Sequent Calculus

from inference rules to logical relations

Problem: "if wishes were horses, beggars would ride"

Attempt: $$\frac{\forall x\left(\frac{\text{wish}(x)}{\text{horse}(x)}\right) \qquad \text{beggar}(b)}{\exists h.\ \text{rides}(b,h) \qquad \text{horse}(h)}$$

Ugh!

"," (comma) $\rightsquigarrow$ $\underbrace{\overset{\text{prop}}{A} \otimes \overset{\text{prop}}{B}}_{\text{prop}}$ $\qquad \Delta, A, B \rightsquigarrow \Delta, A \otimes B$

2nd rule

what about empty? $\qquad 1 \qquad \cdot \rightsquigarrow 1$

$1 \rightsquigarrow \cdot$

"———" (inference rule) $\rightsquigarrow$ $A \multimap B$

w/ tensor, can only consider $\frac{A}{B}$ $\qquad \underbrace{A, A \multimap B}_{\text{wait! this disappears}} \rightsquigarrow B$

$? \rightsquigarrow A \multimap B \quad ?$

we need to generalize our world

(pretending everything is just a prop won't work.)

divergence from Girard

our development is consistent w/ type theory

$$\underbrace{A_1, \cdots A_n}_{\text{resources}} \vdash \underbrace{C}_{\text{goal}} \qquad \text{sequent}$$

labels omitted for now

Δ

cannot have anything in it! no waste

Identity: $\dfrac{}{\cdot, A \vdash A}\ id_A$

exchange OK
weakening/contraction NO

Cut: $\dfrac{\Delta \vdash A \qquad \Delta', A \vdash C}{\Delta, \Delta' \vdash C}\ cut$

Implication $\dfrac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \multimap R$

$$\dfrac{\Delta \vdash A \qquad \Delta', B \vdash C}{\Delta, \Delta' A \multimap B \vdash C}\ \multimap L$$

are these rules good? there is a formal check for this

$$\dfrac{\dfrac{}{A \vdash A}\ id_A \qquad \dfrac{\dfrac{}{B \vdash B}\ id_B \qquad \dfrac{}{C \vdash C}\ id_C}{B, B \multimap C \vdash C}\ \multimap L}{A, B, A \multimap (B \multimap C) \vdash C}\ \multimap L$$

Tensor

$$\frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\ (\otimes L)$$

easy → what about right?

$$\frac{\Delta \vdash A \qquad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B}\ (\otimes R)$$

$$\frac{\Delta \vdash C}{\Delta, 1 \vdash C}\ (1L)$$

$$\frac{}{\cdot \vdash 1}\ (1R)$$

the distinction: can we wait or do we have to do it now?

$+L$ $\quad -R$ → can always apply

$-L$ $\quad +R$? may not be allowed.

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \,\&\, B}\ (\&R)$$

1 test:

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \& B \vdash A} \qquad \dfrac{\overline{B \vdash B}}{A \& B \vdash B}}{A \& B \vdash A \& B}$$

tells us it's negative

but this's not complete

if you have it, you can prove it
vs. if you can prove it, you have it

ignored unless $\&L_2$ used

$$\dfrac{\dfrac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \& B}\,\&R \qquad \dfrac{\Delta', A \vdash C}{\Delta', A \& B \vdash C}\,\&L_1}{\Delta, \Delta' \vdash C}\,cut$$

$\Downarrow$ removes connective from cut

$$\dfrac{\Delta \vdash A \qquad \Delta', A \vdash C}{\Delta, \Delta' \vdash C}$$

cut reduction

Ex) if we had split the assumptions w/ A and B what goes wrong?

harmony

in linear logic, you tend to have very few choices

Generalize to intuitionistic logic

$$\Gamma ; \Delta \vdash C$$

persistent resources (also "valid") — Γ

ephemeral resources — Δ

goal — C

just add Γ to all of the rules

problem: how do we use Γ?

idea: $$\frac{A \in \Gamma}{\Gamma ; \cdot \vdash A} \times$$

doesn't work, violates tests

this doesn't work because it requires cut

$$\frac{\Gamma, A ; \Delta, A \vdash C}{\Gamma, A ; \Delta \vdash C}$$

stronger

because it could be used many times

$$\frac{\Gamma ; \cdot \vdash A \qquad \Gamma, A ; \Delta \vdash C}{\Gamma ; \Delta \vdash C} \text{ cut!}$$

$$\frac{\Gamma ; \cdot \vdash A \text{ true}}{\Gamma \vdash A \text{ valid}}$$

unconstrained on state

$$\frac{\Gamma ; \cdot \vdash A}{\Gamma ; \cdot \vdash !A} \, !R$$

of course it's true

$$\frac{\Gamma, A ; \Delta \vdash C}{\Gamma ; \Delta, !A \vdash C}$$

(eliminating pers causes computational problems)

If wishes were horses, all beggars would ride

beggar stops being a horse

$$!\forall b. \left(\left(!\forall x. \mathrm{wish}(x) \multimap \mathrm{horse}(x)\right) \otimes \mathrm{beggar}\ b\right)$$
$$\multimap \exists h.\ \mathrm{rides}(b,h) \otimes\ !\mathrm{horse}(h)$$