Reynolds' Parametricity

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Based on joint work with Neil Ghani, Fredrik Nordvall Forsberg, Federico Orsanigo, and Tim Revell

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Course Outline

Topic: Reynolds' theory of parametric polymorphism for System F

Goals: - extract the fibrational essence of Reynolds' theory
- generalize Reynolds' construction to very general models

- **Lecture 1: Reynolds' theory of parametricity for System F**
- **Lecture 2: Introduction to fibrations**
- **Lecture 3: A bifibrational view of parametricity**
- **Lecture 4: Bifibrational parametric models for System F**

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- Reynolds' construction is (ignoring size issues) such a model

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- **Re-state Reynolds' construction in terms of the relations fibration on Set**
- **Set up infrastructure needed for our generalization**

The Category **Rel**

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- This can be extended to an equality functor from **Set** to **Rel** in the obvious way

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- $\mathsf{Rel}(X, Y)$ is the fibre over (X, Y)

Reynolds' Semantics of Types, Fibrationally

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- Theorem (Reynolds' Semantics of Types, Fibrationally) Let U be the relations fibration on Set. Every judgement $\Delta \vdash \tau$ induces a fibred functor $[\![\Delta \vdash \tau]\!] : |U|^{|\Delta|} \to U$.

$$\begin{array}{ccc}
|\mathsf{Rel}|^{|\Delta|} & \xrightarrow{[\![\Delta \vdash \tau]\!]_r} & \mathsf{Rel} \\
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|\mathsf{Set}|^{|\Delta|} \times |\mathsf{Set}|^{|\Delta|} & \xrightarrow[{[\![\Delta \vdash \tau]\!]_o \times [\![\Delta \vdash \tau]\!]_o}]{} & \mathsf{Set} \times \mathsf{Set}
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- We use discrete categories in the domain of $[\![\Delta \vdash \tau]\!]$ to reflect the fact that Reynolds did not give a functorial action of types on morphisms

Identity Extension Lemma, Fibrationally

- **If $\Delta \vdash \tau$ then**

$$[\![\Delta \vdash \tau]\!]_r \, (\mathsf{Eq}\, X_1, \ldots, \mathsf{Eq}\, X_{|\Delta|}) = \mathsf{Eq}\, ([\![\Delta \vdash \tau]\!]_o (X_1, \ldots, X_{|\Delta|}))$$

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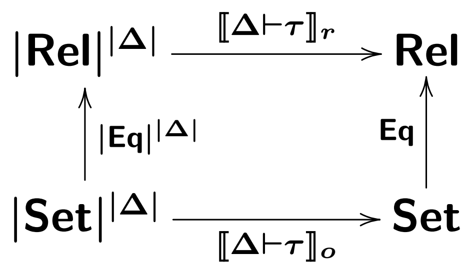

Abstraction Theorem, Fibrationally

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- **Abstraction Theorem Let $\overline{X}, \overline{Y} : \mathsf{Set}^{|\Delta|}$, $\overline{R} : \mathsf{Rel}^{|\Delta|}(\overline{X}, \overline{Y})$, $\overline{A} \in [\![\Delta \vdash \Gamma]\!]_o \overline{X}$, and $\overline{B} \in [\![\Delta \vdash \Gamma]\!]_o \overline{Y}$. For every $\Delta;\Gamma \vdash t:\tau$, if $(\overline{A}, \overline{B}) \in [\![\Delta \vdash \Gamma]\!]_r \overline{R}$, then $([\![\Delta;\Gamma \vdash t:\tau]\!]_o\ \overline{X}\ \overline{A}, [\![\Delta;\Gamma \vdash t:\tau]\!]_o\ \overline{Y}\ \overline{B}) \in [\![\Delta \vdash \tau]\!]_r \overline{R}$.**

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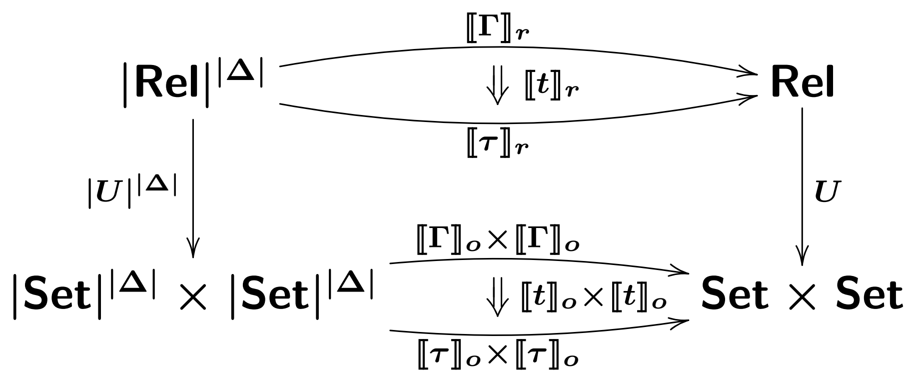
- **Abstraction Theorem** Let $\overline{X},\overline{Y} : \mathsf{Set}^{|\Delta|}$, $\overline{R} : \mathsf{Rel}^{|\Delta|}(\overline{X},\overline{Y})$, $\overline{A} \in [\![\Delta \vdash \Gamma]\!]_o\overline{X}$, and $\overline{B} \in [\![\Delta \vdash \Gamma]\!]_o\overline{Y}$. For every $\Delta;\Gamma \vdash t:\tau$, if $(\overline{A},\overline{B}) \in [\![\Delta \vdash \Gamma]\!]_r\overline{R}$, then $([\![\Delta;\Gamma \vdash t:\tau]\!]_o\ \overline{X}\ \overline{A}, [\![\Delta;\Gamma \vdash t:\tau]\!]_o\ \overline{Y}\ \overline{B}) \in [\![\Delta \vdash \tau]\!]_r\overline{R}$.
- **Theorem** (**Abstraction Theorem, Fibrationally**) Every term $\Delta;\Gamma \vdash t : \tau$ is interpreted as a fibred natural transformation

$$([\![\Delta;\Gamma \vdash t:\tau]\!]_o \times [\![\Delta;\Gamma \vdash t:\tau]\!]_o, [\![\Delta;\Gamma \vdash t:\tau]\!]_r) : [\![\Delta \vdash \Gamma]\!] \to [\![\Delta \vdash \tau]\!]$$



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is a morphism between relations that is over

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- This is the conclusion of Reynolds' original statement of the theorem!!!

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  - opens the way to our generalization of Reynolds' construction
- To generalize $[\![-]\!]_o$ and $[\![-]\!]_r$ in such a way that the Identity Extension Lemma and the Abstraction Theorem hold, we must have sufficient structure to define analogues of all the structure we used in the relations fibration on **Set** for more general fibrations

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- **$\mathsf{Rel}(U)$ is the relations fibration for U**

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\mathsf{Rel}(\mathcal{E}) & \xrightarrow{q} & \mathcal{E} \\
{\scriptstyle \mathsf{Rel}(U)}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\
\mathcal{B} \times \mathcal{B} & \xrightarrow[\_\times\_]{} & \mathcal{B}
\end{array}
$$

- $\mathsf{Rel}(U)$ is the relations fibration for U

- The objects of $\mathsf{Rel}(\mathcal{E})$ are called relations on $\mathcal{B}$

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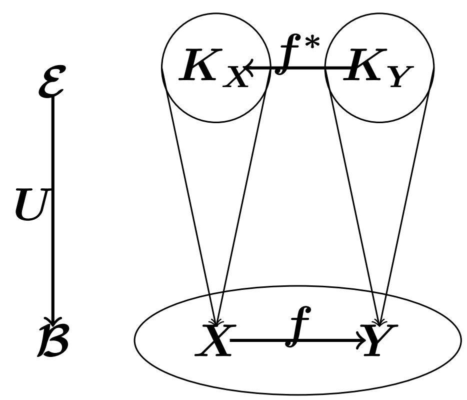

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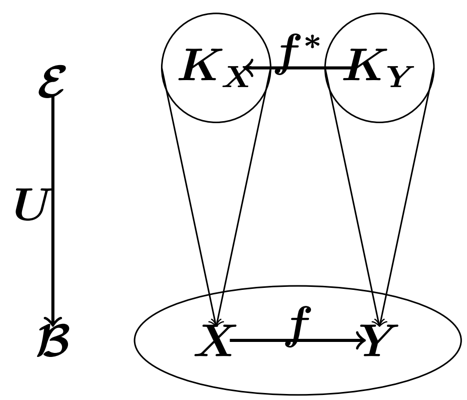

- The map sending each object X of $\mathcal{B}$ to K_X extends to a functor $K : \mathcal{B} \to \mathcal{E}$ called the truth functor for U

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- That is, $\Sigma_{\delta_X}(KX)$ acts like an equality relation on X

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- But these issues will not arise in this course

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- There are reasonable hypotheses on U making $\mathsf{Rel}(U)$ an equality preserving arrow fibration (see MFPS'15 and FoSSaCS'16)

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  - **Morphisms: fibred natural transformations between them**
- **Note the use of discrete categories**

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- $\mathsf{Rel}(U)$ is a $\forall$-fibration if
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- Then for all $F : |\mathsf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ and $G : |\mathsf{Rel}(U)|^{n+1} \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ there is an isomorphism

  $$\varphi_n \; : \; \mathsf{Hom}(F \circ \pi_n, G) \cong \mathsf{Hom}(F, \forall G)$$

  that is natural in n

Coming Up

- **Use relations fibrations that are equality preserving arrow fibrations and ∀-fibrations to interpret System F types as fibred functors and System F terms as fibred natural transformations**

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