

# BATBAMBAMMBA: Boolean and Arithmetic Languages

## Oregon Programming Languages Summer School

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# 1 BA: Boolean Arithmetic Language

## Program Static Syntax

$$\begin{aligned}
 t &\in \text{TERM}, \quad n \in \mathbb{N}, \quad b \in \mathbb{B}, \quad p \in \text{PGM} = \text{TERM} \\
 t &::= \text{true} \mid \text{false} \mid \text{if } t \text{ then } t \text{ else } t \\
 &\quad \mid n \mid \text{succ}(t) \mid \text{pred}(t) \mid \text{zero?}(t) \\
 b &::= \text{true} \mid \text{false}
 \end{aligned}$$

## Program Runtime Syntax

$$\begin{aligned}
 E &\in \text{ECTXT}, \quad v \in \text{VALUE}, \quad r \in \text{REDEX} \subseteq \text{PGM}, \quad f \in \text{FAULTY} \subseteq \text{PGM}, \quad \text{err} \in \text{ERROR}, \quad c \in \text{CONFIG} \quad o \in \text{OBS} \\
 v &::= b \mid n \\
 E &::= \square \mid \text{if } E \text{ then } p \text{ else } p \mid \text{succ}(E) \mid \text{pred}(E) \mid \text{zero?}(E) \\
 r &::= \text{if } v \text{ then } p \text{ else } p \mid \text{succ}(v) \mid \text{pred}(v) \mid \text{zero?}(v) \\
 f &::= \text{if } n \text{ then } p \text{ else } p \mid \text{succ}(b) \mid \text{pred}(b) \mid \text{zero?}(b) \\
 \text{err} &::= \text{mismatch} \mid \text{underflow} \\
 c &::= p \mid \text{err} \\
 o &::= v \mid \text{err}
 \end{aligned}$$
 $\rightsquigarrow \subseteq \text{REDEX} \times \text{PGM}$ 

### Notions of Reduction

$$\begin{aligned}
 \text{if true then } p_2 \text{ else } p_3 &\rightsquigarrow p_2 \\
 \text{if false then } p_2 \text{ else } p_3 &\rightsquigarrow p_3 \\
 \text{pred}(n) &\rightsquigarrow n - 1 \quad \text{if } n > 0 \\
 \text{succ}(n) &\rightsquigarrow n + 1 \\
 \text{zero?}(0) &\rightsquigarrow \text{true} \\
 \text{zero?}(n) &\rightsquigarrow \text{false} \quad \text{if } n > 0
 \end{aligned}$$
 $\longrightarrow \subseteq \text{PGM} \times \text{CONFIG}$ 

### Single-step Reduction

$$\begin{array}{c}
 \frac{r \rightsquigarrow p}{E[r] \longrightarrow E[p]} \\
 \hline
 E[f] \longrightarrow \text{mismatch} \\
 \hline
 E[\text{pred}(0)] \longrightarrow \text{underflow}
 \end{array}$$
 $\longrightarrow^* \subseteq \text{CONFIG} \times \text{CONFIG}$ 

### Multi-step Reduction

$$\begin{array}{ccc}
 (\text{incl}) \frac{c_1 \longrightarrow c_2}{c_1 \longrightarrow^* c_2} & (\text{refl}) \frac{}{c \longrightarrow^* c} & (\text{trans}) \frac{c_1 \longrightarrow^* c_2 \quad c_2 \longrightarrow^* c_3}{c_1 \longrightarrow^* c_3}
 \end{array}$$
 $\text{eval}_{BA} : \text{PGM} \rightarrow \text{OBS}$ 

$$\begin{aligned}
 \text{eval}_{BA}(p) &= b \text{ if } p \longrightarrow^* b \\
 \text{eval}_{BA}(p) &= n \text{ if } p \longrightarrow^* n \\
 \text{eval}_{BA}(p) &= \text{mismatch} \text{ if } p \longrightarrow^* \text{mismatch} \\
 \text{eval}_{BA}(p) &= \text{underflow} \text{ if } p \longrightarrow^* \text{underflow}
 \end{aligned}$$

## Safety (AKA Coherence AKA Definedness)

**Conjecture 1 (Progress).** For all  $p \in \text{PGM}$  one of the following is true:

1.  $p \in \text{VALUE}$ ;
2.  $p \longrightarrow p'$  for some  $p' \in \text{PGM}$ ;
3.  $p \longrightarrow \text{err}$  for some  $\text{err} \in \text{ERROR}$ .

**Conjecture 2 (Preservation (Vacuous)).** If  $p_1 \longrightarrow p_2$  then  $p_2 \in \text{PGM}$ . (uhh...?!?)

## TBA: Typed Boolean Arithmetic Language

### Program Static Syntax

$$t \in \text{TERM}, \quad n \in \mathbb{N}, \quad b \in \mathbb{B}, \quad \text{Same as BA}$$

$$T \in \text{TYPE}, \quad p \in \text{PGM} = \{t \in \text{TERM} \mid \exists T \in \text{TYPE}. \vdash t : T\}$$

$$T ::= \text{Nat} \mid \text{Bool}$$

### Program Runtime Syntax

$$E \in \text{ECTXT}, \quad v \in \text{VALUE}, \quad r \in \text{REDEX} \subseteq \text{PGM}, \quad \text{Same grammar as BA (over updated p)}$$

$$\text{err} \in \text{ERROR}, \quad c \in \text{CONFIG} \quad o \in \text{OBS}$$

$$v ::= b \mid n$$

$$r ::= \text{if } v \text{ then } p \text{ else } p \mid \text{succ}(v) \mid \text{pred}(v) \mid \text{zero?}(v)$$

$$\text{err} ::= \text{underflow}$$

$$c ::= p \mid \text{err}$$

$$o ::= v \mid \text{err}$$

$\rightsquigarrow \subseteq \text{REDEX} \times \text{PGM}$	<b>Same formal schema as BA (over updated p)</b>
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$\longrightarrow \subseteq \text{PGM} \times \text{CONFIG}$	<b>Single-step Reduction</b>
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$$\frac{r \rightsquigarrow p}{E[r] \longrightarrow E[p]} \qquad \frac{}{E[\text{pred}(0)] \longrightarrow \text{underflow}}$$

$\models \dots \subseteq \text{TERM} \times \text{TYPE}$  **Semantic Typing** (Q: Should  $t$  be statically typed? Interesting implications!)

$\models t : \text{Bool}$  if and only if  $t \longrightarrow^* b$  or  $t \longrightarrow^* \text{underflow}$

$\models t : \text{Nat}$  if and only if  $t \longrightarrow^* n$  or  $t \longrightarrow^* \text{underflow}$

$\vdash \dots \subseteq \text{TERM} \times \text{TYPE}$  **Syntactic Typing**

$$\frac{}{\vdash \text{true} : \text{Bool}} \quad \frac{}{\vdash \text{false} : \text{Bool}} \quad \frac{\vdash t_1 : \text{Bool} \quad \vdash t_2 : T \quad \vdash t_3 : T}{\vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T} \quad \frac{}{\vdash n : \text{Nat}} \quad \frac{\vdash t : \text{Nat}}{\vdash \text{succ}(t) : \text{Nat}}$$

$$\frac{\vdash t : \text{Nat}}{\vdash \text{pred}(t) : \text{Nat}} \quad \frac{\vdash t : \text{Nat}}{\vdash \text{zero?}(t) : \text{Bool}}$$

**Evaluator**  $\boxed{\text{eval}_{TBA} : \text{PGM} \rightarrow \text{OBS}}$

$$\text{eval}_{TBA}(p) = b \text{ if } p \longrightarrow^* b$$

$$\text{eval}_{TBA}(p) = n \text{ if } p \longrightarrow^* n$$

$$\text{eval}_{TBA}(p) = \text{underflow} \text{ if } p \longrightarrow^* \text{underflow}$$

### Safety

**Conjecture 3** (Progress). For all  $p \in \text{PGM}$  one of the following is true:

1.  $p \in \text{VALUE}$ ;
2.  $p \longrightarrow p'$  for some  $p' \in \text{PGM}$ ;
3.  $p \longrightarrow \text{underflow}$ .

**Conjecture 4** (Preservation). If  $\vdash p_1 : T$  and  $p_1 \longrightarrow p_2$  then  $\vdash p_2 : T$ .

**Conjecture 5** (Semantic Type Soundness). If  $\vdash t : T$  then  $\models t : T$ .

## 2 MBA: Mixed Boolean and Arithmetic Language

### Program Static Syntax

$s \in \text{SBA}$ ,  $d \in \text{DBA}$ ,  $n, ns, nd \in \mathbb{N}^*$ ,  $b, bs, bd \in \mathbb{B}^*$ ,  $\star$ : A bit sloppy perhaps  
 $ps \in \text{SPGM} = \{s \in \text{SBA} \mid \exists T \in \text{TYPE}. \vdash s : T\}$ ,  $pd \in \text{DPGM} = \{d \in \text{DBA} \mid \vdash d \checkmark\}$ ,  $p \in \text{PGM} = ps$   
 $s ::= \text{true} \mid \text{false} \mid \text{if } s \text{ then } s \text{ else } s$   
 $\quad \mid ns \mid \text{succ}(s) \mid \text{pred}(s) \mid \text{zero?}(s) \mid \lceil d \rceil$   
 $d ::= \text{true} \mid \text{false} \mid \text{if } d \text{ then } d \text{ else } d$   
 $\quad \mid nd \mid \text{succ}(d) \mid \text{pred}(d) \mid \text{zero?}(d) \mid \lfloor s \rfloor$   
 $bs ::= \text{true} \mid \text{false}$   
 $bd ::= \text{true} \mid \text{false}$   
 $b ::= bs \mid bd$

$\vdash \cdot : \cdot \subseteq \text{SBA} \times \text{TYPE}$ $\vdash \cdot \checkmark \subseteq \text{DBA}$	<b>Syntactic Typing</b>
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$\frac{}{\vdash \text{true} : \text{Bool}}$	$\frac{}{\vdash \text{false} : \text{Bool}}$	$\frac{\vdash s_1 : \text{Bool} \quad \vdash s_2 : T \quad \vdash s_3 : T}{\vdash \text{if } s_1 \text{ then } s_2 \text{ else } s_3 : T}$	$\frac{}{\vdash ns : \text{Nat}}$	$\frac{\vdash s : \text{Nat}}{\vdash \text{succ}(s) : \text{Nat}}$	
	$\frac{\vdash s : \text{Nat}}{\vdash \text{pred}(s) : \text{Nat}}$	$\frac{\vdash s : \text{Nat}}{\vdash \text{zero?}(s) : \text{Bool}}$	$\frac{\vdash d \checkmark}{\vdash \lceil d \rceil : T}$		
$\frac{}{\vdash \text{true} \checkmark}$	$\frac{}{\vdash \text{false} \checkmark}$	$\frac{\vdash d_1 \checkmark \quad \vdash d_2 \checkmark \quad \vdash d_3 \checkmark}{\vdash \text{if } d_1 \text{ then } d_2 \text{ else } d_3 \checkmark}$	$\frac{}{\vdash nd \checkmark}$	$\frac{\vdash d \checkmark}{\vdash \text{succ}(d) \checkmark}$	$\frac{\vdash d \checkmark}{\vdash \text{pred}(d) \checkmark}$
		$\frac{\vdash d \checkmark}{\vdash \text{zero?}(d) \checkmark}$		$\frac{\vdash s : T}{\vdash \lfloor s \rfloor \checkmark}$	

### Program Runtime Syntax

$Es \in \text{SECTXT}$ ,  $Ed \in \text{DECTXT}$ ,  $vs \in \text{SVALUE}$ ,  $vd \in \text{DVALUE}$ ,  
 $rs \in \text{SREDEX}$ ,  $rd \in \text{DREDEX}$ ,  $fs \in \text{SFAULTY}$ ,  $fd \in \text{DFAULTY}$ ,  $cs \in \text{SCONFIG}$ ,  $ed \in \text{DCONFIG}$ ,  
 $v \in \text{VALUE}$ ,  $err \in \text{ERROR}$ ,  $o \in \text{OBS}$   
 $Es ::= \square \mid Es[\text{if } \square \text{ then } ps \text{ else } ps] \mid Es[\text{succ}(\square) \mid Es[\text{pred}(\square)] \mid Es[\text{zero?}(\square)] \mid Ed[\lfloor \square \rfloor]]$   
 $Ed ::= Ed[\text{if } \square \text{ then } ps \text{ else } ps] \mid Ed[\text{succ}(\square) \mid Ed[\text{pred}(\square)] \mid Ed[\text{zero?}(\square)] \mid Es[\lceil \square \rceil]]$   
 $vs ::= bs \mid ns \mid \lceil bd \rceil \mid \lceil nd \rceil$   
 $vd ::= bd \mid nd \mid \lfloor bs \rfloor \mid \lfloor ns \rfloor$   
 $rs ::= \text{if } vs \text{ then } ps \text{ else } ps \mid \text{succ}(vs) \mid \text{pred}(vs) \mid \text{zero?}(vs)$   
 $rd ::= \text{if } vd \text{ then } pd \text{ else } pd \mid \text{succ}(vd) \mid \text{pred}(vd) \mid \text{zero?}(vd)$   
 $fs ::= \text{if } \lceil nd \rceil \text{ then } ps \text{ else } ps \mid \text{succ}(\lceil bd \rceil) \mid \text{pred}(\lceil bd \rceil) \mid \text{zero?}(\lceil bd \rceil)$   
 $fd ::= \text{if } nd \text{ then } pd \text{ else } pd \mid \text{succ}(bd) \mid \text{pred}(bd) \mid \text{zero?}(bd)$   
 $\quad \mid \text{if } \lfloor ns \rfloor \text{ then } pd \text{ else } pd \mid \text{succ}(bd) \mid \text{pred}(\lfloor bs \rfloor) \mid \text{zero?}(\lfloor bs \rfloor)$   
 $err ::= \text{mismatch} \mid \text{underflow}$   
 $v ::= b \mid n$   
 $cs ::= ps \mid err$   
 $o ::= v \mid err$

<div style="border: 1px solid black; padding: 2px; display: inline-block; margin-bottom: 5px;"><math>\rightsquigarrow_s \subseteq \text{SREDEX} \times \text{SPGM}</math></div> <b>Static Notions Red.</b>	<div style="border: 1px solid black; padding: 2px; display: inline-block; margin-bottom: 5px;"><math>\rightsquigarrow_d \subseteq \text{DREDEX} \times \text{DPGM}</math></div> <b>Dynamic Notions of Red.</b>
<pre style="margin: 0;"> if true then ps<sub>2</sub> else ps<sub>3</sub> <math>\rightsquigarrow_s</math> ps<sub>2</sub> if false then ps<sub>2</sub> else ps<sub>3</sub> <math>\rightsquigarrow_s</math> ps<sub>3</sub>   pred(ns) <math>\rightsquigarrow_s</math> n - 1   if n &gt; 0   succ(ns) <math>\rightsquigarrow_s</math> n + 1   zero?(0) <math>\rightsquigarrow_s</math> true   zero?(ns) <math>\rightsquigarrow_s</math> false   if n &gt; 0 if [true] then ps<sub>2</sub> else ps<sub>3</sub> <math>\rightsquigarrow_s</math> ps<sub>2</sub> if [false] then ps<sub>2</sub> else ps<sub>3</sub> <math>\rightsquigarrow_s</math> ps<sub>3</sub>   pred([ns]) <math>\rightsquigarrow_s</math> n - 1   if n &gt; 0   succ([ns]) <math>\rightsquigarrow_s</math> n + 1   zero?([0]) <math>\rightsquigarrow_s</math> true   zero?([ns]) <math>\rightsquigarrow_s</math> false   if n &gt; 0   [RG :???</pre>	<pre style="margin: 0;"> if true then pd<sub>2</sub> else pd<sub>3</sub> <math>\rightsquigarrow_s</math> pd<sub>2</sub> if false then pd<sub>2</sub> else pd<sub>3</sub> <math>\rightsquigarrow_s</math> pd<sub>3</sub>   pred(nd) <math>\rightsquigarrow_s</math> n - 1   if n &gt; 0   succ(nd) <math>\rightsquigarrow_s</math> n + 1   zero?(0) <math>\rightsquigarrow_s</math> true   zero?(nd) <math>\rightsquigarrow_s</math> false   if n &gt; 0 if [true] then ps<sub>2</sub> else ps<sub>3</sub> <math>\rightsquigarrow_s</math> ps<sub>2</sub> if [false] then ps<sub>2</sub> else ps<sub>3</sub> <math>\rightsquigarrow_s</math> ps<sub>3</sub>   pred([ns]) <math>\rightsquigarrow_s</math> n - 1   if n &gt; 0   succ([nd]) <math>\rightsquigarrow_s</math> n + 1   zero?([0]) <math>\rightsquigarrow_s</math> true   zero?([ns]) <math>\rightsquigarrow_s</math> false   if n &gt; 0   [RG :???</pre>

$\longrightarrow \subseteq \text{PGM} \times \text{CONFIG}$

**Single-step Reduction**

$\frac{rs \rightsquigarrow ps}{\text{Es}[rs] \longrightarrow \text{Es}[ps]}$	$\frac{rd \rightsquigarrow pd}{\text{Ed}[rd] \longrightarrow \text{Ed}[pd]}$	$\frac{}{\text{Es}[fs] \longrightarrow \text{mismatch}}$	$\frac{}{\text{Ed}[fd] \longrightarrow \text{mismatch}}$
$\frac{}{\text{Es}[\text{pred}(0)] \longrightarrow \text{underflow}}$	$\frac{}{\text{Es}[\text{pred}([0])] \longrightarrow \text{underflow}}$	$\frac{}{\text{Ed}[\text{pred}(0)] \longrightarrow \text{underflow}}$	
$\frac{}{\text{Ed}[\text{pred}([0])] \longrightarrow \text{underflow}}$			

 $\models \dots \subseteq \text{TERM} \times \text{TYPE}$  **Semantic Typing [RG: Exercise! Here's an old one:]**

$\models t : \text{Bool}$  if and only if  $t \longrightarrow^* b$  or  $t \longrightarrow^* \text{underflow}$   
 $\models t : \text{Nat}$  if and only if  $t \longrightarrow^* n$  or  $t \longrightarrow^* \text{underflow}$

$eval_{MBA} : \text{PGM} \rightarrow \text{OBS}$

```

evalMBA(p) = b if p  $\longrightarrow^*$  bs
evalMBA(p) = b if p  $\longrightarrow^*$  [bd]
evalMBA(p) = n if p  $\longrightarrow^*$  ns
evalMBA(p) = n if p  $\longrightarrow^*$  [nd]
evalMBA(p) = mismatch if p  $\longrightarrow^*$  mismatch
evalMBA(p) = underflow if p  $\longrightarrow^*$  underflow

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### Safety

**Conjecture 6 (Progress).** For all  $ps \in \text{SPGM}$  one of the following is true:

1.  $ps \in \text{SVALUE}$ ;
2.  $ps \longrightarrow ps'$  for some  $ps' \in \text{SPGM}$ ;
3.  $ps \longrightarrow err$  for some  $err \in \text{ERROR}$ .

**Conjecture 7 (Preservation).** If  $\vdash p_1 : T$  and  $p_1 \longrightarrow p_2$  then  $\vdash p_2 : T$ .

**Conjecture 8 (Semantic Type Soundness).** If  $\vdash t : T$  then  $\models t : T$ .

### 3 mMBA: Minimal Mixed Boolean and Arithmetic Language

#### Program Static Syntax

$s \in \text{SBA}$ ,  $d \in \text{DBA}$ ,  $n, ns \in \mathbb{N}^*$ ,  $b, bs \in \mathbb{B}^*$ ,  $*$ : A bit sloppy perhaps  
 $ps \in \text{SPGM} = \{s \in \text{SBA} \mid \exists T \in \text{TYPE}. \vdash s : T\}$ ,  $pd \in \text{DPGM} = \{d \in \text{DBA} \mid \vdash d \checkmark\}$ ,  $p \in \text{PGM} = ps$   
 $s ::= \text{true} \mid \text{false} \mid \text{if } s \text{ then } s \text{ else } s$   
 $\quad \mid ns \mid \text{succ}(s) \mid \text{pred}(s) \mid \text{zero?}(s) \mid \lceil d \rceil$   
 $d ::= \lfloor s \rfloor$   
 $bs ::= \text{true} \mid \text{false}$   
 $b ::= bs \mid bd$

$\vdash \cdot : \cdot \subseteq \text{SBA} \times \text{TYPE}$ $\vdash \cdot \checkmark \subseteq \text{DBA}$	<b>Syntactic Typing</b>
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$\frac{}{\vdash \text{true} : \text{Bool}}$	$\frac{}{\vdash \text{false} : \text{Bool}}$	$\frac{\vdash s_1 : \text{Bool} \quad \vdash s_2 : T \quad \vdash s_3 : T}{\vdash \text{if } s_1 \text{ then } s_2 \text{ else } s_3 : T}$	$\frac{}{\vdash ns : \text{Nat}}$	$\frac{\vdash s : \text{Nat}}{\vdash \text{succ}(s) : \text{Nat}}$
	$\frac{\vdash s : \text{Nat}}{\vdash \text{pred}(s) : \text{Nat}}$	$\frac{\vdash s : \text{Nat}}{\vdash \text{zero?}(s) : \text{Bool}}$	$\frac{\vdash d \checkmark}{\vdash \lceil d \rceil : T}$	
		$\frac{\vdash s : T}{\vdash \lfloor s \rfloor \checkmark}$		

#### Program Runtime Syntax

$Es \in \text{SECTXT}$ ,  $Ed \in \text{DECTXT}$ ,  $vs \in \text{SVALUE}$ ,  $vd \in \text{DVALUE}$ ,  
 $rs \in \text{SREDEX}$ ,  $rd \in \text{DREDEX}$ ,  $fs \in \text{SFAULTY}$ ,  $fd \in \text{DFAULTY}$ ,  $cs \in \text{SCONFIG}$ ,  $ed \in \text{DCONFIG}$ ,  
 $v \in \text{VALUE}$ ,  $err \in \text{ERROR}$ ,  $o \in \text{OBS}$   
 $Es ::= \square \mid Es[\text{if } \square \text{ then } ps \text{ else } ps] \mid Es[\text{succ}(\square) \mid Es[\text{pred}(\square)] \mid Es[\text{zero?}(\square)]] \mid Ed[\lfloor \square \rfloor]$   
 $Ed ::= Es[\lfloor \square \rfloor]$   
 $vs ::= bs \mid ns$   
 $vd ::= \lfloor bs \rfloor \mid \lfloor ns \rfloor$   
 $rs ::= \text{if } vs \text{ then } ps \text{ else } ps \mid \text{succ}(vs) \mid \text{pred}(vs) \mid \text{zero?}(vs)$   
 $rd ::= \text{if } vd \text{ then } pd \text{ else } pd \mid \text{succ}(vd) \mid \text{pred}(vd) \mid \text{zero?}(vd)$   
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$\longrightarrow \subseteq \text{PGM} \times \text{CONFIG}$

**Single-step Reduction**

$\frac{rs \rightsquigarrow ps}{\text{Es}[rs] \longrightarrow \text{Es}[ps]}$	$\frac{rd \rightsquigarrow pd}{\text{Ed}[rd] \longrightarrow \text{Ed}[pd]}$	$\frac{}{\text{Es}[fs] \longrightarrow \text{mismatch}}$	$\frac{}{\text{Ed}[fd] \longrightarrow \text{mismatch}}$
$\frac{}{\text{Es}[\text{pred}(0)] \longrightarrow \text{underflow}}$	$\frac{}{\text{Es}[\text{pred}([0])] \longrightarrow \text{underflow}}$	$\frac{}{\text{Ed}[\text{pred}(0)] \longrightarrow \text{underflow}}$	
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 $\models \dots \subseteq \text{TERM} \times \text{TYPE}$  **Semantic Typing [RG: Exercise! Here's an old one:]**

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$eval_{MBA} : \text{PGM} \rightarrow \text{OBS}$

```

evalMBA(p) = b if p  $\longrightarrow^*$  bs
evalMBA(p) = b if p  $\longrightarrow^*$  [bd]
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```

### Safety

**Conjecture 9 (Progress).** For all  $ps \in \text{SPGM}$  one of the following is true:

1.  $ps \in \text{SVALUE}$ ;
2.  $ps \longrightarrow ps'$  for some  $ps' \in \text{SPGM}$ ;
3.  $ps \longrightarrow err$  for some  $err \in \text{ERROR}$ .

**Conjecture 10 (Preservation).** If  $\vdash p_1 : T$  and  $p_1 \longrightarrow p_2$  then  $\vdash p_2 : T$ .

**Conjecture 11 (Semantic Type Soundness).** If  $\vdash t : T$  then  $\models t : T$ .