

# Sequent Calculus

from inference rules to logical relations

Problem: "if wishes were horses, beggars would ride"

Attempt: 
$$\frac{\forall x \left( \frac{\text{wish}(x)}{\text{horse}(x)} \right) \quad \text{beggar}(s)}{\exists h. \text{rides}(s, h) \quad \text{horse}(h)}$$

ugh!

" , " (comma)



$$\frac{A \otimes B}{\text{prop}}$$

↑ prop    ↑ prop

$$\Delta A, B \rightsquigarrow \Delta, A \otimes B$$

and vice versa

what about empty?

$$1$$

$$\begin{aligned} \cdot &\rightsquigarrow 1 \\ 1 &\rightsquigarrow \cdot \end{aligned}$$

" — " (inference rule)  $\rightsquigarrow A \multimap B$

w/ tensor, can only consider  $\frac{A}{B}$

$$A, A \multimap B \rightsquigarrow B$$

wait! this disappears

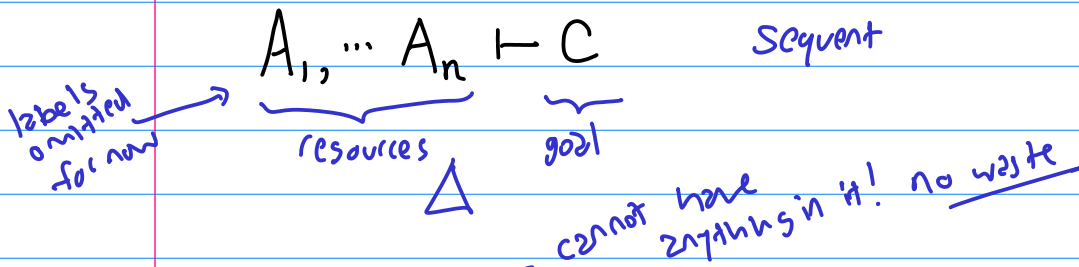
$$(\text{?}) \rightsquigarrow A \multimap B \quad (\text{?})$$

we need to generalize our world

(pretending everything is just 2 prop won't work.)

# divergence from Girard

our development is consistent w/ type theory



Identity:  $\frac{}{A \vdash A} id_A$

exchange OK  
weakening/contraction NO

Cut:  $\frac{\frac{}{\Delta \vdash A} \quad \frac{}{\Delta', A \vdash C}}{\Delta, \Delta' \vdash C} cut$

Implication  $\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B} \multimap R$

$\frac{\frac{}{\Delta \vdash A} \quad \frac{}{\Delta', B \vdash C}}{\Delta, \Delta' \vdash A \multimap B \vdash C} \multimap L$

are these rules good? there is a formal check for this

$\frac{\frac{}{A \vdash A} id_A \quad \frac{\frac{}{B \vdash B} id_B \quad \frac{}{C \vdash C} id_C}{B, B \multimap C \vdash C} \multimap L}{A, B, A \multimap (B \multimap C) \vdash C} \multimap L$

# Tensor

$$\frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C} (\otimes L)$$

easy  $\downarrow$   
what about (R)?

$$\frac{\Delta \vdash A \quad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B} (\otimes R)$$

$$\frac{\Delta \vdash C}{\Delta, 1 \vdash C} (1L)$$

$$\frac{}{\cdot \vdash 1} (1R)$$

The distinction: can we write or do we have to do it now?

+L -R  $\rightarrow$  can always apply  
-L +R ? may not be allowed.

$$\frac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \& B} (\&R)$$

1 test:

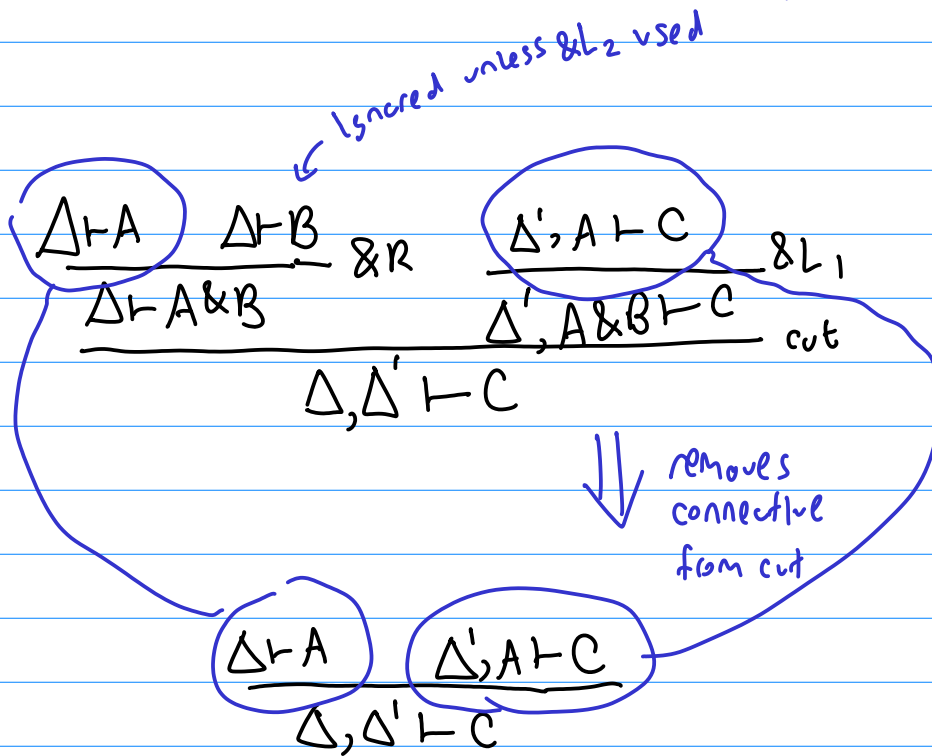
$$\frac{\frac{\overline{A \vdash A}}{A \& B \vdash A} \quad \frac{\overline{B \vdash B}}{A \& B \vdash B}}{A \& B \vdash A \& B}$$



tells us it's negative

but this is not complete

if you have  $\mathcal{H}$ , you can prove it  
 vs. if you can prove  $\mathcal{H}$ , you have it



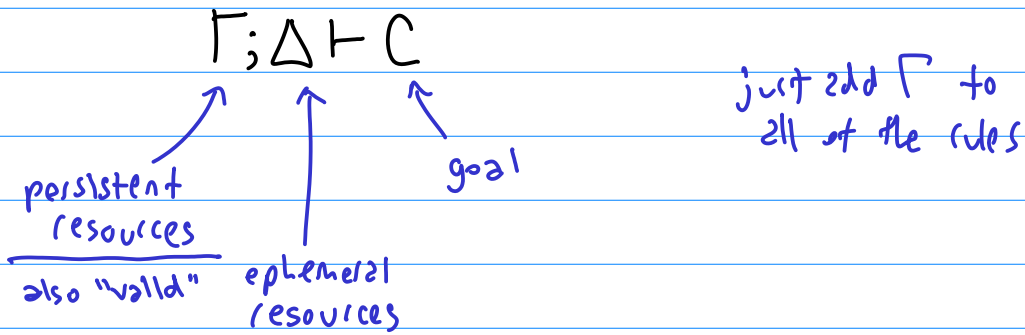
cut reduction

Ex) if we had split the assumptions w/ A and B what goes wrong?

harmony

in linear logic, you tend to have very few choices

# Generalize to intuitionistic logic



problem: how do we use  $\Gamma$ ?

idea:  $\frac{A \in \Gamma}{\Gamma; \vdash A}$  ~~X~~ doesn't work, violates tests

$$\frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}$$

Stronger

this doesn't work because it requires cut

because it could be used many times

$$\frac{\Gamma; \vdash A \quad \Gamma, A; \Delta \vdash C}{\Gamma; \Delta \vdash C} \text{ cut!}$$

$$\frac{\Gamma; \vdash A \text{ true}}{\Gamma \vdash A \text{ valid}} \leftarrow \text{uncontingent on state}$$

$$\frac{\Gamma; \vdash A}{\Gamma; \vdash !A} \text{ !R}$$

of course it's true

$$\frac{\Gamma, A; \Delta \vdash C}{\Gamma; \Delta, !A \vdash C}$$

(eliminating pers cases computational problems)

If wishes were horses, <sup>all</sup> beggars would ride

→ logic stops solving a horse

!  $\forall b. (\forall x. wish(x) \rightarrow horse(x)) \otimes \text{beggar } b$   
 $\sim \exists h. rides(b, h) \otimes ! \text{horse}(h)$

