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Homework 2: Kindom of Kittens

15-819 Homotopy Type Theory
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1 Sequal of Heyting Algebra

This time we will play a little bit with categorical semantics. Although not the absolutely essential part of this course, category provides a nice way to think about various properties diagrammatically. In particular, in this section you have to justify various type-theoretical rules in IPL in a category-theoretical manner.

Similar to Heyting Algebra, a *bicartesian closed category* also models the IPL. Intuitively, a bicartesian closed category is a category with (binary) products, (binary) coproducts and exponentials as described in class. Actually, a Heyting algebra can be viewed as a bicartesian closed category where there is at most one morphism between any two objects, where there is a morphism from A to B iff $A \leq B$. In other words, a Heyting Algebra can only keep track of *provability* that is represented by the sole morphism. Here we are considering a more general case where one can have multiple morphisms between objects, which correspond to different *proofs* of the same proposition.

Again, let $(-)^*$ be the (lifted) translation function from propositions to objects, and $(-)^-$ be the comprehension of this function for Γ. To make your life easier, let's agree that in Homework 2 the function $(-)^+$ does not swap the order of propositions in Γ, which is to say $(\Gamma, x{:}A)^+ = \langle \Gamma^+, A^* \rangle$. M^*_Γ means the translation of the proof M into morphisms. The correspondence, in a nutshell, is that

$$\Gamma \vdash M : A$$

iff the morphism

$$M_\Gamma^* : \Gamma^+ \to A^*$$

exists for any bicartesian closed category and any assignment for atomic propositions,

1.1 Structural Safety

The critical part of the argument is a proper translation of proofs into morphisms. We will not go through the whole construction here; instead, write down the morphisms in terms of these constructs: `id`, $A \circ B$, $\langle A, B \rangle$, `fst`, `snd`, `inl`, `inr`, $\{A, B\}$, $\lambda(A)$ and `ap`.

Task 1. *What morphisms justify these structural properties of the IPL?*

- $x^*_{\Gamma,x:P}$. *This is to implement reflexivity* $\Gamma, x{:}P \vdash x : P$.
- *Suppose* $\Gamma, x{:}P, y{:}Q \vdash M : R$. *Write down* $M^*_{\Gamma,y:Q,x:P}$ *in terms of* $A = M^*_{\Gamma,x:P,y:Q}$. *This is to implement the exchange rule*

$$\frac{\Gamma, x{:}P, y{:}Q \vdash M : R}{\Gamma, y{:}Q, x{:}P \vdash M : R}.$$

- *Suppose* $\Gamma \vdash M : P$ *and* $\Gamma, x{:}P \vdash N : Q$. *Write down a morphism in terms of* $A = M^*_\Gamma$ *and* $B = N^*_{\Gamma,x:P}$ *that is supposed to be equivalent to* $([M/x]N)^*_\Gamma$. *(You do not have to show the equivalence.) This is to implement substitution.*

Solution:

- $\texttt{snd}$.
- $A \circ \langle \langle \texttt{fst} \circ \texttt{fst}, \texttt{snd} \circ \texttt{fst} \rangle, \texttt{snd} \circ \texttt{snd} \rangle$.
- $B \circ \langle \texttt{id}, A \rangle$

1.2 β and η rules

Task 2. *Show that β and η rules are justified by the universal property of exponentials.*

Solution: $\mathtt{ap} \circ \langle \lambda(B), A \rangle = \mathtt{ap} \circ \langle \lambda(B) \circ \mathtt{fst}, \mathtt{snd} \rangle \circ \langle \mathtt{id}, A \rangle = B \circ \langle \mathtt{id}, A \rangle$.
$A = \lambda(\mathtt{ap} \circ \langle A \circ \mathtt{fst}, \mathtt{snd} \rangle)$

2 η for Coproducts in IPL

Solution:

$$M \equiv [z/z]M \equiv [\mathtt{case}(z; x.\mathtt{inl}(x); y.\mathtt{inr}(y))/z]M$$

and then

$$[\mathtt{case}(z; x.\mathtt{inl}(x); y.\mathtt{inr}(y))/z]M \equiv \mathtt{case}(z; x.[\mathtt{inl}(x)/z]M; y.[\mathtt{inr}(y)/z]M)$$