Strong normalisation for focalising system L

Pierre-Louis Curien (CNRS & Université Paris VII)

May 31, 2012

Abstract

Classical realisability is due to Krivine [2]. Here we present (a variant of) its polarised version, due to Guillaume Munch [4], and we apply it to give a proof of strong normalisation of a syntax for focalised classical logic (see also [5], on which this note builds in great part).

We base ourselves on the mono-sided, indirect style version of system L, which we call $\mathsf{LK}_{\downarrow}$, and which we define as follows.

Formulas:

$$P ::= X \mid P \otimes P \mid P \oplus P \mid {\downarrow}N \qquad N ::= \overline{X} \mid N \parr N \mid N \& N \mid {\uparrow}P$$

Judgements:

Commands	Values	Positive terms	Negative terms
$c : (\vdash \Gamma)$	$\vdash V^+ : P \,;\, \Gamma$	$\vdash t^+ : P \mid \Gamma$	$\vdash t^- : N \mid \Gamma$

Terms:

$$\begin{array}{l} c ::= \langle t^+ \mid t^- \rangle \\ x ::= x^+ \mid x^- \\ V^+ ::= x^+ \mid (V_1^+, V_2^+) \mid \mathit{inl}(V^+) \mid \mathit{inr}(V^+) \mid (t^-)^{\downarrow} \\ t^+ ::= V^+ \mid \mu x^-.c \\ t^- ::= x^- \mid \mu x^+.c \mid \mu(x_1^+, x_2^+).c \mid \mu[\mathit{inl}(x_1^+).c_1, \mathit{inr}(x_2^+).c_2] \mid \mu(x^-)^{\downarrow}.c \end{array}$$

Typing rules:

$$\frac{}{\vdash x^+ : P \,;\, x^+ : \overline{P}} \qquad \frac{}{\vdash x^- : N \mid x^- : \overline{N}} \qquad \frac{\vdash t^+ : P \mid \Gamma \quad \vdash t^- : \overline{P} \mid \Delta}{\langle t^+ \mid t^- \rangle : (\vdash \Gamma, \Delta)}$$

$$\frac{\vdash V^+ : P \,;\, \Gamma}{\vdash V^+ : P \mid \Gamma} \quad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x.c : A \mid \Gamma} \quad \frac{c : (\vdash \Gamma)}{c : (\vdash x : A, \Gamma)} \quad \frac{c : (\vdash x_1 : A, x_2 : A, \Gamma)}{c[x_1 \leftarrow x_2] : (\vdash x_2 : A, \Gamma)}$$

$$\frac{\vdash V_1^+ : P_1 \,;\, \Gamma \qquad \vdash V_2^+ : P_2 \,;\, \Delta}{\vdash (V_1^+, V_2^+) : P_1 \otimes P_2 \,;\, \Gamma, \Delta} \qquad \frac{\vdash V_1^+ : P_1 \,;\, \Gamma}{\vdash \mathit{inl}(V_1^+) : P_1 \oplus P_2 \,;\, \Gamma} \qquad \frac{\vdash V_2^+ : P_2 \,;\, \Gamma}{\vdash \mathit{inr}(V_2^+) : P_1 \oplus P_2 \,;\, \Gamma}$$

$$\frac{c : (\vdash x_1^+ : N_1, x_2^+ : N_2, \Gamma)}{\vdash \mu(x_1^+, x_2^+).c : N_1 \parr N_2 \mid \Gamma} \qquad \frac{c_1 : (\vdash x_1^+ : N_1, \Gamma) \quad c_2 : (\vdash x_2^+ : N_2, \Gamma)}{\vdash \mu[inl(x_1^+).c_1, inr(x_2^+).c_2] : N_1 \& N_2 \mid \Gamma}$$

$$\frac{\vdash t^- : N \mid \Gamma}{\vdash (t^-)^\downarrow : \downarrow N \,;\, \Gamma} \qquad \frac{c : (\vdash x^- : P, \Gamma)}{\vdash \mu(x^-)^\downarrow.c : \uparrow P \mid \Gamma}$$

Reduction rules:

$$\begin{array}{l}
\langle V^+ \mid \mu x^+.c\rangle \to c[x^+ \leftarrow V^+] \\
\langle \mu x^-.c \mid t^-\rangle \to c[x^- \leftarrow t^-] \\
\langle (V_1^+, V_2^+) \mid \mu(x_1^+, x_2^+).c\rangle \to c[V_1^+/x_1^+, V_2^+/x_2^+] \\
\langle inl(V_1^+) \mid \mu[inl(x_1^+).c_1, inr(x_2^+).c_2]\rangle \to c_1[x_1^+ \leftarrow V_1^+] \\
\langle inr(V_2^+) \mid \mu[inl(x_1^+).c_1, inr(x_2^+).c_2]\rangle \to c_2[x_2^+ \leftarrow V_2^+] \\
\langle (t^-)^\downarrow \mid \mu(x^-)^\downarrow.c\rangle \to c[x^- \leftarrow t^-]
\end{array}$$

1 Weak and strong normalisation

Notation: $X \subseteq \{\ldots t^+ \ldots\}$ means: X is a set of positive terms, etc... These terms are not typed, and not necessarily closed. (Note that in the absence of base inhabited type constants or of second-order quantification, there are no closed types and a fortiori no closed terms of closed types.)

One chooses a fixed set $\perp\!\!\!\perp \subseteq \{\ldots c \ldots\}$ of commands, closed under backward reduction: if $c \in \perp\!\!\!\perp$ and $c' \to c$, then $c' \in \perp\!\!\!\perp$.

We say that t^+ is orthogonal to t^- (and may use the notation $t^+ \perp t^-$ for this) when $\langle t^+ \mid t^-\rangle \in \perp\!\!\!\perp$. One then defines:

$$\begin{array}{l}
\text{for any } X \subseteq \{\ldots t^+ \ldots\},\ \ X^\perp = \{t^- \mid \forall\, t^+ \in X\ \ \langle t^+ \mid t^-\rangle \in \perp\!\!\!\perp\} \\
\text{for any } Y \subseteq \{\ldots t^- \ldots\},\ \ Y^\perp = \{t^+ \mid \forall\, t^- \in Y\ \ \langle t^+ \mid t^-\rangle \in \perp\!\!\!\perp\}
\end{array}$$

We have the usual properties (for $Z \subseteq \{\ldots t^+ \ldots\}$ or $Z \subseteq \{\ldots t^- \ldots\}$):

$$Z \subseteq Z^{\perp\perp} \qquad Z^\perp = Z^{\perp\perp\perp}$$

(the lattter following from antimonotonicity of $Z \mapsto Z^\perp$). We say that $Z \subseteq \{\ldots t^+ \ldots\}$ (resp. $Z \subseteq \{\ldots t^- \ldots\}$) is a positive (resp. negative) behaviour if $Z = Z^{\perp\perp}$ (this terminology comes from [3]). Note that any $Z^\perp$ is a behaviour.

We then associate with each positive type P a set (not a behaviour) $\mathcal{V}[\![P]\!] \subseteq \{\ldots V^+ \ldots\}$ (parameterised by an environment), as follows, by induction on the syntax of positive formulas:

$$\begin{array}{l}
\mathcal{V}[\![X]\!]\rho = \rho(X) \\
\mathcal{V}[\![P \otimes Q]\!]\rho = \{(V_1^+, V_2^+) \mid V_1^+ \in \mathcal{V}[\![P]\!]\rho \quad \text{and} \quad V_2^+ \in \mathcal{V}[\![Q]\!]\rho\} \\
\mathcal{V}[\![P \oplus Q]\!]\rho = \{inl(V_1^+) \mid V_1^+ \in \mathcal{V}[\![P]\!]\rho\} \ \cup\ \{inr(V_2^+) \mid V_2^+ \in \mathcal{V}[\![Q]\!]\rho\} \\
\mathcal{V}[\![\downarrow N]\!]\rho = \{(t^-)^\downarrow \mid t^- \in [\![N]\!]\rho\}
\end{array}$$

One then sets:

$$[\![P]\!] = \mathcal{V}[\![P]\!]^{\perp\perp} \qquad [\![N]\!] = [\![\overline{N}]\!]^\perp$$

In other words, the $\mathcal{V}[\![P]\!]$'s determine the interpretations $[\![P]\!] \subseteq \{\ldots t^+ \ldots\}$ and $[\![N]\!] \subseteq \{\ldots t^- \ldots\}$, for all positive and negative formulas. (If you are worried by the occurrence of $[\![N]\!]$ in the definition of $\mathcal{V}[\![\downarrow N]\!]$, replace it with $\mathcal{V}[\![\overline{N}]\!]^\perp$.) We have (omitting the ρ from now on, to simplify the notation):

$$\begin{array}{ll} [\![P]\!] = [\![P]\!]^{\perp\perp} \quad \text{and} \quad [\![N]\!] = [\![N]\!]^{\perp\perp} & \quad \text{i.e., } [\![P]\!] \text{ and } [\![N]\!] \text{ are behaviours} \\ [\![P]\!] = [\![\overline{P}]\!]^{\perp} \quad \text{and} \quad [\![N]\!] = [\![\overline{N}]\!]^{\perp} & \\ [\![N]\!] = \mathcal{V}[\![\overline{N}]\!]^{\perp} & \end{array}$$

Lemma 1.1 (Fundamental lemma) *Let*

$$\Delta = \ldots, x^+ : N, \ldots, x^- : P, \ldots \qquad \ldots, V^+ \in \mathcal{V}[\![\overline{N}]\!], \ldots, t^- \in [\![\overline{P}]\!], \ldots$$

Then:

$$\left.\begin{array}{l} c : (\vdash \Delta) \\ \vdash T'^+ : P\,;\,\Delta \\ \vdash t^+ : P \,|\, \Delta \\ \vdash t'^- : N \,|\, \Delta \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c[\ldots, x^+ \leftarrow V^+, \ldots, x^- \leftarrow t^-, \ldots] \in \perp\!\!\!\perp \\ T'^+[\ldots, x^+ \leftarrow V^+, \ldots, x^- \leftarrow t^-, \ldots] \in \mathcal{V}[\![P]\!] \\ t^+[\ldots, x^+ \leftarrow V^+, \ldots, x^- \leftarrow t^-, \ldots] \in [\![P]\!] \\ t'^-[\ldots, x^+ \leftarrow V^+, \ldots, x^- \leftarrow t^-, \ldots] \in [\![N]\!] \end{array}\right.$$

Proof. The proof is elementary, by induction on terms. Note that, say, for the case $\mu[x,y].c$, we need to use that $\perp\!\!\!\perp$ is closed under backward reduction. □

Illustration: If $\Gamma \vdash t^+ : P \oplus Q \,|\,$ and α is fresh, then either $\langle t^+ \mid \alpha\rangle \rightarrow^* \langle \mathit{inl}(V_1^+) \mid \alpha\rangle$ for some V_1^+ or $\langle t^+ \mid \alpha\rangle \rightarrow^* \langle \mathit{inr}(V_2^+) \mid \alpha\rangle$ for some V_2^+.
Proof: Take $\perp\!\!\!\perp = \{t^+ \mid \exists V_1^+ \ \langle t^+ \mid \alpha\rangle \rightarrow^* \langle \mathit{inl}(V_1^+) \mid \alpha\rangle \text{ or } \exists V_2^+ \ \langle t^+ \mid \alpha\rangle \rightarrow^* \langle \mathit{inl}(V_2^+) \mid \alpha\rangle\}$. We have to check that $\langle t^+ \mid \alpha\rangle \in \perp\!\!\!\perp$. But by the fundamental lemma we have $t^+ \in \mathcal{V}[\![P \oplus Q]\!]^{\perp\perp}$. We then conclude noticing that $\alpha \in \mathcal{V}[\![P \oplus Q]\!]^{\perp}$.

Exercise Unroll the definitions of $\mathcal{V}[\![P \to Q]\!]$ and $[\![P \to Q]\!]$ (for the CBV encoding $\downarrow(\overline{P} \parr (\uparrow Q))$ of implication. Compare with what you can find in the literature on call-by-value logical relations (as e.g. in Amal Ahmed's papers).

We move on to an application of the technique to prove strong normalisation. We write `SN` for the set of strongly normalising terms (it will be clear from the context whether we mean commands, values, positive terms, or negative terms). We also write `Var` for the set of variables (positive or negative, depending on the context). Actually, the fundamental lemma as stated above will lead us only to weak normalisation, and we shall have to do some little "patch" to get strong normalisation. Here, we follow closely the normalisation proof in Krivine's book [1].

We take:

- $\perp\!\!\!\perp = \mathtt{WN}$ (the set of *weakly* normalising terms, which one may as well to choose as the set of terms for which the head reduction terminates)
- $\mathcal{V}[\![X]\!] = \mathtt{WN}$ (in fact, any choice of $\mathcal{V}[\![X]\!]$ such that $\emptyset \neq \mathcal{V}[\![X]\!] \subseteq \mathtt{WN}$ will do).

Lemma 1.2 *If $K \subseteq \mathtt{WN}$, then Var $\subseteq K^\perp$.*

Proof. Note that every command of the form $\langle x \mid t\rangle$ is a normal form if t is a normal form. □

Lemma 1.3 *If $\emptyset \neq K$ then $K^\perp \subseteq \mathtt{WN}$.*

PROOF. Choose $s \in K$. Then if $t \in K^\perp$, in particular $\langle s \mid t\rangle \in \perp\!\!\!\perp = \mathtt{WN}$, and we conclude by observing that any subterm of a strongly normalising term is strongly normalising. $\square$

Proposition 1.4 *For the above choice of $\perp\!\!\!\perp, \mathcal{V}[\![X]\!]$, we have, for all P, Q:*

1. $\emptyset \neq \mathcal{V}[\![P]\!] \subseteq \mathtt{WN}$,
2. $\mathtt{Var} \subseteq [\![N]\!]$,
3. $[\![P]\!] \subseteq \mathtt{WN}$,
4. $[\![N]\!] \subseteq \mathtt{WN}$,

PROOF. We have listed the items in the order in which we prove them, together, by induction on the size of the formula. The proof goes as follows.

1. This is our assumption on X. Now, looking at the definition of $\mathcal{V}[\![\ ]\!]$:
   - by induction (property (1) or (4)), we have $\mathcal{V}[\![P]\!] \subseteq \mathtt{WN}$, and
   - by induction (property (1) or (3)), we have $\emptyset \neq \mathcal{V}[\![P]\!]$.
2. By (1) and Lemma 2, since $[\![N]\!] = \mathcal{V}[\![\overline{N}]\!]^\perp$.
3. By (2) and Lemma 3, since $[\![P]\!] = [\![N]\!]^\perp$.
4. By (1) (which implies a fortiori $\emptyset \neq [\![P]\!]$) and Lemma 2, since $[\![P]\!] = \mathcal{V}[\![\overline{P}]\!]^\perp$. $\square$

We need a last lemma to prove our weak normalisation theorem.

Lemma 1.5 *If $c[\vec{x} \leftarrow \vec{s}]$ is weakly normalising, so is c (and the same for t, T).*

PROOF. This follows from the fact that one always has (by rule instantiation) that $c_1 \to c_2$ implies $c_1[\vec{x} \leftarrow \vec{s}] \to c_2[\vec{x} \leftarrow \vec{s}]$. $\square$

Theorem 1.6 *Weak normalisation holds in $\mathsf{LK}_\downarrow$, i.e. any typable expression in this system terminates*

PROOF. . Let us take, say, a positive term $c \vdash t^+ : P \mid \ldots, x^+ : N, \ldots, y^- : P, \ldots$. By Proposition 1 ((1) and (2)) we can pick $\ldots, V^+ \in \mathcal{V}[\![\overline{N}]\!], \ldots, t^- \in [\![\overline{P}]\!], \ldots$, and apply the Fundamental lemma, to get $t^+[\ldots, x^+ \leftarrow V^+, \ldots, x^- \leftarrow t^-, \ldots] \in \mathcal{V}[\![P]\!]$, and conclude by Proposition 1 (3). And similarly for a value, a negative term, or a command (the latter case being even simpler: remember that we chose $\perp\!\!\!\perp = \mathtt{SN}$!. $\square$

We now explain the necessary adjustments we need to do to get actually strong normalisation. First, it is easy to see that the proofs of Lemmas 1, 2, and 3, and of Proposition 1 go through when replacing WN with SN everywhere (including in the initial assumptions, i.e. taking now $\perp\!\!\!\perp = \mathcal{V}[\![X]\!] = \mathtt{SN}$), even if SN does *not* satisfy the backward reduction assumption. This assumption was only used in the proof of the fundamental lemma! Now, what does hold (exercise!) is the following list of properties:

$$\text{If } c[x^+ \leftarrow V^+] \in \mathtt{SN} \text{ and } V^+ \in \mathtt{SN}, \text{ then } \langle V^+ \mid \mu x^+.c\rangle \in \mathtt{SN}$$

$$\vdots$$

$$\text{If } c[V_1^+/x_1^+, V_2^+/x_2^+] \in \mathtt{SN},\ V_1^+ \in \mathtt{SN}, \text{ and } V_2^+ \in \mathtt{SN}, \text{ then } \langle (V_1^+, V_2^+) \mid \mu(x_1^+, x_2^+).c\rangle \in \mathtt{SN}$$

$$\vdots$$

Then thanks to Proposition 1, we can unroll the proof of the fundamental lemma and make it work in this case too, since in the induction, whenever we use a term which sits in the interpretation of a type, we know that is strongly normalisable, so that the above weaker properties of backwards stability suffice.

2 Axiomatisation

But still, strictly speaking, strong normalisation is not a direct application of the fundamental lemma as stated above, since the choice of $\perp\!\!\!\perp = \texttt{SN}$ does not fit into its hypotheses. What we need is a more parameterised version of the whole framework, which we now give.

Definition 2.1 *A realisability structure is given by three sets $\mathbb{T}^+$, $\mathbb{T}^-$ and $\perp\!\!\!\perp$ of positive terms, negative terms, and commands, respectively, satisfying the following properties:*

1. *If $\langle t_1^+ \mid t_2^- \rangle \in \perp\!\!\!\perp$, then $t_1^+ \in \mathbb{T}^+$ and $t_2^- \in \mathbb{T}^-$.*
2. *If $t^+ \in \mathbb{T}^+$, then $\langle t^+ \mid x^- \rangle \in \perp\!\!\!\perp$ (for any variable x^-).*
3. *If $t^- \in \mathbb{T}^-$, then $\langle x^+ \mid t^- \rangle \in \perp\!\!\!\perp$ (for any variable x^+).*
4. *Closure of $\mathbb{T}^+$ under value constructors:*
   - *If $V_1^+, V_2^+ \in \mathbb{T}^+$, then $(V_1, V_2) \in \mathbb{T}^+$.*
   - *If $V \in \mathbb{T}^+$, then $inl(V) \in \mathbb{T}^+$ and $inr(V) \in \mathbb{T}^+$.*
   - *If $t^- \in \mathbb{T}^-$, then $(t^-)^{\downarrow} \in \mathbb{T}^+$.*
5. *Closure of $\perp\!\!\!\perp$ under (careful) backward reduction:*
   - *If $V^+ \in \mathbb{T}^+$ and $c[x^+ \leftarrow V^+] \in \perp\!\!\!\perp$, then $\langle V^+ \mid \mu x^+.c \rangle \in \perp\!\!\!\perp$,*
   - *If $t^- \in \mathbb{T}^-$ and $c[x^- \leftarrow t^-] \in \perp\!\!\!\perp$, then $\langle \mu x^-.c \mid t^- \rangle \in \perp\!\!\!\perp$.*
   - *If $V_1^+, V_2^+ \in \mathbb{T}^+$ and $c[V_1^+/x_1^+, V_2^+/x_2^+] \in \perp\!\!\!\perp$, then $\langle (V_1^+, V_2^+) \mid \mu(x_1^+, x_2^+).c \rangle \in \perp\!\!\!\perp$.*
   - *If $V_1^+ \in \mathbb{T}^+$, $c_2 \in \perp\!\!\!\perp$ and $c_1[x_1^+ \leftarrow V_1^+] \in \perp\!\!\!\perp$, then $\langle inl(V_1^+) \mid \mu[inl(x_1^+).c_1, inr(x_2^+).c_2] \rangle \in \perp\!\!\!\perp$.*
   - *If $V_2^+ \in \mathbb{T}^+$, $c_1 \in \perp\!\!\!\perp$ and $c_1[x_2^+ \leftarrow V_2^+] \in \perp\!\!\!\perp$, then $\langle inl(V_1^+) \mid \mu[inl(x_1^+).c_1, inr(x_2^+).c_2] \rangle \in \perp\!\!\!\perp$.*
   - *If $t^- \in \mathbb{T}^-$ and $c[x^- \leftarrow t^-] \in \perp\!\!\!\perp$, then $\langle (t^-)^{\downarrow} \mid \mu(x^-)^{\downarrow}.c \rangle \in \perp\!\!\!\perp$*
6. *If an instance of c, t^+, t^- is in $\perp\!\!\!\perp, \mathbb{T}^+, \mathbb{T}^-$, respectively, then so is c, t^+, t^-.*

Proposition 1.4, the fundamental lemma, and Theorem 1.6 hold in any realisability structure (and the proposition is used in the proof of the fundamental lemma). One chooses of course $\emptyset \neq \mathcal{V}[\![X]\!] \subseteq \mathbb{T}^+$.

Proposition 2.2 *We have, for all P, Q:*

1. *$\emptyset \neq \mathcal{V}[\![P]\!] \subseteq \mathbb{T}^+$,*
2. *$\texttt{Var} \subseteq [\![N]\!]$,*
3. *$[\![P]\!] \subseteq \mathbb{T}^+$,*

4. $[\![N]\!] \subseteq \mathbb{T}^-$,

Lemma 2.3 (Fundamental lemma) *Same statement as in the previous section.*

Theorem 2.4 *Any typable command, positive term, negative term lies in* $\perp\!\!\!\perp, \mathbb{T}^+, \mathbb{T}^-$*, respectively*

Here are two examples of realisability structures, from which we retrieve the weak normalisation and strong normalisation results, respectively:

- $\perp\!\!\!\perp = \mathtt{WN}$, $\mathbb{T}^+ = \{\dots t^+ \dots\}$, $\mathbb{T}^- = \{\dots t^- \dots\}$
- $\perp\!\!\!\perp = \mathtt{SN}$, $\mathbb{T}^+ = \mathtt{SN}$, $\mathbb{T}^- = \mathtt{SN}$

References

[1] J.-L. Krivine, Lambda-calculus, types and models, Ellis Horwood (1993) (freely available from `http://www.pps.univ-paris-diderot.fr/~krivine`).

[2] J.-L. Krivine, Realizability in classical logic, in Interactive models of computation and program behaviour, Panoramas et synthèses, Société Mathématique de France,197-229 (2009).

[3] J.-Y. Girard, Locus solum: from the rules of logic to the logic of rules, Mathematical Structures in Computer Science 11(3), 301-506 (2001).

[4] G. Munch-Maccagnoni, Focalisation and classical realisability, Proc. CSL 2009 LNCS 5771, 409–423, Springer (long version available from `http://perso.ens-lyon.fr/guillaume.munch/articles`).

[5] G. Munch-Maccagnoni, λ-calcul, machines et orthogonalit, unpublished manuscript (in French), available from `http://www.pps.univ-paris-diderot.fr/~munch/papers`.