Homework 1: Heyting Algebra and IPL

15-819 Homotopy Type Theory

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This is 15-819's first homework assignment!

1 Heyting Meets Boole

The goal of this section is to reason about an algebra, for example a Boolean algebra, through abstract properties, without appealing a particular model, such as `true` and `false`. In particular, meets and joins are defined by their *universal properties* instead of truth tables.

As mentioned in the lectures, with implications one can show distributiveness of any Heyting algebra. The following is one of the most interesting parts in the proof.

Task 1. *Show that* $A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)$ *in any Heyting algebra.*

Solution:

$$\dfrac{\dfrac{}{A \wedge (B \vee C) \leq B \vee C} \qquad \dfrac{\dfrac{\dfrac{}{A \wedge B \leq (A \wedge B) \vee (A \wedge C)}}{B \leq A \supset (A \wedge B) \vee (A \wedge C)} \qquad \dfrac{\dfrac{}{A \wedge C \leq (A \wedge B) \vee (A \wedge C)}}{C \leq A \supset (A \wedge B) \vee (A \wedge C)}}{B \vee C \leq A \supset (A \wedge B) \vee (A \wedge C)}}{A \wedge (B \vee C) \leq A \supset (A \wedge B) \vee (A \wedge C)}$$

and so

$$\dfrac{\dfrac{}{A \wedge (B \vee C) \leq A} \qquad A \wedge (B \vee C) \leq A \supset (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \leq A \wedge (A \supset (A \wedge B) \vee (A \wedge C))}$$

and finally

$$\dfrac{A \wedge (B \vee C) \leq A \wedge (A \supset (A \wedge B) \vee (A \wedge C)) \qquad \dfrac{}{A \wedge (A \supset (A \wedge B) \vee (A \wedge C)) \leq (A \wedge B) \vee (A \wedge C)}}{A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)}.$$

In class we also gave two definitions of negations $\neg A$, one with explicit construction and the other through universal properties. The next task is to show that these two definitions are equivalent.

Task 2. *Show that in any Heyting algebra, $A \supset \bot$ is one of the largest C's inconsistent with A, and is equivalent to any such C.*

Solution: We clearly have $A \wedge (A \supset \bot) \leq \bot$ and also for any B,

$$\dfrac{A \wedge B \leq \bot}{B \leq A \supset \bot}.$$

Suppose there is a C such that $A \wedge C \leq \bot$ and for any B, $A \wedge B \leq \bot$ implies $B \leq C$. We have

$$\dfrac{A \wedge C \leq \bot}{C \leq A \supset \bot}$$

and by picking $B = A \supset \bot$ we get $A \supset \bot \leq C$. Thus $A \supset \bot$ is equivalent to C.

Finally, with the introduction of mighty complements, the value of exponentials becomes questionable. In fact, they become definable if we assume distributiveness. (Please refer to the lecture note for the correct definition of complements. There was a mistake in the definition given in class.)

Task 3. *Show that in any Boolean algebra (complemented distributive lattice), $\overline{A} \vee B$ is a valid implementation of $A \supset B$. That is, it satisfies all properties of $A \supset B$.*

Solution:

- "$A \wedge (A \supset B) \leq B$."

  We have $A \wedge (\overline{A} \vee B) \leq (A \wedge \overline{A}) \vee (A \wedge B)$. For the first branch, $A \wedge \overline{A} \leq \bot \leq B$. For the second branch, $A \wedge B \leq B$. Thus $A \wedge (\overline{A} \vee B) \leq B$.

- "If $A \wedge C \leq B$ then $C \leq A \supset B$."

  Since $C \leq \top \leq A \vee \overline{A}$, $C \leq (A \vee \overline{A}) \wedge C \leq (A \wedge C) \vee (\overline{A} \wedge C)$. For the first branch, $A \wedge C \leq B \leq \overline{A} \vee B$. For the second branch, $\overline{A} \wedge C \leq \overline{A} \leq \overline{A} \vee B$. Therefore $C \leq \overline{A} \vee B$.

2 IPL Structural Engineering

Here we will explore structural properties of IPL, among which the most important one is transitive as shown below.

Task 4. *Show that IPL is transitive, which is to say*

$$\frac{\Gamma, \Gamma' \vdash A \; \textit{true} \qquad \Gamma, A \; \textit{true}, \Gamma' \vdash B \; \textit{true}}{\Gamma, \Gamma' \vdash B \; \textit{true}}$$

is admissible. You only have to consider the case that the last rule applied in the right sub-derivation (of Γ, A true$, \Gamma' \vdash B$ true) is the primitive reflexivity or rules in negative fragment. You may assume weakening and exchange as admissible rules.

Solution: Induction on the right sub-derivation. Consider the last rule applied.

- Reflexivity.

  If B true is proved from that particular A true in the context by reflexivity, then B true $= A$ true and by assumption $\Gamma, \Gamma' \vdash A$ true. Otherwise, that particular A true is irrelevant and one can apply reflexivity to the rest of the context to obtain B true.

- $\top$I.

  $$\overline{\Gamma, A \; \mathsf{true}, \Gamma' \vdash \top \; \mathsf{true}}.$$

  By the same rule $\Gamma, \Gamma' \vdash \top$ true.

- $\wedge$I.
$$\frac{\Gamma, A, \Gamma' \text{ true} \vdash B \text{ true} \qquad \Gamma, A, \Gamma' \text{ true} \vdash C \text{ true}}{\Gamma, A, \Gamma' \text{ true} \vdash B \wedge C \text{ true}}.$$
By inductive hypotheses we have $\Gamma, \Gamma' \vdash B$ true and $\Gamma, \Gamma' \vdash C$ true. Therefore
$$\frac{\Gamma, \Gamma' \vdash B \text{ true} \qquad \Gamma, \Gamma' \vdash C \text{ true}}{\Gamma, \Gamma' \vdash B \wedge C \text{ true}}.$$

- $\wedge E_1$.
$$\frac{\Gamma, A \text{ true}, \Gamma' \vdash B \wedge C \text{ true}}{\Gamma, A \text{ true}, \Gamma' \vdash B \text{ true}}.$$
By inductive hypothesis $\Gamma, \Gamma' \vdash B \wedge C$ true. And so $\Gamma, \Gamma' \vdash B$ true.

- $\wedge E_2$.
$$\frac{\Gamma, A \text{ true}, \Gamma' \vdash B \wedge C \text{ true}}{\Gamma, A \text{ true}, \Gamma' \vdash B \text{ true}}.$$
By inductive hypothesis $\Gamma, \Gamma' \vdash B \wedge C$ true. And so $\Gamma, \Gamma' \vdash B$ true.

- $\supset$I.
$$\frac{\Gamma, A \text{ true}, \Gamma', B \text{ true} \vdash C \text{ true}}{\Gamma, A \text{ true}, \Gamma' \vdash B \supset C \text{ true}}.$$
By weakening Γ, Γ', Btrue $\vdash A$true. By inductive hypothesis Γ, Γ', Btrue $\vdash C$ true and thus $\Gamma, \Gamma' \vdash B \supset C$ true.

- $\supset$E.
$$\frac{\Gamma, A \text{ true}, \Gamma' \vdash C \supset B \text{ true} \qquad \Gamma, A \text{ true}, \Gamma' \vdash C \text{ true}}{\Gamma, A \text{ true}, \Gamma' \vdash B \text{ true}}.$$
By inductive hypotheses $\Gamma, \Gamma' \vdash C \supset B$ true and $\Gamma, \Gamma' \vdash C$ true. Thus from the same rule $\Gamma, \Gamma' \vdash B$ true.

3 Semantical Analysis of IPL

Any Heyting algebra can be a model of IPL. In fact, $\Gamma \vdash A$ true is provable iff $\Gamma^+ \leq A^*$ in any Heyting algebra, where $(-)^*$ is the straightforward lifting of any evaluation function from atomic propositions to objects in the Heyting algebra in question, and Γ^+ is defined by the following equations:

1. $\cdot^+ = \top$.
2. $(A \text{ true})^+ = A^*$.
3. $(\Gamma, A \text{ true}, B \text{ true})^+ = B^* \wedge (\Gamma, A \text{ true})^+$.

(The reason that we handle singleton contexts specially will be clear in the last task.)

Task 5. *Show that if $\Gamma \vdash A$ true then $\Gamma^+ \leq A^*$. You only have to consider the cases in which the last rule applied is $\supset$I or $\supset$E.*

Solution:

- $\supset$I.

  By inductive hypothesis we have

  $$A^* \wedge \Gamma^+ \leq B^*$$

  and thus $\Gamma^+ \leq A^* \supset B^* = (A \supset B)^*$.

- $\supset$E.

  By inductive hypotheses $\Gamma^+ \leq A^* \supset B^*$ and $\Gamma^+ \leq A^*$. Therefore $\Gamma^+ \leq A^* \wedge (A^* \supset B^*) \leq B^*$.

Task 6. *Considering the Lindenbaum algebra of IPL. Show that if $\Gamma^+ \leq A^*$ in that algebra then $\Gamma \vdash A$ true. You only have to consider the case where the last property used is transitivity. You may assume weakening and exchange, or cite previous tasks as lemmas. State clearly your* induction ordering *if you are using induction.* **(Hint)** *By the construction of Lindenbaum algebras, the evaluation function $(-)^*$ is surjective.*

Solution: Induction on the derivation of $\Gamma^+ \leq A^*$.

By inversion we know $\Gamma^+ \leq C^*$ and $C^* \leq A^*$ for some proposition C. By inductive hypothesis $\Gamma \vdash C$ true and C true $\vdash A$ true. By weakening Γ, C true $\vdash A$ true. By transitivity $\Gamma \vdash A$ true.