Algebra of Programming

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cata / ana

1. folds and unfolds
2. paramorphisms
   histomorphisms
   hylomorphisms ...
3. metamorphisms
4. traversals

Zoom: 916 2794 1099, 326749

$odd : \mathbb{N} \to Bool$

$A \times B = \{(a,b) \mid a \in A, b \in B\}$ — explicit ; implicit :

a <u>product</u> of A and B is $(X, fst, \cancel{f}snd)$ where

X is a set

$fst : X \to A$, $snd : X \to B$

st. $\forall\, C, f : C \to A, g : C \to B$

$\exists!\, h : C \to X \bullet f = fst \circ h \wedge g = snd \circ h$

we write $(A \times B, fst, snd)$ for "the" product.

$h = f \vartriangle g \iff f = fst \circ h \wedge g = snd \circ h$

— X

$$\begin{tikzcd} & A \times B \arrow[dl, "fst"'] \arrow[dr, "snd"] & \\ A & C \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, dashed, "h"'] & B \end{tikzcd}$$

$\forall h, f, g$

$$h = f \vartriangle g \iff fst \circ h = f \wedge snd \circ h = g$$

$\forall h, f, g$ $fst \circ (f \vartriangle g) = f \wedge snd \circ (f \vartriangle g) = g$

$$h = (fst \circ h) \vartriangle (snd \circ h)$$

$\forall h, f, g$ $(f \vartriangle g) \circ h = (f \circ h) \vartriangle (g \circ h)$

$$fst \vartriangle snd = id$$

$\forall A, B, C, D,\ h : D \to C,$
$f : C \to A,\ g : C \to B$

$$\begin{tikzcd} A & & \\ & C \arrow[ul, "f"'] \arrow[dl, "g"] & D \arrow[l, "h"'] \\ B & & \end{tikzcd}$$

$$\frac{f : C \to A \quad g : C \to B}{f \vartriangle g : C \to A \times B}$$

coproduct/
sum

$$\begin{tikzcd}
 & A \times B \arrow[dl, "fst"'] \arrow[dr, "snd"] & \\
A & C \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, dashed, "h"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A+B \arrow[d, dashed, "h"] & \\
A \arrow[ur, "inl"] \arrow[r, "f"'] & C & B \arrow[ul, "inr"'] \arrow[l, "g"]
\end{tikzcd}$$

$A+B = \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}$

$inl : A \to A+B$
$inl\ a = (0,a)$
$inr : B \to A+B$
$inr\ b = (1,b)$

$\forall f\, g \cdot \exists!\, h \cdot$ ~~

$h = f \triangledown g \iff h \circ inl = f \wedge h \circ inr = g$

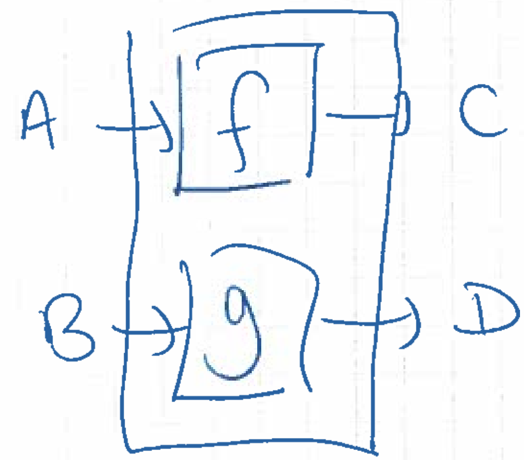
functor

$A \times B$

$f \times g$

$$\frac{f : A \to C \quad g : B \to D}{f \times g : A \times B \to C \times D}$$

$f \times g = (f \circ fst) \vartriangle (g \circ snd)$



polynomial functor

$\times$, $+$, 1,
constants

$() : ()$
$* : 1$

$$N(X) = 1 + X$$
$$L(X) = 1 + \mathbb{N} \times X$$
$$T_A(X) = A + X \times X$$

functor F:

$$\frac{f : A \to B}{Ff : FA \to FB}$$

have least fixed points

$$\mu F \cong F(\mu F)$$
$$in : F(\mu F) \to \mu F$$

$(\mu F, in : F(\mu F) \to \mu F)$ s.t.

$\forall A, f : FA \to A.$

$\exists! h : \mu F \to A.$

$h \circ in = f \circ Fh$

$$\begin{tikzcd} F(\mu F) \arrow[r, "Fh"] \arrow[d, "in"'] & F(A) \arrow[d, "f"] \\ \mu F \arrow[r, dashed, "h"'] & A \end{tikzcd}$$

for example

$LX = 1 + \mathbb{N} \times X$

$add : L\mathbb{N} \to \mathbb{N}$

$add : 1 + \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$add\ (inl\ *) = 0$

$add\ (inr\ (x, y)) = x + y$

$\mu L = List\ \mathbb{N}$

$h = cata\ f$

$\Leftrightarrow$

$h \circ in = f \circ Fh$

in Haskell – algebraic datatype

data L x = Nil | Cons ~~Nat~~ Int x

add :: L Int → Int
add Nil = 0
add (Cons x y) = x + y

~~data ListInt = In (L ListInt)~~ better: generic

data Mu f = In (f (Mu f)) datatype-generic

cata :: Functor f ⇒ (f a → a) → Mu f → a
cata f (In x) = f (fmap (cata f) x)