Homework 5: Pathfinders

15-819 Homotopy Type Theory

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Out: 18/Nov/13
Due: 2/Dec/13

1 Syntax

We have `tlevel` as the type of truncation levels. It is the same as `nat` except that it starts with -2 and is using $\texttt{rec}_{\texttt{tlevel}}$ as the recursor. We overload the syntax `suc` as successors in `tlevel` as well.

To make programming easier, you are allowed to use pattern matching for Σ types in λ. Formally speaking, the syntax is $\lambda\langle x, y\rangle.M$ and it means $\lambda p.[\texttt{fst}(p), \texttt{snd}(p)/x, y]M$.

2 Truncation Levels

Implement the following interface with access to the lemma:

- $\texttt{is-prop-is-set} : \prod_{A:\mathcal{U}} \texttt{isProp}(A) \to \texttt{isSet}(A)$

As usual, you do not have to justify the code.

Task 1. $\texttt{contr-equiv} : \prod_{(A:\mathcal{U})} \prod_{(B:A\to\mathcal{U})} \texttt{isContr}(A) \to \texttt{isContr}(B) \to (A \simeq B)$.

Solution. $\lambda\_\_\langle a, \alpha\rangle.\lambda\langle b, \beta\rangle.\langle \lambda\_.b, \langle \lambda\_.a, \beta\rangle, \langle \lambda\_.a, \alpha\rangle\rangle$

Task 2. *Suppose* $(A \leftrightarrow B) :\equiv (A \to B) \times (B \to A)$. *Implement*

$$\prod_{A,B:\mathcal{U}} \mathtt{isProp}(A) \to \mathtt{isProp}(B) \to (A \leftrightarrow B) \to (A \simeq B).$$

Solution. $\lambda\_\_\alpha\beta\langle f, g\rangle.\langle f, \langle g, \lambda y.\beta(f(g(y)))(y)\rangle, \langle g, \lambda x.\alpha(g(f(x)))(x)\rangle\rangle$

Task 3. $\prod_{A:\mathcal{U}} \mathtt{is}\text{-}(-1)\text{-}\mathtt{truncated}(A) \to \mathtt{isProp}(A).$

Solution. $\lambda\beta x y.\mathtt{fst}(\beta(x)(y))$

Task 4. $\mathtt{prop\text{-}level} : \prod_{A:\mathcal{U}} \mathtt{isProp}(A) \to \mathtt{is}\text{-}(-1)\text{-}\mathtt{truncated}(A)$

Solution. $\lambda\beta x y.\langle\beta(x)(y), \mathtt{is\text{-}prop\text{-}is\text{-}set}(A)(\beta)(x)(y)(\beta(x)(y))\rangle$

Task 5. $\mathtt{is\text{-}contr\text{-}is\text{-}prop} : \prod_{A:\mathcal{U}} \mathtt{isContr}(-1)(A) \to \mathtt{isProp}(A).$

Solution. $\lambda\alpha x y.\mathtt{snd}(\alpha)(x)^{-1} \cdot \mathtt{snd}(\alpha)(y)$

Task 6. $\mathtt{raise\text{-}contr} : \prod_{A:\mathcal{U}} \mathtt{isContr}n(A) \to \mathtt{is}\text{-}(-1)\text{-}\mathtt{truncated}(A)$

Solution. $\lambda A.\mathtt{prop\text{-}level}(A) \circ \mathtt{is\text{-}contr\text{-}is\text{-}prop}(A)$

Task 7. $\mathtt{raise\text{-}level} : \prod_{(A:\mathcal{U})} \prod_{(n:\mathtt{tlevel})} \mathtt{is}\text{-}n\text{-}\mathtt{truncated}(A) \to \mathtt{is\text{-}suc}(n)\text{-}\mathtt{truncated}(A)$

Solution.

$$\lambda A n.\mathtt{rec}_{\mathtt{tlevel}}[n.\prod_{A:\mathcal{U}} \mathtt{is}\text{-}n\text{-}\mathtt{truncated}(A) \to \mathtt{is\text{-}suc}(n)\text{-}\mathtt{truncated}(A)]$$
$$(n, \mathtt{raise\text{-}contr}, \_.f.\lambda A\alpha x y.f(x = y)(\alpha(x)(y)))(A)$$

Task 8. $\mathtt{prop\text{-}any\text{-}suc\text{-}level} : \prod_{(A:\mathcal{U})} \prod_{(n:\mathtt{tlevel})} \mathtt{isProp}(A) \to \mathtt{is\text{-}suc}(n)\text{-}\mathtt{truncated}(A)$

Solution.

$$\lambda A n.\mathtt{rec}_{\mathtt{tlevel}}[n.\mathtt{isProp}(A) \to \mathtt{is\text{-}suc}(n)\text{-}\mathtt{truncated}(A)]$$
$$(n, \mathtt{prop\text{-}level}(A), n.f.\lambda\alpha.\mathtt{raise\text{-}level}(A)(\mathtt{suc}(n))(f(\alpha)))$$

3 Trunc-verses

Definition 3.1.

$$\mathcal{U}_n :\equiv \sum_{A:\mathcal{U}} \texttt{is-}n\texttt{-truncated}(A)$$

- `funext` as the inverse of `happly`.

$$\texttt{funext} : \left(\prod_{x:A} f\,x = g\,x\right) \to (f = g)$$

- $$\texttt{is-trunc-is-prop}(A)(n) : \texttt{isProp}(\texttt{is-}n\texttt{-truncated}(A))$$

- Being an equivalences is a proposition. Suppose $f : A \to B$.

$$\texttt{is-equiv-is-prop}(f) : \texttt{isProp}(\mathsf{isEquiv}(f))$$

- Closed under $\sum$.

$$\texttt{sigma-level} : \texttt{is-}n\texttt{-truncated}(A) \to \left(\prod_{x:A} \texttt{is-}n\texttt{-truncated}(B(x))\right) \to \texttt{is-}n\texttt{-truncated}\left(\sum_{x:A} B(x)\right)$$

- Closed under $\prod$.

$$\texttt{product-level} : \left(\prod_{x:A} \texttt{is-}n\texttt{-truncated}(B(x))\right) \to \texttt{is-}n\texttt{-truncated}\left(\prod_{x:A} B(x)\right)$$

Lemma 3.1. *For any $A, B : \mathcal{U}$ and $n : \texttt{tlevel}$,*

$$(A \simeq B) \to \texttt{is-}n\texttt{-truncated}(A) \to \texttt{is-}n\texttt{-truncated}(B)$$

Solution. $\lambda e.\lambda\alpha.\texttt{tr}[x.\texttt{is-}n\texttt{-truncated}(x)](\texttt{ua}(e))(\alpha)$

Task 9. *We can show that for any $A : \mathcal{U}$, $B : A \to \mathcal{U}$ and $\beta : \prod_{x:A} \mathtt{isProp}(B(x))$,*

$$\prod_{m,n:\sum_{x:A} B(x)} (m = n) \simeq (\mathtt{fst}(m) = \mathtt{fst}(n)).$$

One function is simply $\mathtt{ap}_{\mathtt{fst}}$. Implement an inverse map of it. That is, prove that

$$\mathtt{subtype\text{-}path}(A)(B)(\beta) : \prod_{m,n:\sum_{x:A} B(x)} (\mathtt{fst}(m) = \mathtt{fst}(n)) \to (m = n).$$

You do not have to show your code is really an inverse function.

Solution.

$$\lambda\langle m_1, m_2\rangle\langle n_1, n_2\rangle p.\mathsf{J}[m_1.n_1.\_.\prod_{m_2:B(m_1)}\prod_{n_2:B(n_1)} \langle m_1, m_2\rangle = \langle n_1, n_2\rangle]$$
$$(p, x_1.\lambda m_2 n_2.\mathtt{ap}_{\lambda x_2.\langle x_1, x_2\rangle}(\beta(x_1)(m_2)(n_2)))(m_2)(n_2)$$

Task 10.

$$\mathtt{equiv\text{-}contr} : \prod_{A,B:\mathcal{U}} \mathtt{isContr}(A) \to \mathtt{isContr}(B) \to \mathtt{isContr}(A \simeq B)$$

Solution.

$$\lambda AB\alpha\beta.\langle \mathtt{contr\text{-}equiv}(A)(B)(\alpha)(\beta),$$
$$\lambda\gamma.\mathtt{subtype\text{-}path}(A \to B)(\mathsf{isEquiv})(\mathtt{is\text{-}equiv\text{-}is\text{-}prop})$$
$$(\mathtt{contr\text{-}equiv}(A)(B)(\alpha)(\beta))(\gamma)(\mathtt{funext}(\lambda a.\mathtt{snd}(\beta)(\mathtt{fst}(\gamma)(a))))\rangle$$

Task 11.

$$\mathtt{subtype\text{-}level} : \prod_{(A:\mathcal{U})}\prod_{(B:A\to\mathcal{U})}\prod_{(n:\mathtt{tlevel})} \mathtt{is\text{-}suc}(n)\mathtt{\text{-}truncated}(A) \to \left(\prod_{x:A} \mathtt{isProp}(B(x))\right)$$
$$\to \mathtt{is\text{-}suc}(n)\mathtt{\text{-}truncated}\left(\sum_{x:A} B\right)$$

Solution.

$$\lambda A B n \alpha \beta.\texttt{sigma-level}(\alpha)(\lambda x.\texttt{prop-any-suc-level}(B(x))(\texttt{suc}(n))(\beta(x)))$$

Task 12.

$$\prod_{(A,B:\mathcal{U})} \prod_{(n:\texttt{tlevel})} \texttt{is-}n\texttt{-truncated}(A) \to \texttt{is-}n\texttt{-truncated}(B) \to \texttt{is-}n\texttt{-truncated}(A \simeq B)$$

Solution.

$$\lambda A B n.\texttt{rec}_{\texttt{tlevel}}[n.\texttt{is-}n\texttt{-truncated}(A) \to \texttt{is-}n\texttt{-truncated}(B) \to \texttt{is-}n\texttt{-truncated}(A \simeq B)]$$
$$(n, \texttt{equiv-contr}(A)(B), n.\_.\lambda\_\beta.$$
$$\texttt{subtype-level}(A \to B)(\mathsf{isEquiv})(n)(\texttt{product-level}(\lambda\_.\beta))(\texttt{is-equiv-is-prop}))$$

Theorem 3.1. *For any* $n : \texttt{tlevel}$*,* $\texttt{is-suc}(n)\texttt{-truncated}(U_n)$*.*