Reynolds' Parametricity

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Based on joint work with Neil Ghani, Fredrik Nordvall Forsberg, Federico Orsanigo, and Tim Revell

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Course Outline

Topic: Reynolds' theory of parametric polymorphism for System F

Goals: - extract the fibrational essence of Reynolds' theory
- generalize Reynolds' construction to very general models

- **Lecture 1: Reynolds' theory of parametricity for System F**
- **Lecture 2: Introduction to fibrations**
- **Lecture 3: A bifibrational view of parametricity**
- **Lecture 4: Bifibrational parametric models for System F**

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- This is the main inspiration for the bifibrational model of parametricity for System F we will develop
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- Generalize Reynolds' constructions to bifibrational models of System F for which we can prove (bifibrational versions of) the IEL and Abstraction Theorem
- Reynolds' construction is (ignoring size issues) such a model

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- We are interested in indexing because Reynolds' interpretations are type-indexed

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  - $\mathcal{B}$ **is a set** I **of indices,**
  - $\mathcal{E}$ **is** $X = \bigcup_{i \in I} X_i$**, where** $(X_i)_{i \in I}$ **is a (wlog, disjoint) family of sets**
  - $U : X \to I$ **maps each** $x \in X$ **to the index** $i \in I$ **such that** $x \in X_i$

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- The set

$$X_i = U^{-1}(i) = \{x \in X \mid Ux = i\}$$

  is called the fibre of X over i

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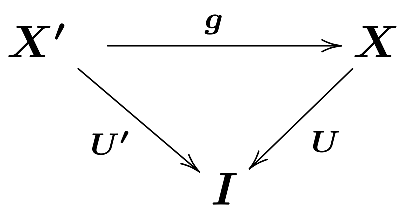

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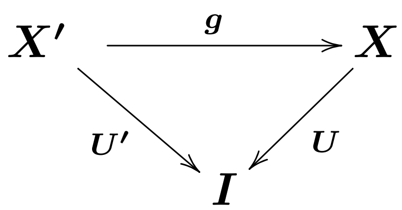

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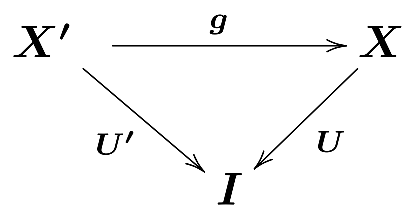

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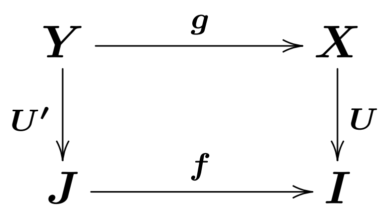

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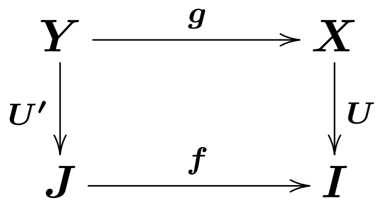

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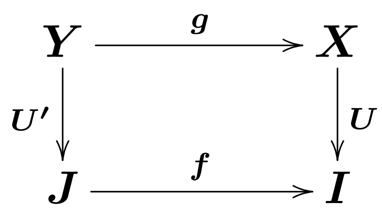

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- Identities and composition are componentwise inherited from Set.
- $\mathsf{Set}^{\rightarrow}$ induces a codomain functor $cod : \mathsf{Set}^{\rightarrow} \rightarrow \mathsf{Set}$ mapping

$$U : X \rightarrow I \text{ to } I \quad \text{and} \quad (g, f) \text{ to } f$$

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- We usually write $f^*(U)$ for the display map U'

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- **Thus $Y = \bigcup_{j \in \{*\}} Y_j = Y_* = X_i$**
- **So substituting along a particular element of I selects the fibre of X over that element**

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- **So $Y = \bigcup_{j \in J} Y_j = J \times X$ (since the Y_j are disjoint)**

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Best Substitution Morphisms - Part I

- **The pair (g, f) in the pullback diagram**

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle f^*(U)}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ J & \xrightarrow[f]{} & I \end{array}$$

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- **We call (g, f) a substitution morphism from $f^*(U)$ to U**

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- (g, f) is such that if
  - $U'' : Z \to K$ is any object in $\mathsf{Set}^{\to}$
  - $(g', f') : U'' \to U$ is a morphism in $\mathsf{Set}^{\to}$
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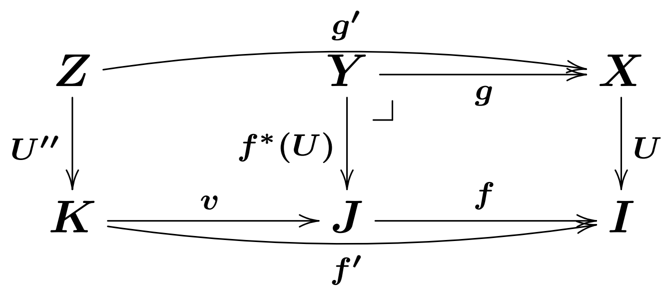

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  then there exists a unique $h : Z \to Y$ in $\mathsf{Set}^{\to}$ such that
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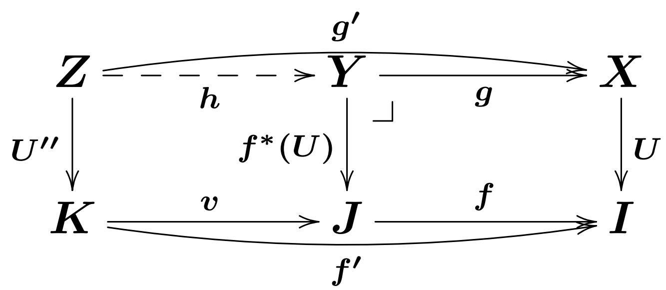

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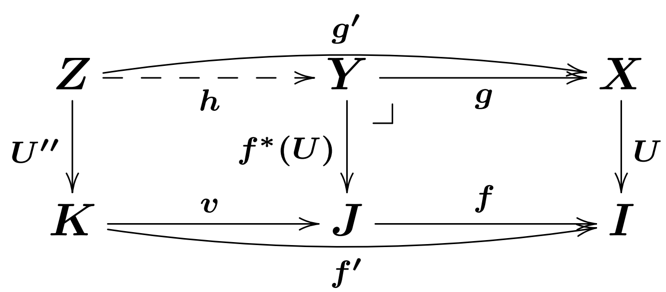

- That is, (g, f) is the best substitution morphism from $f^*(U)$ to U

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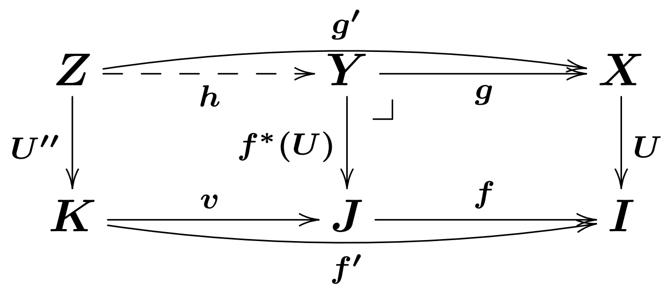
  - $(g', f') : U'' \to U$ is a morphism in $\mathsf{Set}^{\to}$
  - $f' : K \to I$ factors through $f : J \to I$ via $v : K \to J$ (i.e., $f' = f \circ v$)

  then there exists a unique $h : Z \to Y$ in $\mathsf{Set}^{\to}$ such that
  - $cod(h, v) = v$ for $cod : \mathsf{Set}^{\to} \to \mathsf{Set}$
  - $g \circ h = g'$



- That is, (g, f) is the best substitution morphism from $f^*(U)$ to U
- The existence of such best substitution morphisms is what makes cod : $\mathsf{Set}^{\to} \to \mathsf{Set}$ a fibration

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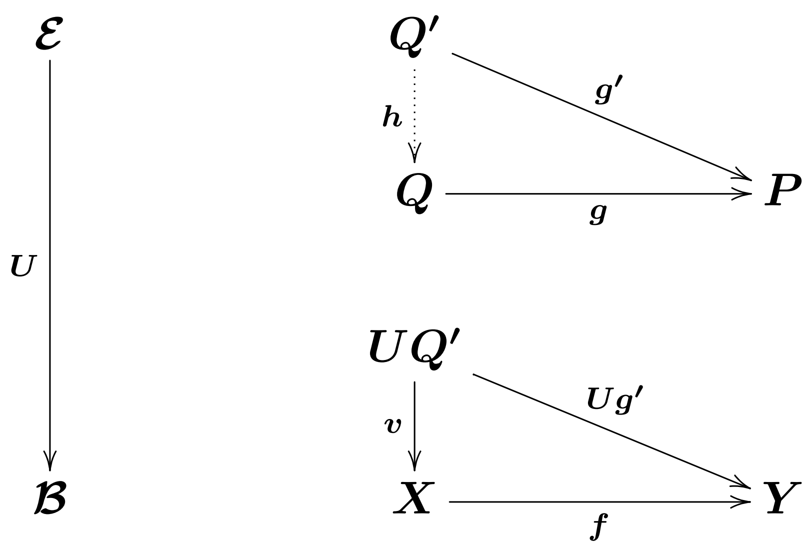

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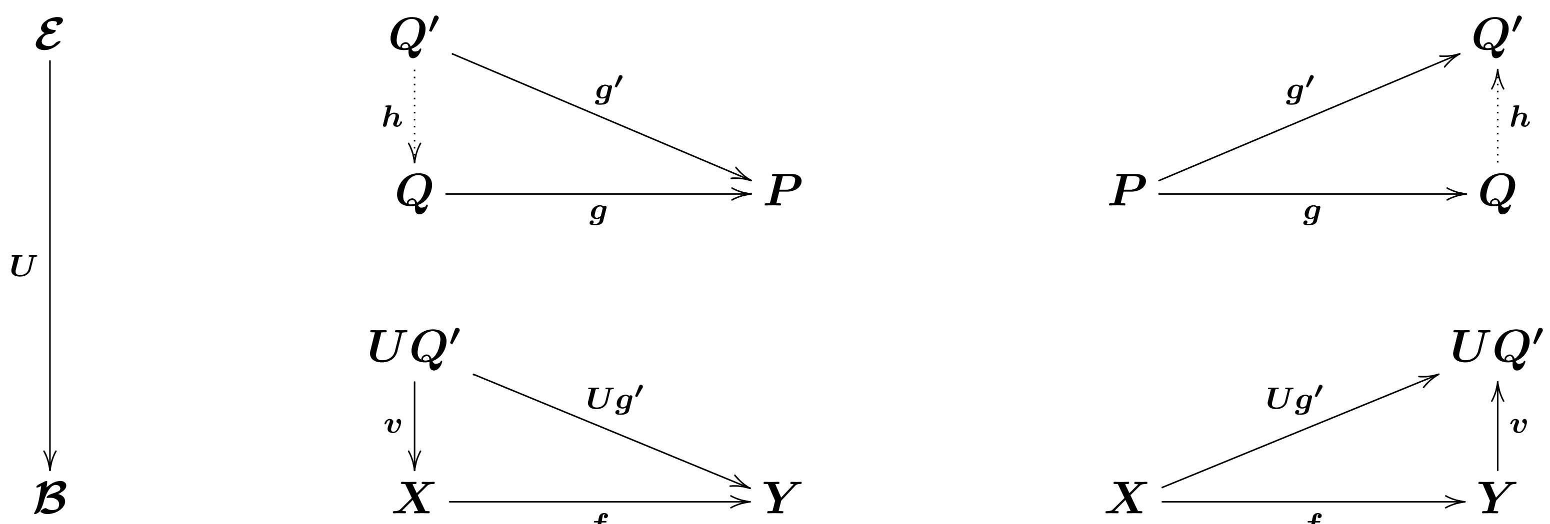

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- The fibre $\mathcal{E}_X$ over an object X in $\mathcal{B}$ is the subcategory of $\mathcal{E}$ of objects over X and morphisms over id_X

Indexing and Reindexing Functors

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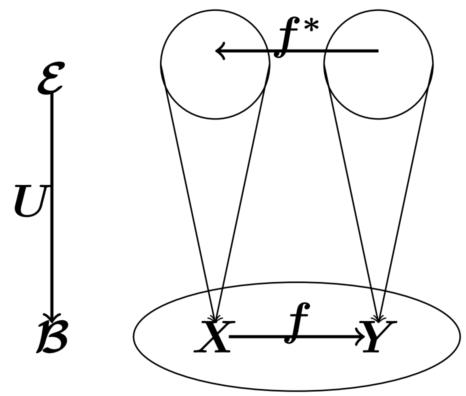

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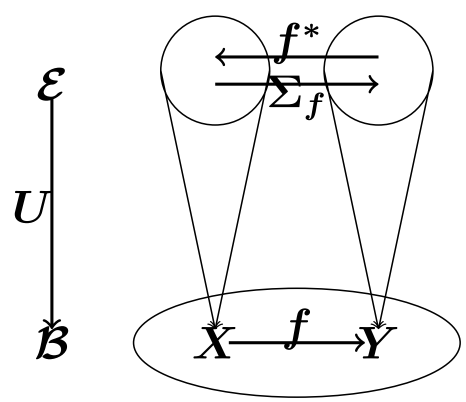

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- Lemma
  1. If $U : \mathcal{E} \to \mathcal{B}$ is a functor, then $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is a bifibration, called the discrete fibration for U
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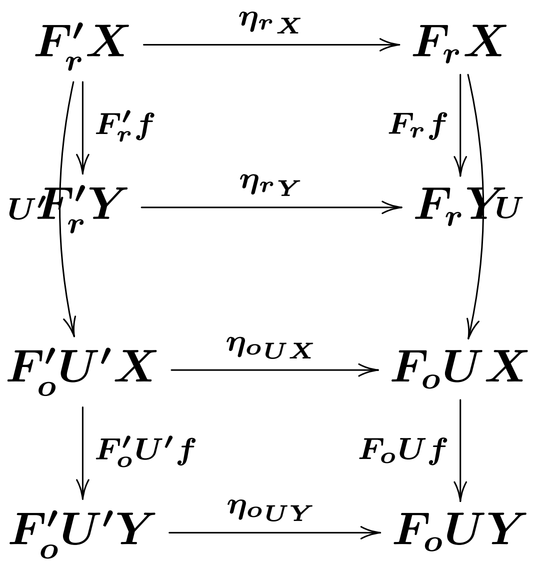

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- **Reynolds' construction is (ignoring size issues) such a model**

References

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