Homework 4: Second Identity Crisis

15-819 Homotopy Type Theory
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Due: 16/Nov/13

1 Application

Task 1. *Assume $\Gamma \vdash f : A \to B$ and $\Gamma \vdash g : B \to C$. Help Favonia to fill in these blanks.*

$$\Gamma \vdash \_\_\_\_ : \prod_{m:A} \mathtt{ap}_f(\mathtt{refl}_A(m)) = \mathtt{refl}_B(f\,m)$$

$$\Gamma \vdash \_\_\_\_ : \prod_{m,n:A} \prod_{p:m=n} \mathtt{ap}_f(p^{-1}) = \mathtt{ap}_f(p)^{-1}$$

$$\Gamma \vdash \_\_\_\_ : \prod_{l,m,n:A} \prod_{p:l=m} \prod_{q:m=n} \mathtt{ap}_f(p \cdot q) = \mathtt{ap}_f(p) \cdot \mathtt{ap}_f(q)$$

$$\Gamma \vdash \_\_\_\_ : \prod_{m,n:A} \prod_{p:m=n} \mathtt{ap}_{\mathtt{id}_A}(p) = p$$

$$\Gamma \vdash \_\_\_\_ : \prod_{m,n:A} \prod_{p:m=n} \mathtt{ap}_{g \circ f}(p) = \mathtt{ap}_g(\mathtt{ap}_f(p))$$

Solution:

$\lambda m.\mathtt{refl}_{f(m)=f(m)}(\mathtt{refl}_B(f(m)))$

$\lambda mnp.\mathsf{J}[m.n.p.\mathtt{ap}_f(p^{-1}) = \mathtt{ap}_f(p)^{-1}](p, x.\mathtt{refl}_{f\,m=f\,m}(\mathtt{refl}_B(f(m))))$

$\lambda lmnpq.\mathsf{J}[l.m.p.\prod_q \mathtt{ap}_f(p \cdot q) = \mathtt{ap}_f(p) \cdot \mathtt{ap}_f(q)](p, m.\lambda q.\mathtt{refl}_{f(m)=f(n)}(\mathtt{ap}_f(q)))(q)$

$\lambda mnp.\mathsf{J}[m.n.p.\mathtt{ap}_{\mathtt{id}_A}(p) = p](p, m.\mathtt{refl}_{m=m}(\mathtt{refl}_A(m)))$

$\lambda mnp.\mathsf{J}[m.n.p.\mathtt{ap}_{g \circ f}(p) = \mathtt{ap}_g(\mathtt{ap}_f(p))](p, m.\mathtt{refl}_{g(f(m))=g(f(m))}(\mathtt{refl}_C(g(f(m)))))$

2 Over the Paths

Define paths over paths by using J.

Solution:

$$\texttt{PathOver}_{x.B}(m, n, p, a, b) :\equiv J[\_.\_.\_.\mathcal{U}](p, m.\texttt{Id}_{[m/x]B}(a, b))$$

3 Application Binary Interface

Solution:

$$\texttt{ap}^2_f(p, q) = \texttt{ap}_{f(l)}(q) \cdot \texttt{ap}_{\lambda m.f(m)(o)}(p)$$
$$\texttt{ap}'^2_f(p, q) = \texttt{ap}_{\lambda l.f(l)(n)}(p) \cdot \texttt{ap}_{f(m)}(q)$$

$$\begin{aligned} q \cdot p &= \texttt{ap}_{\texttt{id}}(q) \cdot \texttt{ap}_{\_\cdot\texttt{refl}(m)}(p) \\ &= \texttt{ap}_{\texttt{refl}(m)\cdot\_}(q) \cdot \texttt{ap}_{\_\cdot\texttt{refl}(m)}(p) \\ &= \texttt{ap}^2_{\_\cdot\_}(p, q) \\ &= \texttt{ap}'^2_{\_\cdot\_}(p, q) \\ &= \texttt{ap}_{\_\cdot\texttt{refl}(m)}(p) \cdot \texttt{ap}_{\texttt{refl}(m)\cdot\_}(q) \\ &= \texttt{ap}_{\_\cdot\texttt{refl}(m)}(p) \cdot \texttt{ap}_{\texttt{id}}(q) \\ &= p \cdot q \end{aligned}$$

Task 2. *Equivalence:* (f, g, α, h, β). *Quasi-inverse:* (f, g, α, β).

Solution:

- Reflexivity.

$$\begin{aligned} f &= \texttt{id}_A \\ g &= \texttt{id}_A \\ \alpha &= \lambda a.\texttt{refl}_A(a) \\ h &= \texttt{id}_A \\ \beta &= \lambda a.\texttt{refl}_A(a) \end{aligned}$$

- Symmetry.

$$\begin{aligned} f &= g \\ g &= f \\ \alpha &= \lambda b.(\mathtt{ap}_{f\circ g}(\beta(b)^{-1}) \cdot \mathtt{ap}_f(\alpha(h(b)))) \cdot \beta(b) \\ h &= f \\ \beta &= \alpha \end{aligned}$$

$$\begin{aligned} f &= h \\ g &= f \\ \alpha &= \beta \\ h &= f \\ \beta &= \lambda a.(\mathtt{ap}_{f\circ h}(\alpha(a)^{-1}) \cdot \mathtt{ap}_f(\beta(h(a)))) \cdot \alpha(a) \end{aligned}$$

$$\begin{aligned} f &= g \\ g &= f \\ \alpha &= \beta \\ h &= f \\ \beta &= \alpha \end{aligned}$$

- Transitivity

$$\begin{aligned} f &= f_2 \circ f_1 \\ g &= g_1 \circ g_2 \\ \alpha &= \lambda b.\mathtt{ap}_{f_2}(\alpha_1(g_2(b))) \cdot \alpha_2(b) \\ h &= h_1 \circ h_2 \\ \beta &= \lambda a.\mathtt{ap}_{h_1}(\beta_2(f_1(a))) \cdot \beta_1(a) \end{aligned}$$

4 Paths in Types

$$\prod_{x,y} \mathtt{Id}_{\top}(x, y) \simeq \top$$

Solution:

- f: $\lambda\_.\langle\rangle$
- g: $\lambda\_.\texttt{refl}_{\top}(\langle\rangle)$
- α: $\lambda\_.\texttt{refl}(\langle\rangle)$
- h: $\lambda\_.\texttt{refl}_{\top}(\langle\rangle)$
- β: $\lambda\_.\texttt{refl}(\texttt{refl}(\langle\rangle))$

$$\prod_{x,y} \texttt{Id}_{\perp}(x,y) \simeq \perp$$

$$\prod_{x,y} \texttt{Id}_{A+B}(x,y) \simeq F(x,y)$$