac{\vdash t^- : N \mid \Gamma}{\vdash (t^-)^{\downarrow} : {\downarrow}N \,;\, \Gamma} \quad \frac{c : (\vdash x^- : P, \Gamma)}{\vdash \mu(x^-)^{\downarrow}.c : {\uparrow}P \mid \Gamma}$$

Reduction rules for $\mathsf{LL}_{\downarrow}$

$$\langle V^{+} \mid \mu x^{+}.c\rangle \rightarrow c[V^{+}/x^{+}]$$
$$\langle \mu x^{-}.c \mid t^{-}\rangle \rightarrow c[t^{-}/x^{-}]$$
$$\langle (V_1^{+}, V_2^{+}) \mid \mu(x_1^{+}, x_2^{+}).c\rangle \rightarrow c[V_1^{+}/x_1^{+}, V_2^{+}/x_2^{+}]$$
$$\langle inl(V_1^{+}) \mid \mu[inl(x_1^{+}).c_1, inr(x_2^{+}).c_2]\rangle \rightarrow c_1[V_1^{+}/x_1^{+}]$$
$$\langle \mu(x^{+})^{!}.c \mid (V^{+})^{!}\rangle \rightarrow c[V^{+}/x^{+}]$$
$$\langle (t^{-})^{\downarrow} \mid \mu(x^{-})^{\downarrow}.c\rangle \rightarrow c[t^{-}/x^{-}]$$

Decomposing the exponentials : $!N = \downarrow\sharp N$ ...

Confronting the rules for the exponentials above with the rules for shifts, one may be tempted by the following decomposition :

$$!N = \downarrow\sharp N$$

with the following syntax of formulas : $P ::= \ldots \parallel \flat P \quad N ::= \ldots \parallel \sharp N$

(with ... as before, minus the "normal" exponentials ! and ?), and with the following rules :

$$\frac{c : (\vdash x^+ : N, \uparrow\flat\Gamma)}{\vdash \mu(x^+)^\flat.c : \sharp N \mid \uparrow\flat\Gamma} \qquad \frac{\vdash V^+ : P \,;\, \Gamma}{\vdash (V^+)^\flat : \flat P \,;\, \Gamma}$$

$$\langle (V^+)^\flat \mid \mu(x^+)^\flat.c\rangle \to c[V^+/x^+]$$

This decomposition of the exponential modality appeared also in works on proof nets and on light linear logic (Girard and/or folklore). These rules make good sense in terms of focalised proof search (dereliction is irreversible, promotion is reversible).But the first typing rule with its side condition involving the old ? after all does not make it really convincing.

... or the other way around : $!N = !_e \downarrow N$

One can also decompose "of course" (as in tensor logic) as $!N = !_e \downarrow N$:

$$P ::= \ldots \parallel !_e P \quad N ::= \ldots \parallel ?_e N$$

with the following rules :

$$\frac{c : (\vdash x^- : P, ?_e \Gamma)}{\vdash \mu(x^-)^{!_e}.c : !_e P\, ;\ ?_e \Gamma} \qquad \frac{\vdash t^- : N \mid \Gamma}{\vdash (t^-)^{!_e} : ?_e N \mid \Gamma}$$

$$\langle \mu(x^-)^{!_e}.c \mid (t^-)^{!_e} \rangle \to c[t^- / x^-]$$

It is easy to see that conversely, if one keeps $!, ?$ as primitive and if one defines $!_e P = ! \uparrow P$ and $?_e$ dually, then the above rules are derivable.

Translating LL_{foc} into $\mathsf{LL}_{\downarrow}$ (types)

Translation of types (the translation goes the same for $\oplus$ as for $\otimes$ and the same for $\&$ as for $\parr$) :

$$
\begin{array}{ll}
[\![X]\!]_{\downarrow} = X & [\![\overline{X}]\!]_{\downarrow} = \overline{X} \\
[\![P_1 \otimes P_2]\!]_{\downarrow} = [\![P_1]\!]_{\downarrow} \otimes [\![P_2]\!]_{\downarrow} & [\![N_1 \parr N_2]\!]_{\downarrow} = [\![N_1]\!]_{\downarrow} \parr [\![N_2]\!]_{\downarrow} \\
[\![N_1 \otimes P_2]\!]_{\downarrow} = {\downarrow}[\![N_1]\!]_{\downarrow} \otimes [\![P_2]\!]_{\downarrow} & [\![N_1 \parr P_2]\!]_{\downarrow} = [\![N_1]\!]_{\downarrow} \parr {\uparrow}[\![P_2]\!]_{\downarrow} \\
[\![P_1 \otimes N_2]\!]_{\downarrow} = [\![P_1]\!]_{\downarrow} \otimes {\downarrow}[\![N_2]\!]_{\downarrow} & \\
[\![N_1 \otimes N_2]\!]_{\downarrow} = {\downarrow}[\![N_1]\!]_{\downarrow} \otimes {\downarrow}[\![N_2]\!]_{\downarrow} & \vdots \\
[\![!P]\!]_{\downarrow} = !{\uparrow}[\![P]\!]_{\downarrow} & \\
[\![!N]\!]_{\downarrow} = ![\![N]\!]_{\downarrow} & [\![?P]\!]_{\downarrow} = ?[\![P]\!]_{\downarrow}
\end{array}
$$

Translating LL_{foc} into $\mathsf{LL}_{\downarrow}$ (terms)

Variables are translated to themselves. We give only the cases where the translation does not commute with the constructors :

$$[\![(t_1^-, V_2^+)]\!]_{\downarrow} = (([\![t_1^-]\!]_{\downarrow})^{\downarrow}, [\![V_2^+]\!]_{\downarrow}) \quad \text{(idem for } [\![(V_1^+, t_2^-)]\!]_{\downarrow})$$
$$[\![(t_1^-, t_2^-)]\!]_{\downarrow} = (([\![t_1^-]\!]_{\downarrow})^{\downarrow}, ([\![t_2^-]\!]_{\downarrow})^{\downarrow})$$
$$[\![inl(t^-)]\!]_{\downarrow} = inl(([\![t^-]\!]_{\downarrow})^{\downarrow})$$
$$[\![\mu(x^-)^!.c]\!]_{\downarrow} = \mu(y^+)^!.\langle y^+ \mid \mu(x^-)^{\downarrow}.[\![c]\!]_{\downarrow}\rangle$$

$$[\![\mu(x_1^-, x_2^+).c]\!]_{\downarrow} = \mu(y_1^+, x_2^+).\langle y_1^+ \mid \mu(x_1^-)^{\downarrow}.[\![c]\!]_{\downarrow}\rangle \quad \text{(idem for } [\![\mu(x_1^+, x_2^-).c]\!]_{\downarrow})$$
$$[\![\mu(x_1^-, x_2^-).c]\!]_{\downarrow} = \mu(y_1^+, y_2^+).\langle y_2^+ \mid \mu(x_2^-)^{\downarrow}\langle y_1^+ \mid \mu(x_1^-)^{\downarrow}.[\![c]\!]_{\downarrow}\rangle\rangle$$
$$[\![\mu[inl(x_1^-).c_1, inr(x_2^+).c_2]]\!]_{\downarrow} = \mu[inl(y_1^+).\langle y_1^+ \mid \mu(x_1^-)^{\downarrow}.[\![c_1]\!]_{\downarrow}\rangle, inr(x_2^+).c_2]$$
$$\vdots$$
$$[\![(t^-)^!]\!]_{\downarrow} = (([\![t^-]\!]_{\downarrow})^{\downarrow})^!$$

The translation is reduction-preserving.

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect		$\mathsf{LL}_{\downarrow} \supseteq \boxed{\mathsf{TL}_{foc}}$

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		

Translating into Melliès' tensor logic

Morally, tensor logic is the intuitionistic restriction of $\text{LL}_{\downarrow}$, where sequents admit at most one positive formula (see below for a more detailed analysis). More precisely, we shall consider a *focalised* subsystem of tensor logic, which we call TL_{foc}. The formulas are :

$$P ::= X \parallel P \otimes P \parallel P \oplus P \parallel \,!_e P \parallel \downarrow N \qquad N ::= \overline{X} \parallel N \parr N \parallel N \& N \parallel \,?_e N \parallel \uparrow P$$

There are only three kinds of judgements (Γ consists of negative formulas only) :

Commands	Values	Negative terms
$c : (\vdash \Gamma)$	$\vdash V^{+} : P \,;\, \Gamma$	$\vdash t^{-} : N \mid \Gamma$

But we have to move to different rules for the shifts and (polarity-keeping) exponentials : there is no space anymore to form $\mu(x^{-})^{!_e}.c$ and $\mu(x^{-})^{\downarrow}.c$ and (see also the discussion on syntactic adjunctions below).

The relation of $\text{LL}_{\downarrow}$ to tensor logic is the same as the relation of $\text{LK}_{\downarrow}$ to Laurent's LLP.

Syntax for TL_{foc}

Terms :

$$
\begin{aligned}
c &::= \langle V^{+} \mid t^{-} \rangle \\
V^{+} &::= x^{+} \parallel (V_1^{+}, V_2^{+}) \parallel inl(V^{+}) \parallel inr(V^{+}) \parallel (V^{+})^{\boldsymbol{!}_e} \parallel \mu(x^{+})^{\downarrow}.c \\
t^{-} &::= \mu x^{+}.c \parallel \mu(x_1^{+}, x_2^{+}).c \parallel \mu[inl(x_1^{+}).c_1, inr(x_2^{+}).c_2] \\
&\quad \parallel \mu(x^{+})^{\boldsymbol{!}_e}.c \parallel (V^{+})^{\downarrow}
\end{aligned}
$$

Typing rules for TL$_{foc}$ (negative contexts only !)

$$\frac{}{\vdash x^+ : P \,;\, x^+ : \overline{P}} \qquad \frac{c : (\vdash x^+ : N, \Gamma)}{\vdash \mu x^+.c : N \mid \Gamma} \qquad \frac{\vdash V^+ : P \,;\, \Gamma_1 \quad \vdash t^- : \overline{P} \mid \Gamma_2}{\langle V^+ \mid t^- \rangle : (\vdash \Gamma_1, \Gamma_2)}$$

$$\frac{\vdash V_1^+ : P_1 \,;\, \Gamma_1 \qquad \vdash V_2^+ : P_2 \,;\, \Gamma_2}{\vdash (V_1^+, V_2^+) : P_1 \otimes P_2 \,;\, \Gamma_1, \Gamma_2} \qquad \frac{\vdash V_1^+ : P_1 \,;\, \Gamma}{\vdash inl(V_1^+) : P_1 \oplus P_2 \,;\, \Gamma}$$

$$\frac{c : (\vdash x_1^+ : N_1, x_2^+ : N_2, \Gamma)}{\vdash \mu(x_1^+, x_2^+).c : N_1 \parr N_2 \mid \Gamma} \qquad \frac{c_1 : (\vdash x_1^+ : N_1, \Gamma) \quad c_2 : (\vdash x_2^+ : N_2, \Gamma)}{\vdash \mu[inl(x_1^+).c_1, inr(x_2^+).c_2] : N_1 \& N_2 \mid \Gamma}$$

$$\frac{\vdash V^+ : P \,;\, ?\Gamma}{\vdash (V^+)^{!} : !P \,;\, ?\Gamma} \qquad \frac{c : (\vdash x^+ : N, \Gamma)}{\vdash \mu(x^+)^{!}.c : ?N \mid \Gamma}$$

$$\frac{c : (\vdash x^+ : N, \Gamma)}{\vdash \mu(x^+)^{\downarrow}.c : \downarrow N \,;\, \Gamma} \qquad \frac{\vdash V^+ : P \,;\, \Gamma}{\vdash (V^+)^{\downarrow} : \uparrow P \mid \Gamma}$$ and contraction and weakening

Reduction rules for TL_{foc}

$$
\begin{array}{l}
\langle V^+ \mid \mu x^+.c\rangle \to c[V^+/x^+] \\
\langle (V_1^+, V_2^+) \mid \mu(x_1^+, x_2^+).c\rangle \to c[V_1^+/x_1^+, V_2^+/x_2^+] \\
\langle inl(V_1^+) \mid \mu[inl(x_1^+).c_1, inr(x_2^+).c_2]\rangle \to c_1[V_1^+/x_1^+] \\
\langle (V^+)^{\overset{!}{e}} \mid \mu(x^+)^{\overset{!}{e}}.c\rangle \to c[V^+/x^+] \\
\langle \mu(x^+)^{\downarrow}.c \mid (V^+)^{\downarrow}\rangle \to c[V^+/x^+]
\end{array}
$$

Melliès authorises (at most) one positive formula in the non-value judgements (in particular, his system is not focalised). But the target of our translation does not need this liberality.

Translating $\mathsf{LL}_{\downarrow}$ to TL_{foc}

Expand $!N$ as $\underset{e}{!}\downarrow N$.

For judgements (Γ negative, $\Delta = x^- : P, \ldots$ positive, $\uparrow\Delta = k^+_{x^-} : \uparrow P, \ldots$) :

$$\begin{array}{cccc} c : (\vdash \Gamma, \Delta) & \vdash V^+ : P\,;\, \Gamma, \Delta & \vdash t^+ : P \mid \Gamma, \Delta & \vdash t^- : N \mid \Gamma, \Delta \\ [\![c]\!]_{\mathsf{TL}} : (\vdash \Gamma, \uparrow\Delta) & \vdash [\![V^+]\!]_{\mathsf{TL}} : P\,;\, \Gamma, \uparrow\Delta & \vdash [\![t^+]\!]_{\mathsf{TL}} : \uparrow P \mid \Gamma, \uparrow\Delta & \vdash [\![t^-]\!]_{\mathsf{TL}} : N \mid \Gamma, \uparrow\Delta \end{array}$$

The only cases where the translation does not commute with the constructors are the following :

$$\begin{array}{l}
[\![x^-]\!]_{\mathsf{TL}} = \mu y^+.\langle k^+_{x^-} \mid (y^+)^{\downarrow}\rangle \\
[\![\langle t^+ \mid t^-\rangle]\!]_{\mathsf{TL}} = \langle \mu(x^+)^{\downarrow}.\langle x^+ \mid [\![t^-]\!]_{\mathsf{TL}}\rangle \mid [\![t^+]\!]_{\mathsf{TL}}\rangle \\
[\![t^+]\!]_{\mathsf{TL}} = \begin{cases} [\![V^+]\!]^{\downarrow}_{\mathsf{TL}} & \text{if } t^+ = V^+ \\ \mu k^+_{x^-}.[\![c]\!]_{\mathsf{TL}} & \text{si } t^+ = \mu x^-.c \end{cases} \\
[\![\mu(x^+)^!.c]\!]_{\mathsf{TL}} = (\mu(x^+)^{\downarrow}.[\![c]\!]_{\mathsf{TL}})^{\underset{e}{!}} \\
[\![(V^+)^!]\!]_{\mathsf{TL}} = \mu(y^+)^{\underset{e}{!}}.\langle y^+ \mid ([\![V^+]\!]_{\mathsf{TL}})^{\downarrow}\rangle \\
[\![(t^-)^{\downarrow}]\!]_{\mathsf{TL}} = \mu(y^+)^{\downarrow}.\langle y^+ \mid [\![t^-]\!]_{\mathsf{TL}}\rangle \\
[\![\mu(x^-)^{\downarrow}.c]\!]_{\mathsf{TL}} = \mu k^+_{x^-}.[\![c]\!]_{\mathsf{TL}}
\end{array}$$

Note that we can optimize the translation of $\langle t^+ \mid t^-\rangle$ when t^+ is a value :

$$[\![\langle V^+ \mid t^-\rangle]\!]_{\mathsf{TL}} = \langle [\![V^+]\!]_{\mathsf{TL}} \mid [\![t^-]\!]_{\mathsf{TL}}\rangle$$

Two levels of indirection

We have that

- $\mathsf{LL}_{\downarrow}$ is more direct than LL_{foc}

- TL_{foc} is more direct than $\mathsf{LL}_{\downarrow}$: the translation of sequents introduces further shifts !

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect	$\boxed{\text{“Indirect LL”} \supseteq \mathrm{TL}}$	$\mathrm{LL}_{\downarrow} \supseteq \mathrm{TL}_{foc}$

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		

Non focalised indirect style

We could have presented LL rather than LL_{foc} in indirect style, and the resulting system would genuinely have the original tensor logic as a subsystem.

In the linear setting, focalisation and indirect style commute.

In the classical setting, translating to indirect style first a priori does not make sense, because of the non-confluence problem, but it does make sense if one always restricts contractions and weakening to negative formulas (cf. discussion concluding part II), and one then recovers LLP as a subsystem (in which the invariant "at most one positive formula" is enforced by further restricting the context to be negative in the $\downarrow$ rule).

About natural deduction...

One may give a natural deduction presentation of (our restriction of) tensor logic, and a corresponding λ-calculus style syntax, typing and reduction rules, in which the new connective

$\neg_{\downarrow} P$ (the negation of tensor logic) stands for $\downarrow\overline{P}$:

$$c ::= t_1 t_2$$
$$t ::= x \mid (t_1, t_2) \mid inl(t) \mid inr(t) \mid t^{!} \mid \lambda x.c \mid \lambda(x_1, x_2).c \mid \lambda z.\textsf{case}\ z\ [inl(x_1) \mapsto c_1, inr(x_2) \mapsto c_2)] \mid \lambda x^{!}.c$$

$$\frac{\Gamma_1 \vdash t_1 : \neg_{\downarrow} P \quad \Gamma_2 \vdash t_2 : P}{\Gamma \vdash t_1 t_2} \qquad \frac{\Gamma, x : P \vdash c}{\Gamma \vdash \lambda x.c : \neg_{\downarrow} P} \qquad \ldots \qquad \frac{!\Gamma \vdash t : P}{!\Gamma \vdash t^{!} : !P}$$

$$\ldots \qquad \frac{\Gamma, x_1 : P_1 \vdash c_1 \quad \Gamma, x_2 : P_2 \vdash c_2 : P_2}{\Gamma \vdash \lambda z.\textsf{case}\ z\ [inl(x_1) \mapsto c_1, inr(x_2) \mapsto c_2)] : \neg_{\downarrow}(P_1 \oplus P_2)} \qquad \frac{\Gamma, x : P \vdash c}{\Gamma \vdash \lambda x^{!}.c : \neg_{\downarrow}(!P)}$$

$$(\lambda x.c)t \to c[t/x] \qquad (\lambda(x_1, x_2).c)(t_1, t_2) \to c[t_1/x_1, t_2/x_2] \qquad \ldots \qquad (\lambda x^{!}.c)(t^{!}) \to c[t/x]$$

... and focalisation

and design reduction-preserving translations between this natural deduction system and the sequent calculus system TL_{foc}.

$$
\begin{array}{rcl}
\text{sequent calculus} & & \text{natural deduction} \\[1em]
\vdash V^{+} : P \,;\, \Gamma & \begin{array}{c}\longmapsto \\ \longleftarrow\!\dashv\end{array} & \tilde{\Gamma} \vdash t : P \\[1em]
c : (\vdash \Gamma) & \begin{array}{c}\longmapsto \\ \longleftarrow\!\dashv\end{array} & \tilde{\Gamma} \vdash c \\[1em]
\vdash t^{-} : N \mid \Gamma & \longmapsto & \tilde{\Gamma} \vdash t : \neg_{\downarrow}\overline{N}
\end{array}
$$

where $\tilde{\Gamma}$ is the variation of $\overline{\Gamma}$ which maps $\uparrow P$ to $\neg_{\downarrow} P$.

These correspondences can be shown to be inverse, making use of η-rules (for the third line, starting from t^{-}, one returns to $\mu(x^{+})^{\downarrow}.\langle x^{+} \mid t^{-}\rangle$).

This can be seen as an additional motivation for focalisation. We prefer to see it as a bonus.

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect		$\mathsf{LL}_{\downarrow} \supseteq \mathsf{TL}_{foc}$

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		$\boxed{\text{(monolateral) } \mathsf{LK}_{\downarrow}}$

Monolateral LK$_{\downarrow}$

This system is obtained from (monolateral) LL$_{\downarrow}$ by removing the exponential rules, and by allowing contraction rules on all formulas, like in LK$_{foc}$.

Encoding call-by-value and call-by-name λ-calculus (indirect style)

Call-by-value implication : $P \to_v Q = {\downarrow}(\overline{P} \parr ({\uparrow}Q))$

λ-terms are translated to positive terms (and λ-abstractions to values).

Judgements $(\ldots, x^+ : P, \ldots \vdash t^+ : Q)$ are encoded as : $(\vdash t^+ : Q \mid \ldots, x^+ : \overline{P}, \ldots)$

$$\left.\begin{array}{l} \lambda x^+.t^+ \\ t_1^+ t_2^+ \end{array}\right\} \text{are encoded as} \left\{\begin{array}{l} (\mu(x^+, (y^-)^{\downarrow}).\langle t^+ \mid y^- \rangle)^{\downarrow} \\ \mu y^-.\langle t_1^+ \mid \mu(z^-)^{\downarrow}.\langle (t_2^+, (y^-)^{\downarrow}) \mid z^- \rangle \rangle \end{array}\right.$$

where

$\mu(x^+, (y^-)^{\downarrow}).c = \mu(x^+, z^+).\langle z^+ \mid \mu(y^-)^{\downarrow}.c \rangle$ (compound pattern-matching)

$(t^+, V^+) = \mu z^-.\langle t^+ \mid \mu(x^+).\langle (x^+, V^+) \mid z^- \rangle \rangle$ (encoding of a non-focalised proof, cf. slide 35)

Call-by-name implication : $M \to_n N = ({\uparrow}\overline{M}) \parr N$

$$\left.\begin{array}{l} \lambda x^-.t^- \\ t_1^- t_2^- \end{array}\right\} \text{are encoded as} \left\{\begin{array}{l} \mu((x^-)^{\downarrow}, y^+).\langle y^+ \mid t^- \rangle \\ \mu y^+.\langle ((t_2^-)^{\downarrow}, y^+)) \mid t_1^- \rangle \end{array}\right.$$

These encodings extend straightforwardly to the CBV and CBN $\lambda\mu$-calculi.

What about direct style encodings ?

It is tempting to return to direct style. In the encodings

$$P \to_v Q = {\downarrow}(\overline{P} \parr ({\uparrow}Q)) \quad \text{and} \quad M \to_n N = ({\uparrow}\overline{M}) \parr N$$

the second and third shifts can be reconstructed, but not the first, which forces hereditary positivity of the translation. Compare (omitting the shifts that can be reconstructed, and using a new notation for the remaining ones)

$$\begin{array}{rcl} P_1 \to_v (P_2 \to_v P_3) & = & {\Downarrow}(\overline{P_1} \parr ({\Downarrow}(\overline{P_2} \parr P_3)) \\ N_1 \to_n (N_2 \to_n N_3) & = & \overline{N_1} \parr \overline{N_2} \parr N_3 \end{array}$$

Conclusion : we need also shifts in direct style !

Shifts in direct style

$$P ::= \ldots \parallel {\Downarrow}A \qquad N ::= \ldots \parallel {\Uparrow}A$$

$$\frac{\vdash V : A \parallel \Gamma}{\vdash V^{\Downarrow} : {\Downarrow}A \,;\, \Gamma} \qquad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x^{\Downarrow}.c : {\Uparrow}A \mid \Gamma}$$

$$\langle V^{\Downarrow} \mid \mu x^{\Downarrow}.c\rangle \to c[V/x]$$

Encoding call-by-value and call-by-name λ-calculus (direct style)

$M \to_n N = \overline{M} \parr N$ $\qquad$ $P \to_v Q = \Downarrow(\overline{P} \parr Q)$

$\lambda x^-.t^- = \mu(x^-, y^+).\langle y^+ \mid t^- \rangle$ $\qquad$ $\lambda x^+.t^+ = (\mu(x^+, y^-).\langle t^+ \mid y^- \rangle)^{\Downarrow}$

$t_1^- t_2^- = \mu y^+.\langle (t_2^-, y^+) \mid t_1^- \rangle$ $\qquad$ $t_1^+ t_2^+ = \mu y^-.\langle t_1^+ \mid \mu(z^-)^{\Downarrow}.\langle (t_2^+, y^-) \mid z^- \rangle \rangle$

We have :

$$
\begin{array}{lcl}
(\lambda x^-.t^-)t_2^- & = & \mu y^+.\langle (t_2^-, y^+)) \mid \mu(x^-, y^+).\langle y^+ \mid t^- \rangle \rangle \\
 & \to & \mu y^+.\langle y^+ \mid t^-[t_2^-/x^-] \rangle \qquad (t_2^- \text{ is a } V!)
\end{array}
$$

$$
\begin{array}{lcl}
(\lambda x^+.t^+)t_2^+ & = & \mu y^-.\langle (\mu(x^+, y^-).\langle t^+ \mid y^- \rangle)^{\Downarrow} \mid \mu(z^-)^{\Downarrow}.\langle (t_2^+, y^-) \mid z^- \rangle \rangle \\
 & \to & \mu y^-.\langle (t_2^+, y^-) \mid \mu(x^+, y^-).\langle t^+ \mid y^- \rangle \rangle \qquad (\Downarrow \text{ reduction}) \\
 & = & \mu y^-.\langle \mu z^-.\langle t_2^+ \mid \mu x^+.\langle (x^+, y^-) \mid z^- \rangle \rangle \mid \mu(x^+, y^-).\langle t^+ \mid y^- \rangle \rangle \\
 & \to & \mu y^-.\langle t_2^+ \mid \mu x^+.\langle (x^+, y^-) \mid \mu(x^+, y^-).\langle t^+ \mid y^- \rangle \rangle \rangle \\
 & \to & \mu y^-.\langle t_2^+ \mid \mu x^+.\langle t^+ \mid y^- \rangle \rangle
\end{array}
$$

The point is that t_2^+ is not a V, and that (t_2^+, y^-) is a macro for $\mu z^-.\langle t_2^+ | \mu x^+.\langle (x^+, y^-) | z^- \rangle \rangle$.

System L reduction as an abstract machine

System L does not only account for reduction axioms, but for reduction in context (**Felleisen**) :

$$E[(\lambda x.t)t_2] \to E[t[t_2/x]]$$

Using E for V^+, in CBN, we can "read off"

$$\begin{array}{lll} t_1^- t_2^- = \mu y^+.\langle (t_2^-, y^+) \mid t_1^- \rangle & \text{as} & \langle E \mid t_1^- t_2^- \rangle \to \langle (t_2^-, E) \mid t_1^- \rangle \\ \lambda x^-.t^- = \mu(x^-, y^+).\langle y^+ \mid t^- \rangle & \text{as} & \langle (t_2^-, E) \mid \lambda x^-.t^- \rangle \to \langle E \mid t^-[t_2^-/x^-] \rangle \end{array}$$

Krivine abstract machine !

In CBV, setting $(t^+)^\Uparrow = \mu(z^-)\Downarrow.\langle t^+ \mid z^- \rangle$, and using e^+ for t^-, we read

$$\begin{array}{lll} t_1^+ t_2^+ = \mu y^-.\langle t_1^+ \mid \mu(z^-)^\Downarrow.\langle (t_2^+, y^-) \mid z^- \rangle\rangle & \text{as} & \langle t_1^+ t_2^+ \mid e^+ \rangle \to \langle t_1^+ \mid (t_2^+, e^+)^\Uparrow \rangle \\ \lambda x^+.t^+ = (\mu(x^+, y^-).\langle t^+ \mid y^- \rangle)^\Downarrow & \text{as} & \langle \lambda x^+.t^+ \mid (t_2^+, e^+)^\Uparrow \rangle \to^* \langle t_2^+ \mid \mu x^+.\langle t^+ \mid e^+ \rangle\rangle \end{array}$$

But it is odd to view a context E, e^+ as a term V^+, t^-. This will be repaired in a bilateral system.

Translating to intuitionistic logic

One translates positive formulas :

$$X_{cps} = X \qquad (P \otimes Q)_{cps} = P_{cps} \times Q_{cps} \qquad P \oplus Q_{cps} = P_{cps} + Q_{cps}$$

$$\downarrow\overline{P}_{cps} = R^{P_{cps}}$$

We set (Γ, Δ contexts of positive formulas) :

$$\Gamma_{cps} = \{x : P_{cps} \mid x : P \in \Gamma\} \qquad R^{\Delta_{cps}} = \{k_\alpha : R^{P_{cps}} \mid \alpha : P \in \Delta\}$$

We have :

$c : (\vdash \mathcal{P}, \mathcal{N})$ $\Downarrow$ $\overline{\mathcal{N}}_{cps}\, ,\ R^{\mathcal{P}_{cps}} \vdash c_{cps} : R$	$\vdash V^+ : P\, ;\, \mathcal{P}, \mathcal{N}$ $\Downarrow$ $\overline{\mathcal{N}}_{cps}\, ,\ R^{\mathcal{P}_{cps}} \vdash V_{cps} : P_{cps}$
$\vdash t^+ : P \mid \mathcal{P}, \mathcal{N}$ $\Downarrow$ $\overline{\mathcal{N}}_{cps}\, ,\ R^{\mathcal{P}_{cps}} \vdash t^+_{cps} : R^{R^{P_{cps}}}$	$\vdash t^- : N \mid \mathcal{P}, \mathcal{N}$ $\Downarrow$ $\overline{\mathcal{N}}_{cps}\, ,\ R^{\mathcal{P}_{cps}} \vdash t^-_{cps} : R^{\overline{N}_{cps}}$

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect		$\mathsf{LL}_{\downarrow} \supseteq \mathsf{TL}_{foc}$

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		(monolateral) $\mathsf{LK}_{\downarrow} \supseteq \boxed{\mathsf{LLP}_{foc}}$

(A focalised restriction of) LLP as a retract of $\mathsf{LK}_\downarrow$

Just as $\mathsf{LL}_\downarrow$ translates to TL_{foc}, $\mathsf{LK}_\downarrow$ translates to a focalised fragment LLP_{foc} of LLP (obtained by removing the rules for the exponentials from TL_{foc}).

Conversely, one can easily embed LLP_{foc} as a fragment of $\mathsf{LK}_\downarrow$, by expanding

$$\left.\begin{array}{l} (V^+)^\downarrow \\ \mu(x^+)^\downarrow.c \end{array}\right\} \text{as} \left\{\begin{array}{l} \mu(x^-)^\downarrow.\langle V^+ \mid x^-\rangle \\ (\mu x^+.c)^\downarrow \end{array}\right.$$

so as to exhibit LLP_{foc} as a retract of $\mathsf{LK}_\downarrow$.

(The same does not seem to hold for TL_{foc} with respect to $\mathsf{LL}_\downarrow$ because of the different styles of exponentials in the two systems. We cannot adopt $¡P, ¿N$ as primitive for the syntax of $\mathsf{LL}_\downarrow$ as this would hinder the translation to TL_{foc}).

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect		$\mathsf{LL}_{\downarrow} \supseteq \mathsf{TL}_{foc}$

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		$\boxed{\text{(bilateral) } \mathsf{LK}_{\downarrow}}$

Two notions of symmetry

In bilateral sequents, we can account for two kinds of symmetry / duality :

> the symmetry left-right corresponds to the symmetry input - output
> the duality positive-negative corresponds the duality eager-lazy

We shall illustrate this later with CBPV.

Different flavours of negation

Bilateral sequents are not only convenient to express further symmetries, but are also needed if we want an explicit **involutive negation**, rather than an implicit one (the overlining in our notation).

We should not confuse this involutive negation (explicit or implicit) with the negations $\neg_{\downarrow} P = \downarrow\overline{P}$ and $\neg_{\uparrow} N = \uparrow\overline{N}$, which are the ones involved in (the encoding of) call-by-value and call-by-name λ-calculus, and in tensor logic and LLP, as we have seen.

Formulas and judgements of bilateral LK$_{\downarrow}$

$$P ::= X \mid P \otimes Q \mid P \oplus Q \mid \neg N \mid {\downarrow}N$$
$$N ::= \overline{X} \mid N \parr N \mid N \,\&\, N \mid \neg P \mid {\uparrow}P$$
$$A ::= P \mid N$$

In sequents, Γ stands for $\ldots, x^+ : P, \ldots, x^- : N, \ldots$, and Δ stands for $\ldots, \alpha^+ : P, \ldots, \alpha^- : N, \ldots$. (Note that there may be positive and negative formulas both on the left and on the right of sequents)

There are now five kinds of judgements (we'll stop there, don't worry !) :

Commands	Values	Expressions	Covalues	Contexts
$c : (\Gamma \vdash \Delta)$	$\Gamma \vdash V^+ : P \,;\, \Delta$	$\Gamma \vdash v : A \mid \Delta$	$\Gamma \,;\, E^- : N \vdash \Gamma$	$\Gamma \mid e : A \vdash \Gamma$

(We could have done this bilateral extension keeping shifts implicit, cf. Munch-Maccagnoni's bilateral version of LK$_{pol}$.)

Syntax for bilateral $\mathsf{LK}_{\downarrow}$

(For the rest of the talk, we write V, E rather than V^+, E^-, for short)

Commands	$c ::= \langle v^+ \mid e^+ \rangle \parallel \langle v^- \mid e^- \rangle$
Expressions	$v^+ ::= V \parallel \mu\alpha^+.c$
	$v^- ::= x^- \parallel \mu\alpha^-.c \parallel \mu(\alpha^+)^{\uparrow}.c$
	$\parallel \mu[\alpha_1^-, \alpha_2^-].c \parallel \mu(\alpha_1^-[fst].c_1, \alpha_2^-[snd].c_2) \parallel \mu(x^+)^{\neg}.c$
Values	$V ::= x^+ \parallel (V, V) \parallel inl(V) \parallel inr(V) \parallel (v^-)^{\downarrow}$
Contexts	$e^- ::= E \parallel \tilde{\mu}x^-.c$
	$e^+ ::= \alpha^+ \parallel \tilde{\mu}x^+.c \parallel \tilde{\mu}(x^-)^{\downarrow}.c$
	$\parallel \tilde{\mu}(x^+, y^+).c \parallel \tilde{\mu}[inl(x_1^+).c_1, inr(x_2^+).c_2]$
Covalues	$E ::= \alpha^- \parallel [E, E] \parallel E[fst] \parallel E[snd] \parallel (e^+)^{\uparrow} \parallel V^{\neg}$

We can factorise a few rules using the following mergings :

$$v ::= v^+ \parallel v^- \qquad \alpha ::= \alpha^+ \parallel \alpha^- \qquad e ::= e^+ \parallel e^- \qquad x ::= x^+ \parallel x^-$$

Removing contraction and weakening rules altogether

For the rest of this talk,

– we push weakening to the axioms,

– we merge cut and contraction in an additive contraction rule, and

– we give an additive formulation of the right tensor rule.

The resulting system has no explicit weakening nor contraction rule.

Typing rules for bilateral $\mathsf{LK}_{\downarrow}$

$$\dfrac{}{\Gamma, x^+ : P \vdash x^+ : P \,;\, \Delta} \qquad \dfrac{}{\Gamma \,|\, \alpha^+ : P \vdash \alpha^+ : P, \Delta} \qquad \dfrac{}{\Gamma \,;\, \alpha^- : N \vdash \alpha^- : N, \Delta} \qquad \dfrac{}{\Gamma, x^- : N \vdash x^- : N \,|\, \Delta}$$

$$\dfrac{\Gamma \vdash v : A \,|\, \Delta \qquad \Gamma \,|\, e : A \vdash \Delta}{\langle v \,|\, e \rangle : (\Gamma \vdash \Delta)} \qquad \dfrac{c : (\Gamma \vdash \alpha : A \,,\, \Delta)}{\Gamma \vdash \mu\alpha.c : A \,|\, \Delta} \qquad \dfrac{c : (\,\Gamma, x : A \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}x.c : A \vdash \Delta} \qquad \dfrac{\Gamma \vdash V : P \,;\, \Delta}{\Gamma \vdash V : P \,|\, \Delta} \qquad \dfrac{\Gamma \,;\, E : N \vdash \Delta}{\Gamma \,|\, E : N \vdash \Delta}$$

$$\dfrac{\Gamma \vdash v^- : N \,|\, \Delta}{\Gamma \vdash (v^-)^{\downarrow} : {\downarrow}N \,;\, \Delta} \qquad \dfrac{\Gamma \vdash V_1 : P_1 \,;\, \Delta \qquad \Gamma \vdash V_2 : P_2 \,;\, \Delta}{\Gamma \vdash (V_1, V_2) : P_1 \otimes P_2 \,;\, \Delta} \qquad \dfrac{\Gamma \vdash V_1 : P_1 \,;\, \Delta}{\Gamma \vdash \mathit{inl}(V_1) : P_1 \oplus P_2 \,;\, \Delta} \qquad \dfrac{\Gamma \vdash V : P \,;\, \Delta}{\Gamma \,;\, V^{\neg} : \neg P \vdash \Delta}$$

$$\dfrac{c : (\Gamma \vdash \alpha^+ : P, \Delta)}{\Gamma \vdash \mu(\alpha^+)^{\uparrow}.c : {\uparrow}P \,|\, \Delta} \qquad \dfrac{c : (\Gamma \vdash \alpha_1^- : N_1 \,,\, \alpha_2^- : N_2 \,,\, \Delta)}{\Gamma \vdash \mu[\alpha_1^-, \alpha_2^-].c : N_1 \parr N_2 \,|\, \Delta}$$

$$\dfrac{c_1 : (\Gamma \vdash \alpha_1^- : N_1 \,,\, \Delta) \qquad c_2 : (\Gamma \vdash \alpha_2^- : N_2 \,,\, \Delta)}{\Gamma \vdash \mu(\alpha_1^-[\mathit{fst}].c_1, \alpha_2^-[\mathit{snd}].c_2) : N_1 \,\&\, N_2 \,|\, \Delta} \qquad \dfrac{c : (\Gamma, x : P \vdash \Delta)}{\Gamma \vdash \mu(x^+)^{\neg}.c : \neg P \,|\, \Delta}$$

$$\dfrac{\Gamma \,|\, e^+ : P \vdash \Delta}{\Gamma \,;\, (e^+)^{\uparrow} : {\uparrow}P \vdash \Delta} \qquad \dfrac{\Gamma \,;\, E_1 : N_1 \vdash \Delta \qquad \Gamma \,;\, E_2 : N_2 \vdash \Delta}{\Gamma \,;\, [E_1, E_2] : N_1 \parr N_2 \vdash \Delta} \qquad \dfrac{\Gamma \,;\, E_1 : N_1 \vdash \Delta}{\Gamma \,;\, E_1[\mathit{fst}] : N_1 \,\&\, N_2 \vdash \Delta} \qquad \dfrac{\Gamma \,;\, E : N \vdash \Delta}{\Gamma \vdash E^{\neg} : \neg N \,;\, \Delta}$$

$$\dfrac{c : (\Gamma, x^- : N \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}(x^-)^{\downarrow}.c : {\downarrow}N \vdash \Delta} \qquad \dfrac{c : (\Gamma, x_1^+ : P_1, x_2^+ : P_2 \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}(x_1^+, x_2^+).c : P_1 \otimes P_2 \vdash \Delta}$$

$$\dfrac{c_1 : (\Gamma, x_1^+ : P_1 \vdash \Delta) \qquad c_2 : (\Gamma, x_2^+ : P_2 \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}[\mathit{inl}(x_1^+).c_1, \mathit{inr}(x_2^+).c_2] : P_1 \oplus P_2 \vdash \Delta} \qquad \dfrac{c : (\Gamma \vdash \alpha^- : N, \Delta)}{\Gamma \,|\, \mu(\alpha^-)^{\neg}.c : \neg N \vdash \Delta}$$

Reduction rules for bilateral $\mathsf{LK}_{\downarrow}$

$$\langle V \mid \tilde{\mu}x^{+}.c\rangle \rightarrow c[V/x^{+}]$$
$$\langle \mu\alpha^{-}.c \mid E\rangle \rightarrow c[E/\alpha^{-}]$$
$$\langle v^{-} \mid \tilde{\mu}x^{-}.c\rangle \rightarrow c[v^{-}/x^{-}]$$
$$\langle \mu\alpha^{+}.c \mid e^{+}\rangle \rightarrow c[e^{+}/\alpha^{+}]$$
$$\langle (V_1, V_2) \mid \tilde{\mu}(x_1^{+}, x_2^{+}).c\rangle \rightarrow c[V_1/x_1^{+}, V_2/x_2^{+}]$$
$$\langle \mu[\alpha_1^{-}, \alpha_2^{-}].c \mid [E_1, E_2]\rangle \rightarrow c[E_1/\alpha_1^{-}, E_2/\alpha_2^{-}]$$
$$\langle inl(V_1) \mid \tilde{\mu}[inl(x_1^{+}).c_1, inr(x_2^{+}).c_2]\rangle \rightarrow c_1[V_1/x_1^{+}]$$
$$\langle \mu(\alpha_1^{-}[fst].c_1, \alpha_2^{-}[snd].c_2)) \mid E_1[fst]\rangle \rightarrow c_1[E_1/\alpha_1^{-}]$$
$$\langle (v^{-})^{\downarrow} \mid \tilde{\mu}(x^{-})^{\downarrow}.c\rangle \rightarrow c[v^{-}/x^{-}]$$
$$\langle \mu(\alpha^{+})^{\uparrow}.c \mid (e^{+})^{\uparrow}\rangle \rightarrow c[e^{+}/\alpha^{+}]$$
$$\langle \tilde{\mu}(x^{+})^{\neg}.c \mid V^{\neg}\rangle \rightarrow c[V/x^{+}]$$

Involutive negation is ... involutive !

First we exhibit

$$x^+ : P \vdash V : \neg\neg P \, ; \qquad \text{and} \qquad | \, e : \neg\neg P \vdash \alpha^+ : P$$
$$V = (x^+)^{\neg\neg} \qquad\qquad e = \mu(\beta^-)^{\neg}.\langle \mu(x^+)^{\neg}.\langle x^+ \mid \alpha^+\rangle \mid \beta^-\rangle$$

It is easily seen that $\langle V \,|\, e\rangle \to^* \langle x^+ \,|\, \alpha^+\rangle$. The converse direction is a bit trickier. We have to check that

$$\langle \mu\alpha^+.\langle z^+ \mid e\rangle \mid \tilde{\mu}x^+.\langle V \mid \gamma^+\rangle\rangle = \langle z^+ \mid \gamma^+\rangle$$

The left-hand side reduces to $\langle z^+ \,|\, \mu(\beta^-)^{\neg}.c\rangle$, where $c = \langle \mu(x^+)^{\neg}.\langle ((x^+)^{\neg\neg} \,|\, \gamma^+\rangle \,|\, \beta^-\rangle$. We have to show $\mu(\beta^-)^{\neg}.c = \gamma^+$. After η-expanding γ^+, this goal rephrases as $c = \langle (\beta^-)^{\neg} \,|\, \gamma^+\rangle$. Indeed, we have :

$$\begin{array}{lll} \langle (\beta^-)^{\neg} \mid \gamma^+\rangle & \leftarrow & \langle \mu\beta^-.\langle (\beta^-)^{\neg} \mid \gamma^+\rangle \mid \beta^-\rangle \\ & =_\eta & \langle \mu(x^+)^{\neg}.\langle \mu\beta^-.\langle (\beta^-)^{\neg} \mid \gamma^+\rangle \mid (x^+)^{\neg}\rangle \mid \beta^-\rangle \quad \to \quad c \end{array}$$

(Reference : annex A.2 of Munch-Maccagnoni's long version of "Focalisation and classical realisability")

Full deployment of CBV and CBN implications

We can now revisit CBV and CBN implications as compound connectives, exploiting bilaterality. (cf. Curien-Herbelin 2000)

$$P \to_v Q = {\downarrow}((\neg P) \parr ({\uparrow} Q)) \qquad M \to_n N = ({\uparrow}(\neg M)) \parr N$$

$$\frac{\Gamma \vdash V : P\,;\,\Delta \quad \Gamma \,|\, e^+ : Q \vdash \Delta}{\Gamma \,|\, V \cdot e^+ : P \to_v Q \vdash} \qquad \frac{\Gamma \vdash v^- : M \,|\, \Delta \quad \Gamma\,;\, E : N \vdash \Delta}{\Gamma\,;\, v^- \cdot E : M \to_n N \vdash}$$

$$\frac{\Gamma, x^+ : P \vdash v^+ : Q \,|\, \Delta}{\Gamma \vdash \lambda x^+.v^+ : P \to_v Q\,;\,\Delta} \qquad \frac{\Gamma, x^- : M \vdash v^- : N \,|\, \Delta}{\Gamma \vdash \lambda x^-.v^- : M \to_n N \,|\, \Delta}$$

$$\langle \lambda x^+.v^+ \mid V_2 \cdot e^+ \rangle \to \langle v^+[V_2/x^+] \mid e^+ \rangle \qquad \langle \lambda x^-.v^- \mid v_2^- \cdot E \rangle \to \langle v^-[v_2^-/x^-] \mid E \rangle$$

One encodes $v^- \cdot E$ and $V \cdot e^+$ as $[(\mu(\alpha^-)^\neg.\langle v^- | \alpha^- \rangle)^\uparrow, E]$ and $\mu(x^-)^\downarrow.\langle x^- | [V^\neg, (e^+)^\uparrow] \rangle$, respectively.

A second type-free aside (bilateral, indirect)

A general connective is defined as

a pair $(\{(s_1^i, \ldots, s_{n_i}^i) \mid i \in I\}, s)$, where all s, s_k^i range over $\{R^+, R^-, L^+, L^-\}$

We let $S ::= V \parallel v^- \parallel e^+ \parallel E$, and

$$\left\{\begin{array}{lll} S^{R^+}, S^{R^-}, S^{L^+}, S^{L^-} & \text{stand for} & V, v^-, e^+, E \\ \kappa^{R^+}, \kappa^{R^-}, \kappa^{L^+}, \kappa^{L^-} & \text{stand for} & x^+, x^-, \alpha^+, \alpha^- \\ t^{R^+}, t^{R^-}, t^{L^+}, t^{L^-} & \text{stand for} & v^+, v^-, e^+, e^- \end{array}\right.$$

With the connective we associate terms :

- $S^s ::= \ldots \parallel \iota_i(S_1^{s_1^i}, \ldots, S_{n_i}^{s_{n_i}^i}) \parallel \ldots$, for each $i \in I$ (constructor),
- $t^{\overline{s}} ::= \ldots \parallel \mu(\ldots, \iota_i(\kappa_1^{s_1^i}, \ldots, \kappa_{n_i}^{s_{n_i}^i}).c_i, \ldots) \parallel \ldots$ (co-constructor) (if we care, in fact μ or $\tilde{\mu}$ depending on whether s is a L or a R)

(where $\overline{s}$ is the dual of s : $\overline{R^+} = L^-$, etc...), and the reduction rules

$$\langle \iota_i(S_1, \ldots, S_{n_i}) | \mu(\ldots, \iota_i(\kappa_1, \ldots, \kappa_{n_i}).c_i, \ldots) \rangle \to c_i[S_1/\kappa_1, \ldots, S_{n_i}/\kappa_{n_i}]$$

Dotted connectives

To accommodate, say CBV and CBN implications in their usual formuation, we can "customise" connectives, by "dotting" at most one element in each list $(s^i_1, \ldots, s^i_{n_i})$.

– The constructor associated with $(s^i_1, \ldots, \dot{s}^i_j, \ldots, s^i_{n_i})$ is

$$\iota_i(S_1, \ldots, S_{n_i})$$

– The co-constructor is

$$\mu(\ldots, \iota_i(\kappa_1, \ldots, \kappa_{j-1}, \kappa_{j+1}, \ldots, \kappa_{n_i}).t_i^{\overline{s_i^j}}, \ldots)$$

And the corresponding customised reduction rule is

$$\langle \iota_i(S_1, \ldots, S_j, \ldots S_{n_i}) \mid \mu(\ldots, \iota_i(\kappa_1, \ldots, \kappa_{j-1}, \kappa_{j+1}, \ldots, \kappa_{n_i}).t_i, \ldots) \rangle$$
$$\to \langle S_j \mid t_i[S_1/\kappa_1, \ldots, S_{j-1}/\kappa_{j-1}, S_{j+1}/\kappa_{j+1}, \ldots, S_{n_i}/\kappa_{n_i}] \rangle$$

(we tolerate $\langle e^+ \mid v^+ \rangle$ and $\langle e^- \mid v^- \rangle$, meaning $\langle v^+ \mid e^+ \rangle$ and $\langle v^- \mid e^- \rangle$)

(adapted from unpublished notes of Herbelin)

Classifying the bestiary of connectives

$\otimes$	$\{(R^{+}, R^{+})\}R^{+}$
$\oplus$	$\{(R^{+}), (R^{+})\}R^{+}$
$\downarrow$	$\{(R^{-})\}R^{+}$
$⅋$	$\{(L^{-}, L^{-})\}L^{-}$
$\&$	$\{(L^{-}), (L^{-})\}L^{-}$
$\uparrow$	$\{(L^{+})\}L^{-}$
$P \mapsto \neg P$	$\{(R^{+})\}L^{-}$
$N \mapsto \neg N$	$\{(L^{-})\}^{R}$
$\rightarrow_{v}$	$\{(R^{+}, \dot{L}^{+})\}L^{+}$
$\rightarrow_{n}$	$\{(R^{-}, \dot{L}^{-})\}L^{-}$

A third type-free aside (bilateral, non polarised/focalised)

Herbelin had in fact something a bit different in mind, following the original philosophy of the duality of computation paper.

A general connective in his sense is not polarised, but only lateralised. The connectives (dotted or not) have the same form, but with s, s_k^i now ranging over $\{R, L\}$ (we simplify the syntactic categories S^s, κ^s, t^s accordingly). Herbelin's computation rule is

$$\begin{array}{l}\langle \iota_i(S_1, \ldots, S_{n_i}) \mid \mu(\ldots, \iota_i(\kappa_1, \ldots, \kappa_{n_i}).c_i, \ldots)\rangle \\ \quad \rightarrow \langle S_1 \mid \mu\kappa_1. \ldots \langle S_{n_i} \mid \mu\kappa_{n_i}.c_i\rangle\rangle\end{array}$$

where $\mu\kappa^s$ reads as $\mu\alpha$ when $s = R$ and and $\tilde{\mu}x$ when $s = L$.
And similarly in the dotted case.

This yields a non-deterministic, non-confluent system, which has two well-behaved strategies obtained by giving priority to $\tilde{\mu}$ (CBN) or to μ (CBV).

In the typed setting, we are back to three judgements only :

$$c : (\Gamma \vdash \Delta) \quad \Gamma \vdash v : A \,|\, \Delta \quad \Gamma \,|\, e : A \vdash \Delta$$

A unique non polarised/focalised implication

Applying the "forgetful" map that retains only laterality, the two implications $\rightarrow_v = \{(R^+, \dot{L}^+)\}L^+$ and $\rightarrow_n = \{(R^-, \dot{L}^-)\}L^-$ merge into

$$\rightarrow = \{(R, \dot{L})\}L \text{ , with the rule } \langle \lambda x.v \mid v_2 \cdot e \rangle \rightarrow \langle v_2 \mid \tilde{\mu}x.\langle v \mid e \rangle \rangle$$

This is the rule in Curien-Herbelin 2000, which depending on the $\mu/\tilde{\mu}$ priority discipline yields the respective rules above for $\rightarrow_v$ and $\rightarrow_n$.

Conversely, every non polarised general connective γ gives rise to two connectives γ^-+ (resp. γ^-) in the polarised sense, by replacing everywhere L, R with L^+, R^+, (resp. with L^-, R^-). Accordingly, in the $+$ case (resp. $-$ case), we have the syntactic categories c, v^+, e^+, V (resp. c, v^-, e^-, E). This results in respective subsyntaxes of the unpolarised syntax, which are stable under the respective reduction strategies (next slide).

Note that the polarised world is much richer, leaving space for mixed $+/-$.

Polarised syntax as subsyntax of unpolarised one

Let $\Sigma = \{\gamma_1, \ldots\}$ be a signature of unpolarised general connectives, à la Herbelin. Let $\Sigma^- = \{\gamma_1^-, \ldots\}$ be its negative polarisation (cf. previous slide). Then Σ, Σ^- induce sets of terms (commands, etc...) T_Σ and T_{Σ^-}. The latter can be considered a (strict) subsyntax of the former (e.g., in the case of implication, $v^+ \cdot e^+$ is in T_{Σ^-} only if v^+ is a V).

Proposition : T_{Σ^-}, as a subset of T_Σ, is closed under CBN reduction (priority to $\tilde{\mu}$ in all redexes of the form $\langle \mu\alpha^-.c \mid \tilde{\mu}x^-.c' \rangle$).

Idem for Σ^+ and CBV reduction

Different versions of general connectives

- bilateral, indirect : $s, s^i_j ::= R^+ \parallel R^- \parallel L^+ \parallel L^-$ (our preferred one !)
- bilateral, direct : $s := R^+ \parallel R^- \parallel L^+ \parallel L^- \quad , \quad s^i_j ::= L \parallel R$
- monolateral, indirect : $s, s^i_j ::= + \parallel -$
- monolateral, direct : $s ::= + \parallel -$ (and $(s^i_1, \ldots, s^i_{n_i})$ replaced by n_i) (slightly more expressive than in our first aside, where s was $+$)
- forgetting signs, i.e. forgetting focalisation : $s ::= R \parallel L$: preserves the elegance of CBV / CBN as orientation of a critical pair (Herbelin)

Two syntactic adjunctions

We have, in (bilateral) $\mathsf{LL}_{\downarrow}$ as well as in $\mathsf{LK}_{\downarrow}$:

$\downarrow \dashv \uparrow$ at the level of positive contexts and negative terms

$\uparrow \dashv \downarrow$ at the level of covalues and values

The adjunctions are mediated by command judgements :

$$\Gamma, N \vdash {\uparrow}P \mid \Delta \quad \cong \quad \Gamma, N \vdash P, \Delta \quad \cong \quad \Gamma \mid {\downarrow}N \vdash P, \Delta$$

$$\Gamma, P \vdash {\downarrow}N \,;\, \Delta \quad \cong \quad \Gamma, P \vdash N, \Delta \quad \cong \quad \Gamma \,;\, {\uparrow}P \vdash N, \Delta$$

We exhibit the inverse syntactic isomorphisms. We need two η-rules (which express invertibility) :

$$v^- = \mu\alpha^{+\uparrow}.\langle v^- \mid \alpha^{+\uparrow}\rangle \qquad \text{(for } \Gamma \vdash v^- : {\uparrow}P \mid \Delta\text{)}$$
$$e^+ = \tilde{\mu}x^{-\downarrow}.\langle x^{-\downarrow} \mid e^+\rangle \qquad \text{(for } \Gamma \mid e^+ : {\downarrow}N \vdash \Delta\text{)}$$

As we shall see, the first adjunction is "more primitive" than the second.

The first syntactic adjunction

We have

$$
\begin{array}{ccc}
\Gamma \vdash v^{-} : {\uparrow}P \mid \Delta & v^{-} & \mu\alpha^{+\uparrow}.c \\
\updownarrow & \downarrow & \uparrow \\
c : (\Gamma \vdash \alpha^{+} : P, \Delta) & \langle v^{-} \mid \alpha^{+\uparrow} \rangle & c
\end{array}
$$

and

$$
\begin{array}{ccc}
\Gamma \mid e^{+} : {\downarrow}N \vdash \Delta & e^{+} & \tilde{\mu}x^{-\downarrow}.c \\
\updownarrow & \downarrow & \uparrow \\
c : (\Gamma, x^{-} : N \vdash \Delta) & \langle x^{-\downarrow} \mid e^{+} \rangle & c
\end{array}
$$

so that putting these isos together we obtain isos between

$$\Gamma, x^{-} : N \vdash v^{-} : {\uparrow}P \mid \Delta \ , \ c : (\Gamma, x^{-} : N \vdash \alpha^{+} : P, \Delta) \ , \ \Gamma \mid e^{+} : {\downarrow}N \vdash \alpha^{+} : P, \Delta$$

Preparation for the second syntactic adjunction

We define macros :

$$\begin{array}{ll} \mu(\alpha^-)_{\downarrow}.c = (\mu\alpha^-.c)^{\downarrow} & E^{\downarrow} = \tilde{\mu}(x^-)^{\downarrow}.\langle x^- \mid E\rangle \\ \tilde{\mu}(x^+)^{\uparrow}.c = (\tilde{\mu}x^+.c)^{\uparrow} & V^{\uparrow} = \mu(\alpha^+)^{\uparrow}.\langle V \mid \alpha^+\rangle \end{array}$$

with the following derived typing rules :

$$\frac{c : (\Gamma \vdash \alpha^- : N, \Delta)}{\Gamma \vdash \mu\alpha^{-\downarrow}.c : {\downarrow}N \,;\, \Delta} \qquad \frac{\Gamma \,;\, E : N \vdash \Delta}{\Gamma \mid E^{\downarrow} : {\downarrow}N \vdash \Delta}$$

$$\frac{c : (\Gamma, x^+ : P \vdash \Delta)}{\Gamma \,;\, \tilde{\mu}x^{+\uparrow}.c : {\uparrow}P \vdash \Delta} \qquad \frac{\Gamma \vdash V : P \,;\, \Delta}{\Gamma \vdash V^{\uparrow} : {\uparrow}P \mid \Delta}$$

We need two new η-rules, which are “by value” (cf. $\lambda x.Vx$ in CBV λ-calculus) :

$$\begin{array}{ll} V = \mu\alpha^{-\downarrow}.\langle V \mid \alpha^{-\downarrow}\rangle & (\text{for } \Gamma \vdash V : {\downarrow}N \,;\, \Delta) \\ E = \tilde{\mu}x^{+\uparrow}.\langle x^{+\uparrow} \mid E\rangle & (\text{for } \Gamma \,;\, E : {\uparrow}P \vdash \Delta) \end{array}$$

Second syntactic adjunction

We have

$$
\begin{array}{ccc}
\Gamma \vdash V : {\downarrow}N \,;\, \Delta & V & \mu\alpha^{-\downarrow}.c \\
\Updownarrow & \downarrow & \uparrow \\
c : (\Gamma \vdash \alpha^{-} : N, \Delta) & \langle V \mid \alpha^{-\downarrow} \rangle & c
\end{array}
$$

and

$$
\begin{array}{ccc}
\Gamma \,;\, E : {\uparrow}P \vdash \Delta & E & \tilde{\mu}x^{+\uparrow}.c \\
\Updownarrow & \downarrow & \uparrow \\
c : (\Gamma, x^{+} : P \vdash \Delta) & \langle x^{+\uparrow} \mid E \rangle & c
\end{array}
$$

so that putting these isos together we obtain isos between

$\Gamma, x^{+} : P \vdash V : {\downarrow}N \,;\, \Delta$, $c : (\Gamma, x^{+} : P \vdash \alpha^{-} : N, \Delta)$, $\Gamma \,;\, E : {\uparrow}P \vdash \alpha^{-} : N, \Delta$

Conversely, taking the macros as primitive, we cannot recover the first adjunction.

Roadmap

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect		$\mathsf{LL}_{\downarrow} \supseteq \mathsf{TL}_{foc}$

Classical :

	Non focalised	Focalised
Direct		LK_{foc}
Indirect		(bilateral) $\mathsf{LK}_{\downarrow} \supseteq \boxed{\mathsf{CBPV}}$

From $\mathsf{LK}_{\downarrow}$ to CBPV (in sequent calculus style)

By cutting down $\mathsf{LK}_{\downarrow}$ to intuitionistic judgements of the respective forms (with Γ a *context of positive formulas*) :

Commands	Values / Covalues	Expressions / Contexts
$c : (\Gamma \vdash [.] : N)$	Values: $\Gamma \vdash V : P \,;$	Expressions: $\Gamma \vdash v : N \,|$
	Covalues: $\Gamma \,;\, E : N_1 \vdash [.] ; N_2$	Contexts: $\Gamma \,|\, e : P \vdash [.] : N$

we arrive to a sequent calculus discussed by Pfenning in his course notes on focalisation, and which is exactly a sequent calculus version of **Levy**'s CBPV.

This raises the question of the relation between shifts and monads in general and with the continuation monad in particular. This will be discussed at the end of the talk.

System L **style syntax for** CBPV

Formulas :

$$P ::= P \oplus P \parallel P \otimes P \parallel {\downarrow}N$$
$$N ::= N \,\&\, N \parallel P \to N \parallel {\uparrow}P$$

Values	$V ::= x \parallel (V,V) \parallel inl(V) \parallel inr(V) \parallel \mu[.]^{\downarrow}.c$
Negative terms	$v := \mu[.].c \parallel \mu([fst].c_1,[snd].c_2) \parallel \mu[x \cdot [.]].c \parallel V^{\uparrow}$
Covalues	$E ::= [.] \parallel E[fst] \parallel E[snd] \parallel [V \cdot E] \parallel \tilde{\mu}x^{\uparrow}.c$
Positive contexts	$e ::= \tilde{\mu}x.c \parallel \tilde{\mu}(x,y).c \parallel \tilde{\mu}[inl(x_1).c_1, inr(x_2).c_2] \parallel E^{\downarrow}$
Commands	$c ::= \langle v \mid E\rangle \parallel \langle V \mid e\rangle$

Dictionary wrt to P.B. Levy's notation

value	computation	Σ	$\times$	UN	FP	Π	$P \to N$
positive	negative	$\oplus$	$\otimes$	$\downarrow N$	$\uparrow P$	$\&$	$P \to N$ $(= \overline{P} \invamp N)$

V (value)	M (computation)	K (stack)
V (value)	v (negative term)	E (covalue)

(no counterpart for e, c).

System L style typing rules for CBPV

$$\frac{}{\Gamma, x : P \vdash x : P\,;} \qquad \frac{}{\Gamma\,;\ [.] : N \vdash [.] : N} \qquad \frac{\Gamma \vdash v : N \mid \quad \Gamma\,;\ E : N \vdash [.] : N'}{\langle v \mid E\rangle : (\Gamma \vdash [.] : N')} \qquad \frac{\Gamma \vdash V : P\,; \quad \Gamma \mid e : P \vdash [.] : N}{\langle V \mid e\rangle : (\Gamma \vdash [.] : N)}$$

$$\frac{c : (\Gamma \vdash [.] : N)}{\Gamma \vdash \mu[.].c : N \mid} \qquad \frac{c : (\Gamma, x : P \vdash [.] : N)}{\Gamma \mid \tilde{\mu}x.c : P \vdash [.] : N}$$

$$\frac{c : (\Gamma \vdash [.] : N)}{\Gamma \vdash \mu[.]^{\downarrow}.c : {\downarrow}N\,;} \qquad \frac{\Gamma \vdash V_1 : P_1\,; \qquad \Gamma \vdash V_2 : P_2\,;}{\Gamma \vdash (V_1, V_2) : P_1 \otimes P_2\,;} \qquad \frac{\Gamma \vdash V_1 : P_1\,;}{\Gamma \vdash \mathit{inl}(V_1) : P_1 \oplus P_2\,;}$$

$$\frac{\Gamma \vdash V : P\,;}{\Gamma \vdash V^{\uparrow} : {\uparrow}P \mid} \qquad \frac{c : (\Gamma, x : P \vdash [.] : N)}{\Gamma \vdash \mu[x \cdot [.]].c : P \to N \mid} \qquad \frac{c_1 : (\Gamma \vdash [.] : N_1) \quad c_2 : (\Gamma \vdash [.] : N_2)}{\Gamma \vdash \mu([\mathit{fst}].c_1, [\mathit{snd}].c_2) : N_1 \,\&\, N_2 \mid}$$

$$\frac{\Gamma, x : P \vdash [.] : N}{\Gamma\,;\ \tilde{\mu}x^{\uparrow}.c : {\uparrow}P \vdash [.] : N} \qquad \frac{\Gamma \vdash V : P\,; \qquad \Gamma\,;\ E : N \vdash [.] : N'}{\Gamma\,;\ [V \cdot E] : P \to N \vdash [.] : N'} \qquad \frac{\Gamma\,;\ E_1 : N_1 \vdash [.] : N}{\Gamma\,;\ E_1[\mathit{fst}] : N_1 \,\&\, N_2 \vdash [.] : N}$$

$$\frac{\Gamma\,;\ E : N \vdash [.] : N'}{\Gamma \mid E^{\downarrow} : {\downarrow}N \vdash [.] : N'} \qquad \frac{c : (\Gamma, x_1 : P_1, x_2 : P_2 \vdash [.] : N)}{\Gamma \mid \tilde{\mu}(x_1, x_2).c : P_1 \otimes P_2 \vdash [.] : N} \qquad \frac{c_1 : (\Gamma, x_1 : P_1 \vdash [.] : N) \quad c_2 : (\Gamma, x_2 : P_2 \vdash [.] : N)}{\Gamma \mid \tilde{\mu}[\mathit{inl}(x_1).c_1, \mathit{inr}(x_2).c_2] : P_1 \oplus P_2 \vdash [.] : N}$$

Only the second adjunction is available for CBPV

What happens when cutting down to intuitionistic systems such as CBPV, LLP, or, in the linear case, TL, is that there is no space to express the first adjunction.

In the CBPV case : there are no sequents in which there is a variable x of negative type in the (left) context, and similarly no sequents with a variable α of positive type on the right (only $[.] : N$ is available).

System L style reduction rules for CBPV

$$\langle V \mid \tilde{\mu}x.c\rangle \to c[V/x]$$
$$\langle \mu[.].c \mid E\rangle \to c[E/[.]]$$
$$\langle (V_1, V_2) \mid \tilde{\mu}(x_1, x_2).c\rangle \to c[V_1/x_1, V_2/x_2]$$
$$\langle \mu[x \cdot \alpha].c \mid [V, E]\rangle \to c[V/x, E/\alpha]$$
$$\langle inl(V_1) \mid \tilde{\mu}[inl(x_1).c_1, inr(x_2).c_2]\rangle \to c_1[V_1/x_1]$$
$$\langle \mu([fst].c_1, [snd].c_2)) \mid E_1[fst]\rangle \to c_1[E_1/[.]]$$
$$\langle \mu[.]^{\downarrow}.c \mid E^{\downarrow}\rangle \to c[E/[.]]$$
$$\langle V^{\uparrow} \mid \tilde{\mu}x^{\uparrow}.c\rangle \to c[V/x]$$

Translation from CBPV **to** L **style**

(read "let V (resp. v, v_1, E,...) be the translation of V (resp M, M_1, K,...)")

x	$\rightsquigarrow$	x
return V	$\rightsquigarrow$	$V^{\uparrow}$
thunk M	$\rightsquigarrow$	$\mu[.]^{\downarrow}.\langle v \mid [.]\rangle$
Σ introduction	$\rightsquigarrow$	inl, inr
(V, V')	$\rightsquigarrow$	(V, V')
$\lambda\{1.M_1, 2.M_2\}$	$\rightsquigarrow$	$\mu([fst].\langle v_1 \mid [.]\rangle, [snd].\langle v_2 \mid [.]\rangle)$
$\lambda x.M$	$\rightsquigarrow$	$\mu[x \cdot [.]].\langle v \mid [.]\rangle$
let V be $x.M$	$\rightsquigarrow$	$\mu[.].\langle V \mid \tilde{\mu}x.\langle v \mid [.]\rangle\rangle$
M_1 to $x.M_2$	$\rightsquigarrow$	$\mu[.].\langle v_1 \mid \tilde{\mu}x^{\uparrow}.\langle v_2 \mid [.]\rangle\rangle$
force V	$\rightsquigarrow$	$\mu[.].\langle V \mid [.]^{\downarrow}\rangle$
pm V as $\{(1, x_1).M_1, (2, x_2).M_2\}$	$\rightsquigarrow$	$\mu[.].\langle V \mid \tilde{\mu}[inl(x_1).\langle v_1 \mid [.]\rangle, inr(x_2).\langle v_2 \mid [.]\rangle]\rangle$
pm V as $(x, y).M$	$\rightsquigarrow$	$\mu[.].\langle V \mid \tilde{\mu}(x, y).\langle v \mid [.]\rangle\rangle$
$\widehat{ß}$‘M	$\rightsquigarrow$	$\mu[.].\langle v \mid [.][fst]\rangle$
V‘M	$\rightsquigarrow$	$\mu[.].\langle (v \mid [V \cdot [.]]\rangle$
nil	$\rightsquigarrow$	$[\cdot]$
$[\cdot]$ to $x.M :: K$	$\rightsquigarrow$	$\tilde{\mu}x^{\uparrow}.\langle v \mid E\rangle$
$\widehat{1} :: K$	$\rightsquigarrow$	$E[fst]$ (idem $\widehat{2}, snd$)
$V :: K$	$\rightsquigarrow$	$[V \cdot E]$

Translation from L style to CBPV

The three categories e, c, v are translated to computations M, while V, E of course translate to values and stacks. The translation of contexts e is parameterised by a variable x (the place-holder of e in the sequent). The translation makes use of the dismantling $M \bullet K$ (or read-back) of a state (M, K) as a computation.

$x^\dagger = x$
$(inl(V))^\dagger = (\hat{1}, V^\dagger)$ (idem inr)
$(V_1, V_2)^\dagger = ((V_1)^\dagger, (V_2)^\dagger)$
$(\mu[.]^\downarrow.c)^\dagger = \text{thunk } c^\dagger$

$(\mu[.].c)^\dagger = c^\dagger$
$(\mu([fst].c_1, [snd].c_2))^\dagger = \lambda\{1.(c_1)^\dagger, 2.(c_2)^\dagger\}$
$(\mu[x \cdot [.]].c)^\dagger = \lambda x.c^\dagger$
$(V^\uparrow)^\dagger = \text{return } V^\dagger$

$[.]^\dagger = \text{nil}$
$E[fst]^\dagger = \hat{1} :: E^\dagger$ (idem snd)
$[V \cdot E]^\dagger = V^\dagger :: E^\dagger$
$(\tilde{\mu}x^\uparrow.c)^\dagger = [\cdot] \text{ to } x.c^\dagger :: \text{nil}$

$(\tilde{\mu}x.c)^\dagger_x = c^\dagger$
$(\tilde{\mu}(x_1, x_2).c)^\dagger_x = \text{pm } x \text{ as } (x_1, x_2).c^\dagger$
$(\tilde{\mu}[inl(x_1).c_1, inr(x_2).c_2])^\dagger_x = \text{pm } x \text{ as } \{(1, x_1).(c_1)^\dagger_x, (2, x_2).(c_2)^\dagger_x\}$
$(E^\downarrow)^\dagger_x = (\text{force } x) \bullet E^\dagger$

$\langle v \mid E \rangle^\dagger = v^\dagger \bullet E^\dagger$
$\langle V \mid e \rangle^\dagger = e^\dagger_x[V/x]$

Equivalence

One checks easily that the two systems simulate each other.

To get that the translations are inverse to each other, we need the η-rules (cf. above).

Of course, this is the old story of inter-translating natural deduction and sequent calculus, refined to the equivalence between natural deduction and *focused* sequent calculus (cf. above), but the fact that the target of the translation of CBPV is exactly the projection of a larger, symmetric picture reinforces its relevance.

(Analogy : saying that a Böhm tree is a strategy is not that interesting, what is most interesting is to characterise the strategies arising in this way (innocence).)

V) Perspective

on the

monadic reading of shifts

Roadmap

Linear :

	Non focalised	Focalised
Direct		
Indirect		

Classical :

	Non focalised	Focalised
Direct		
Indirect		$\mathsf{LK}_{\downarrow} \supseteq \boxed{\mathsf{LLP}_{foc}\ ,\ \mathsf{CBPV}}$

What the hell ... ?

We have two quite different (intuitionistic) fragments of $\mathsf{LK}_{\downarrow}$:

• LLP_{foc}, which is complete for the interpretation in response categories, where $\downarrow\uparrow$ is the double negation continuation monad

• CBPV, which is complete for Levy's notion of adjunction models, in which $\downarrow\uparrow$ can be *any* monad

Strange, but true... Why is that so ?

Fragments

We call GNU a fragment of system BLA if
– the formulas of GNU are derived formulas of BLA (i.e., macros, or clusters),
– the sequents are BLA are sequents of GNU possibly satisfying some restrictions (like “at most seven negative formulas in the right contexts”), and
– the typing and reduction rules are derivable rules

We also require that a fragment of $\mathsf{LK}_{\downarrow}$ has at least the connectives $\downarrow$ and $\uparrow$ with their rules as in LLP_{foc}.

Self-duality

A fragment of bilateral $\mathrm{LK}_{\downarrow}$ is called self-dual if

– the sets of positive formulas and of negative formulas are exchanged by duality (in particular, to each positive macro corresponds a dual negative macro),

– the set of allowed sequents is closed under duality, where the dual of $(\Gamma \vdash \Delta)$ is $(\overline{\Delta} \vdash \overline{\Gamma})$, the dual of $(\Gamma \mid A \vdash \Delta)$ is $(\overline{\Delta} \vdash \overline{A} \mid \overline{\Gamma})$, ...

Folding

One may define the folding of a self-dual fragment in four different flavours : right folding (which we shall call “folding” for short), left folding, positive folding, and negative folding.

The right folding consists in mapping sequents $\Gamma \vdash \Delta$ to $\vdash \overline{\Gamma}, \Delta$ (note that here the change of polarity is by duality, not by shifts). This divides the total number of rules by two (only right introduction rules). The left folding places all formulas on the left.

The positive folding requires in addition a precooking of the connectives : e.g. if $\dagger(P_1, N_2, P_3)$ is a ternary connective, replace it with $\dagger'(P_1, P_2, P_3) = \dagger(P_1, \overline{P_2}, P_3)$. Then one moves negative formulas on the other side of the $\vdash$ (resulting in a homogeneous bilateral sequent of positive formulas, hence the set of formulas is also divided by two !). The negative unfolding is dual.

Properties and examples of folding

The target of the folding transformation is a fragment, and the transformation preserves reductions. It collapses laterality and polarity distinctions (cf. our discussion of bilaterality above).

Examples : the folding of $\mathrm{LK}_{\downarrow}$ is what we called monolateral $\mathrm{LK}_{\downarrow}$, its positive and negative foldings are known as LKQ and LKT.

Properties of self-dual fragments

In a self dual fragment, in a provable sequent, one may alter each formula by introducing even blocks of $\neg$ anywhere inside the formula (including the top level), and odd blocks of $\neg$ at the top level of a formula but then moving it on the other side of the sequent, without altering provability and the status of formulas (by status, we mean : in a context, active, or under focus).

Proof by a huge mutual induction...

In other words, in the right or left folding, nothing is lost, apart from the information on laterality, and in the positive or negative folding, nothing is lost, apart from the information on polarity.

$\downarrow\uparrow$ is the continuation monad in $\mathsf{LK}_{\downarrow}$

Recall the notation $\neg_{\downarrow} P = \downarrow\overline{P}$. To exhibit $\downarrow\uparrow = \neg_{\downarrow}\neg_{\downarrow}$ as the continuation monad, we just have to show that there is one-to-one correspondence between

$$x^{+} : P \vdash V : \neg_{\downarrow} Q\,; \qquad \text{and} \qquad |\, e : P \otimes Q \vdash$$

which thanks to the (first half of) the second adjunction and the reversibility of $\otimes$ rephrases as a one-to-one correspondence between

$$c : (x^{+} : P \vdash \alpha^{-} : \overline{Q}) \qquad \text{and} \qquad c' : (x : P, y : Q \vdash)$$

This holds in fact in all self-dual fragments of $\mathsf{LK}_{\downarrow}$ (and their foldings).

Note that the syntax of CBPV formulas is not self-dual. It does not even make sense to write $\neg_{\downarrow} P$ as there is no such thing as $\overline{P}$! It is this breaking of the symmetry that frees $\downarrow\uparrow$ from being the continuation monad !

$\downarrow\uparrow$ **is the continuation monad in** LLP$_{foc}$

LLP$_{foc}$ can be seen as the folded version of a self-dual bilateral system (the union with its “dual intuitionistic” mirror), whence the result.

The direct proof is simpler, since in the (positive) bilateral presentation of LLP$_{foc}$, there is a one-to-one correspondence between

$$x^{+} : P \vdash V : \neg_{\downarrow} Q \,; \qquad \text{and} \qquad c : (x : P, y : Q \vdash)$$

$\downarrow\uparrow$ **in** CBPV

As noticed above, the symmetry (i.e., the duality between the positive and negative formulas) is broken :

$$
\begin{array}{l}
P ::= P \oplus P \parallel P \otimes P \parallel P \oplus P \parallel \downarrow N \\
N ::= N \mathbin{\&} N \parallel P \to N \parallel \uparrow P
\end{array}
$$

See Levy's book and papers for a wealth of examples of concrete monads that fit into the CBPV framework.

LLP$_{foc}$ **as a fragment in** CBPV

We have identified CBPV as a non-symmetric fragment of a large self-dual system. One can also go "the other way around", and recover the folded system LLP$_{foc}$ (for which Levy gives a natural deduction style presentation JWA) as a fragment of CBPV. For this, pick an arbitrary, fixed, negative formula N and define

$$\neg_{\downarrow} P \quad \text{as} \quad {\downarrow}(P \to N)$$

Then the expected adjunction holds, i.e., there is a one-to-one correspondence between

$$x : P \otimes Q \vdash v : N \,| \qquad \text{and} \qquad y : P \vdash V : \neg_{\downarrow} Q \,;$$

Summary

Linear :

	Non focalised	Focalised
Direct	LL	LL_{foc}
Indirect	"Indirect LL" $\supseteq$ TL	$\mathsf{LL}_{\downarrow} \supseteq \mathsf{TL}_{foc}$, LCBPV

Classical :

	Non focalised	Focalised
Direct	"LK"	LK_{foc}
Indirect	"Indirect LK" $\supseteq$ LLP	$\mathsf{LK}_{\downarrow} \supseteq \mathsf{LLP}_{foc}$, CBPV

where LCBPV is a linear version of CBPV, and where the systems within quotes have been only suggested here (slide 53).

What else ?

Adding delimited control : see Guillaume's paper From delimited CPS to polarisation (available from `http://www.pps.jussieu.fr/~munch`).

What next ?

We just ask one question : what is the categorical structure of which System L would be the *internal language* ? (We started to work on this with Marcelo Fiore, taking LCBPV as test-bed.)