Proof Theory – Lecture 4

Cut Elim; Linear Logic Part 1

Questions from the homework:

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... and at least seven consultants

Q1.)

$$\dfrac{A \vee B\downarrow \qquad \begin{matrix} \overline{A\downarrow}^{\,x} \\ \vdots \\ C\downarrow \end{matrix} \qquad \begin{matrix} \overline{B\downarrow}^{\,y} \\ \vdots \\ C\downarrow \end{matrix}}{C\downarrow}\vee E'^{\,x,y}$$

$$\begin{matrix} C\downarrow \\ \vdots \\ D\uparrow \end{matrix}$$

$$\dfrac{\Gamma, A\vee B, A \Rightarrow \overset{\text{CAD}}{} \Gamma, A\vee B, B \Rightarrow D}{\Gamma, A \vee B \Longrightarrow D}$$

means... ∄ sequent for this, so, there is no rule in sequent calculus (this rule is bad), so we'd have to generalize the sequent calculus.

useful for:

- limiting (and counting) the number of proofs of any given theorem.
- proof searches. (∃ elaboration)

Q2.) Notation... why no turnstile in seq. calc.?

$\Gamma \vdash A$ in Natural Deduction;
$\Gamma \Rightarrow A$ in Sequent Calculus.

... b/c "convention", and to separate it from ND.

chain $\underset{\text{up (down)}}{\_\_}$ from $\underset{\text{up (down)}}{\_\_}$ arrow.

Q3.)

$$\dfrac{A @ t+1 \uparrow}{\bigcirc A @ t}\bigcirc I_{\text{Intro}}$$

up arrow, b/c intro rule

Similarly,

$$\dfrac{\overline{\bigcirc A @ t\downarrow}}{A @ t+1\downarrow}\bigcirc E_{\text{lim}}$$

down arrow, b/c elim rule

?: the arrows point in the direction of the "smaller terms"

↑: want to show, "to prove"...
↓: can assume, are given, can use, if we know this, ...
(if working through a proof)

else (i.e., if done proving sthg),
↑: represents our belief that we have a proof.
↓:

Note: today is Picture Day.

$A_1\downarrow \ldots A_n\downarrow$

$\vdots\ \mathcal{D}$

$C\uparrow$

ND $\Longleftrightarrow$ SEQ

$\longrightarrow$

$$\boxed{\begin{array}{ll} \overset{\mathcal{D}}{\Gamma\downarrow \vdash A\uparrow} & \text{(in ND)} \\ \Gamma \Rightarrow A & \text{(in SEQ)} \end{array}}$$

Theorem (version 1)

(i) if $\Gamma\downarrow \vdash A\uparrow$,

then $\Gamma \Rightarrow A$.

Case

intro ~ R

elim ~ (L turned upside-down)

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma\downarrow \vdash A_1\uparrow} \quad \overset{\mathcal{D}_2}{\Gamma\downarrow \vdash A_2\uparrow}}{\Gamma\downarrow \vdash A_1 \wedge A_2\uparrow}\wedge I$$

$\Gamma \Rightarrow A_1$ by Ind. Hyp. on $\mathcal{D}_1$

$\Gamma \Rightarrow A_2$ — " — $\mathcal{D}_2$

$\Gamma \Rightarrow A_1 \wedge A_2$ by $\wedge R$

Case

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Gamma\downarrow \vdash A\downarrow}}{\Gamma\downarrow \vdash A\uparrow}\downarrow\uparrow$$

Thm (revised):

(i) if $\Gamma\downarrow \vdash A\uparrow$, then $\Gamma \Rightarrow A$.

(ii) if $\Gamma\downarrow \vdash A\downarrow$, and $\Gamma, A \Rightarrow C$

then $\Gamma \Rightarrow C$.

ex 1

$$\dfrac{\dfrac{}{\Gamma, A \vdash A}\,id}{}$$

$$\dfrac{E\downarrow}{I\uparrow} \quad \downarrow\uparrow \cong \quad \dfrac{}{E^{-1} = L\uparrow \Rightarrow \uparrow R = I}\,id$$

ex 2

Case $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ A \wedge B\end{matrix}}{A\downarrow}\wedge E_1$

$\Gamma, A \Rightarrow C$ by Assumption

$\Gamma, A \wedge B, A \Rightarrow C$ by Weakening (Lemma)

$\Gamma, A \wedge B, \Rightarrow C$ by $\wedge L_1$

$\Gamma \Rightarrow C$ by Ind. Hyp. on $\mathcal{D}_1$

Case

$\mathcal{D} = \dfrac{}{\underbrace{\Gamma\downarrow, A\downarrow}_{} \vdash A\downarrow}$ in ND, hyp ~ concl'n

$\Gamma, A, A \Rightarrow C$ by Assumption

Weakening: use throughout

Proof! how? by Induction!

$\Gamma, A \Rightarrow C$ by contraction (lemma)

Now, back to the original program...

A true iff $A\uparrow$

$\dfrac{A \supset B\uparrow \quad A\uparrow}{B\uparrow}$ X

hypothetically, consider the "rule"...

$\dfrac{A\uparrow}{A\downarrow}\uparrow\downarrow$ (which we can't actually do b/c it would render our arrows meaningless)

Case

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\downarrow \vdash A \wedge B\end{array}}{\Gamma\downarrow \vdash A\downarrow} \wedge E_1$$

$\Gamma, A \Rightarrow C$ Assumption

$\Gamma, A\wedge B, A \Rightarrow C$ Weakening (Lemma)

$\Gamma, A \wedge B \Rightarrow C$ by $\wedge L_1$

$\Gamma \Rightarrow C$ by Ind. Hyp. on $\mathcal{D}_1$

Case

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma\downarrow \vdash A\uparrow\end{array}}{\Gamma\downarrow \vdash A\downarrow} \uparrow\downarrow$$

how Gentzen proceeded: "Cut is redundant" "Hauptsatz" (just the main rule)

$\Gamma, A \Rightarrow C$ assumption trim (ii)

$\Gamma \Rightarrow A$ by I.H. on $\mathcal{D}'$

to show: $\Gamma \Rightarrow C$

"Hauptsatz"

$$\dfrac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C} \text{Cut}$$

<u>Thm</u> (Admissibility of Cut):

if $\Gamma \overset{\mathcal{D}}{\Longrightarrow} A$

and $\Gamma, A \overset{\mathcal{E}}{\Longrightarrow} C$

then $\Gamma \overset{\mathcal{F}}{\Longrightarrow} C$

by induction first on $\underbrace{A, \text{ second on}}_{?}$ $\underset{\leq}{\mathcal{D}}\ \underset{=}{\mathcal{E}}$, <u>or</u> ? does this "or" belong here?

$\underset{=}{\mathcal{D}}\ \underset{\leq}{\mathcal{E}}$

<u>Case</u>

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow A_1} \qquad \overset{\mathcal{D}_2}{\Gamma \Rightarrow A_2}}{\Gamma \Rightarrow A_1 \wedge A_2}$$

$$\dfrac{\overset{\mathcal{E}_1}{\Gamma, A_1 \wedge A_2, A_1 \Rightarrow C}}{\Gamma, A_1 \wedge A_2 \Rightarrow C}\wedge L_1 = \mathcal{E}$$

$\Gamma, A_1 \Rightarrow C$ by I.H. on $A_1 \wedge A_2$, $\mathcal{D}^{\wedge A_1}$, $\mathcal{E}_1$

$\Gamma \Rightarrow C$ by "I.H." on A_1, $\mathcal{D}_1$, $\mathcal{F}_1$ what gets smaller?

<u>To show:</u> $\Gamma \Rightarrow C$

$$\dfrac{\Gamma, x : A_1 \wedge A_2, y : A_1 \vdash M : C}{\Gamma, x : A_1 \wedge A_2 \vdash \text{let } y = \pi_1 x \text{ in } M : C}$$

Frank Pfenning | OPLSS'15 notes | θ∅ aanag | 6/7

Linear Logic

GIRARD '87

$$\overbrace{\underbrace{A_1, \ldots, A_n}_{\text{resources}}}^{\Delta} \vdash C$$

= must use each exactly once in a proof

at most 1ce → affine logic.

Girard → affine / linear from geometry

!A "of course, A" — you can secure that = can do in ND

unary "operator" ⊸ ?

⊸ lolli ~ lollipop (not loli) → asia, Lolita

Lollypop

Lolly

classical Logic	intuitionistic	linear
$A \supset B \equiv \neg A \vee B$	⊥ L	⊸ ? ? ← next lecture
$\neg(A \vee \neg B) \wedge \neg$	∧ ∨ ¬ ⊃	⊗ ? ?
SEQ	ND	Proof Nets
	lacking 2 connectives	

⊗ = "tensor"

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B} \multimap R \text{ right} \qquad \frac{\Delta \vdash A \quad \Delta', B \vdash C}{\Delta, \Delta', A \multimap B \vdash C} \multimap L \text{ left}$$

Linear logic takes away weakening & contraction (use each $a \in \Delta$ exactly once) → resource

$$\frac{\Delta \vdash A \quad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B} \otimes R \qquad \frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C} \otimes L$$

$$\overline{A \vdash A}\ ID$$

$\vdash A \multimap A$

$\nvdash A \multimap (A \multimap A)$

c →

i → ctns l.o.e.m

l → i + only once

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Delta \vdash A} \qquad \overset{\mathcal{E}}{\Delta' \vdash B}}{\Delta, \Delta' \vdash A \otimes B}\otimes R \qquad \dfrac{\overset{\mathcal{F}}{\Delta'', A, B \vdash C}}{\Delta'', A \otimes B \vdash C}\otimes L}{\Delta, \Delta', \Delta'' \vdash C}\,cut_{A\otimes B}$$

$(!A) \multimap B$
$\cong A \supset B$
$!(A \multimap B)$

"no slack"

$$\dfrac{\overset{\mathcal{E}}{\Delta' \vdash B} \qquad \dfrac{\overset{\mathcal{D}}{\Delta \vdash A} \qquad \overset{\mathcal{F}}{\Delta'', A, B \vdash C}}{\Delta, \Delta'', B \vdash C}\,cut_A}{\Delta, \Delta', \Delta'' \vdash C}\,cut_B$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash {}^{x:}A}\,ID \qquad \dfrac{}{y{:}A \vdash {}^{y:}A}\,ID}{x{:}A, y{:}A \vdash A \otimes A}\otimes R}{A \vdash A \multimap (A \otimes A)}\multimap R}{\vdash A \multimap (A \multimap A \otimes A)}\multimap R$$

weakening
& contraction
are not valid