Sequent Calculus

from inference rules to logical relations

Problem: "if wishes were horses, beggars would ride"

Attempt:

$$\frac{\forall x\left(\dfrac{wish(x)}{horse(x)}\right) \qquad beggar(b)}{\exists h.\ rides(b,h) \qquad horse(h)}$$

Ugh!

"," (comma) $\rightsquigarrow$ $\underbrace{\overset{prop}{A} \otimes \overset{prop}{B}}_{prop}$ $\qquad \Delta, A, B \rightsquigarrow \Delta, A \otimes B$

and vv.

what about empty? $\qquad 1 \qquad \cdot \rightsquigarrow 1$

$1 \rightsquigarrow \cdot$

"———" (inference rule) $\rightsquigarrow$ $A \multimap B$

w/ tensor, can only consider $\dfrac{A}{B}$ $\qquad \underbrace{A, A \multimap B}_{\text{wait! this disappears}} \rightsquigarrow B$

? $\rightsquigarrow A \multimap B$?

we need to generalize our world

(pretending everything is just a prop won't work.)

divergence from Girard

our development is consistent w/ type theory

$$\underbrace{A_1, \cdots A_n}_{\text{resources } \Delta} \vdash \underbrace{C}_{\text{goal}}$$ sequent

labels omitted for now

Identity:

cannot have anything in it! no waste

$$\frac{}{\cdot, A \vdash A}\ id_A$$

exchange OK

weakening/contraction NO

Cut:

$$\frac{\Delta \vdash A \qquad \Delta', A \vdash C}{\Delta, \Delta' \vdash C}\ cut$$

Implication

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \multimap R$$

$$\frac{\Delta \vdash A \qquad \Delta', B \vdash C}{\Delta, \Delta' A \multimap B \vdash C}\ \multimap L$$

are these rules good? there is a formal check for this

$$\dfrac{\dfrac{}{A \vdash A}\ id_A \qquad \dfrac{\dfrac{}{B \vdash B}\ id_B \qquad \dfrac{}{C \vdash C}\ id_C}{B, B \multimap C \vdash C}\ \multimap L}{A, B, A \multimap (B \multimap C) \vdash C}\ \multimap L$$

Tensor

$$\frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C} \ (\otimes L)$$

easy → what about right?

$$\frac{\Delta \vdash A \qquad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B} \ (\otimes R)$$

$$\frac{\Delta \vdash C}{\Delta, 1 \vdash C} \ (1L)$$

$$\frac{}{\cdot \vdash 1} \ (1R)$$

the distinction: can we wait or do we have to do it now?

+L   −R → can always apply

−L   +R ? may not be allowed.

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \,\&\, B} \ (\&R)$$

1 test:

$$\frac{\dfrac{\overline{A \vdash A}}{A \& B \vdash A} \qquad \dfrac{\overline{B \vdash B}}{A \& B \vdash B}}{A \& B \vdash A \& B}$$

← tells us it's negative

but this's not complete

if you have it, you can prove it

vs. if you can prove it, you have it

ignored unless $\&L_2$ used

$$\dfrac{\dfrac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \& B}\,\&R \qquad \dfrac{\Delta', A \vdash C}{\Delta', A \& B \vdash C}\,\&L_1}{\Delta, \Delta' \vdash C}\,\text{cut}$$

$\Downarrow$ removes connective from cut

$$\dfrac{\Delta \vdash A \qquad \Delta', A \vdash C}{\Delta, \Delta' \vdash C}$$

cut reduction

Ex) if we had split the assumptions w/ A and B what goes wrong?

harmony

in linear logic, you tend to have very few choices

Generalize to intuitionistic logic

$$\Gamma ; \Delta \vdash C$$

persistent resources

also "valid"

ephemeral resources

goal

just add Γ to all of the rules

problem: how do we use Γ?

idea: $$\frac{A \in \Gamma}{\Gamma ; \cdot \vdash A} \; \times$$ doesn't work, violates tests

this doesn't work because it requires cut

$$\frac{\Gamma, A ; \Delta, A \vdash C}{\Gamma, A ; \Delta \vdash C}$$

stronger

because it could be used many times

$$\frac{\Gamma ; \cdot \vdash A \qquad \Gamma, A ; \Delta \vdash C}{\Gamma ; \Delta \vdash C} \text{ cut!}$$

$$\frac{\Gamma ; \cdot \vdash A \text{ true}}{\Gamma \vdash A \text{ valid}}$$ uncontingent on state

$$\frac{\Gamma ; \cdot \vdash A}{\Gamma ; \cdot \vdash !A} \; !R$$

of course it's true

$$\frac{\Gamma, A ; \Delta \vdash C}{\Gamma ; \Delta, !A \vdash C}$$

(eliminating pers causes computational problems)

If wishes were horses, all beggars would ride

beggar stops being a horse

$$!\forall b.\ ((!\forall x. \mathit{wish}(x) \multimap \mathit{horse}(x)) \otimes \mathit{beggar}\ b) \multimap \exists h.\ \mathit{rides}(b,h) \otimes !\mathit{horse}(h)$$