OREGON PROGRAMMING LANGUAGES SUMMER SCHOOL

Modal Logic: Proof Theory, Semantics and Applications

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1 Modal Logic Basics

1.1 Modal Logic K

All normal modal logics are constructed from the basic modal logic **K**, located in the bottom-left corner of figure 1. The logic **K** extends *classical* propositional logic with $\Box$ ('necessarily') and $\Diamond$ ('possibly'):

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A.$$

The two modalities are dual by De Morgan's laws: $\neg\Box A = \Diamond\neg A$, which means that one can be defined by the other, and hence it is sufficient to only consider $\Box$ in the system.

In Hilbert systems, in addition to all the axioms and rules from classical propositional logic, **K** includes the distribution axiom and the necessitation rule:

$$k : \Box(A \to B) \to (\Box A \to \Box B) \qquad\qquad \frac{A}{\Box A}.$$

1.1.1 Semantics: Relational (Kripke) Models, Revisited

A Kripke model is a triple $\mathcal{M} = \langle W, R, V \rangle$ where $W \neq \emptyset, R \subseteq W \times W$ and $V : At \to \mathcal{P}(W)$. Kripke models serve as relational models for the modal logics we describe here.

Kripke models can also serve as models for intuitionistic propositional logic. In this case, we constrain the accessibility relation to be a partial order $\leq$ and require the truth of propositional variables to be monotone with respect to $\leq$.

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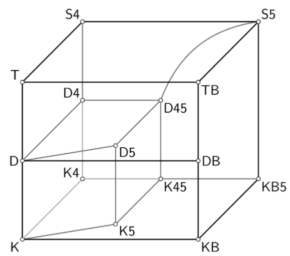


Figure 1: Relationships between modal logics

$\mathcal{M}, w \models A$ **iff**	**Intuitionistic**	**Modal (K)**
$\mathcal{M}, w \models p$	$w \in V(p)$	
$\mathcal{M}, w \models \bot$	never holds	
$\mathcal{M}, w \models \neg A$	$\forall v \geq w.\ \mathcal{M}, v \not\models A$	$\mathcal{M}, w \not\models A$
$\mathcal{M}, w \models A \wedge B$	$\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$	
$\mathcal{M}, w \models A \vee B$	$\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$	
$\mathcal{M}, w \models A \to B$	$\forall v \geq w.\ \mathcal{M}, v \models A \Rightarrow \mathcal{M}, v \models B$	$\mathcal{M}, w \not\models A$ or $\mathcal{M}, w \models B$
$\mathcal{M}, w \models \Box A$		$\forall v.\ wRv \Rightarrow \mathcal{M}, v \models A$
$\mathcal{M}, w \models \Diamond A$		$\exists v.\ wRv \wedge \mathcal{M}, v \models A$

Table 1: Satisfiability relation with Kripke models for intuitionistic and modal (**K**) logics

Definition 1.1. *A modal formula A is* ***satisfiable*** *if there exists a model $\mathcal{M}$ and $w \in W$ such that $\mathcal{M}, w \vDash A$.*

Definition 1.2 ($\vDash A$)**.** *A modal formula A is* ***valid*** *if for every model $\mathcal{M}$ and $w \in W$, $\mathcal{M}, w \vDash A$.*

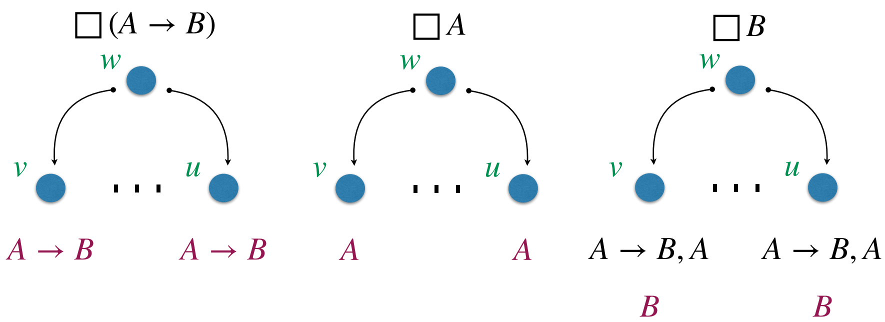
Definition 1.3 ($\Gamma \vDash \Delta$)**.** *The argument from a set of formulas Γ to a set of formulas Δ is* ***valid*** *if, for every model $\mathcal{M}, w \vDash B$ for each $B \in \Gamma$, then $\mathcal{M}, w \vDash A$ for some $A \in \Delta$.*

Beyond **K**, Table 2 shows a few common axioms that can be added to **K**, forming stronger modal logics. We will now analyze the behavior of some of these axioms for arbitrary models.

Example 1.1. $\mathsf{k} : \Box(A \to B) \to (\Box A \to \Box B)$

We will show that k is valid. Consider an arbitrary model $\mathcal{M}$ and world w in $\mathcal{M}$. We need to show that if $\mathcal{M}, w \models \Box(A \to B)$ and $\mathcal{M}, w \models \Box A$ then $\mathcal{M}, w \models \Box B$.

$\Box(A \to B)$ and $\Box A$ being true in w means in all accessible worlds $A \to B$ and A are true. But this means, in these accessible worlds, B is also true, and therefore $\Box B$ in w.



Example 1.2. T : $\Box A \to A$

Consider $\mathcal{M} = \langle \{w\}, \emptyset, \{p \mapsto \emptyset\} \rangle$. $\Box p$ is vacuously true at w, but p is not true at w.

We will use the Modal Logic Playground for visualization. Figure 2 shows the generated counter-model.

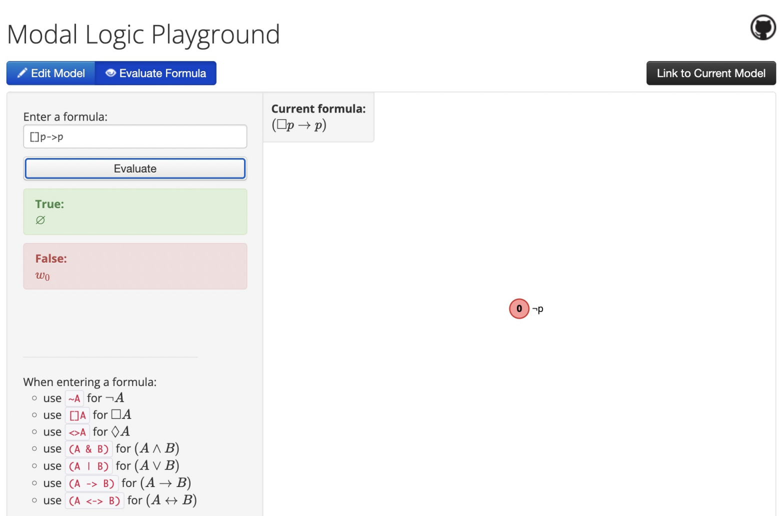


Figure 2: Counter-model

Example 1.3. 4 : $\Box A \to \Box\Box A$

Consider $\mathcal{M} = \langle \{w_0, w_1, w_2\}, \{(w_0, w_1), (w_1, w_1), (w_1, w_2)\}, \{p \mapsto \{w_1\}\} \rangle$. $\Box p$ is true at w_0, since p is true at w_1. However, $\Box\Box p$ is not true at w_0, since $\Box p$ is not true at w_1. Both w_1 and w_2 are accessible from w_1 and p is not true at w_2.

In Modal Logic Playground, this looks like Figure 3.

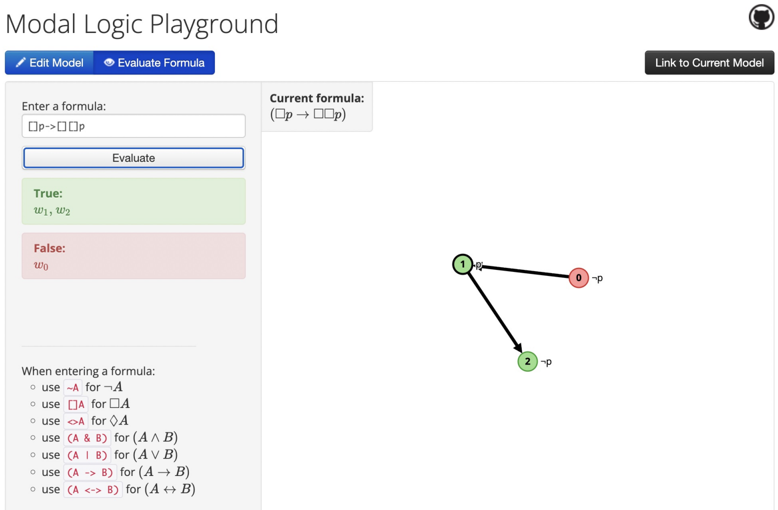


Figure 3: Counter-model

It turns out that these counter-models are not accidental. The axioms from Table 2 actually restrict the accessibility relation in predictable ways. We prove the correspondences in §3.

2 From Semantics to Syntax: Proof Search *vs* Counter-Model

2.1 Classical Propositional Logic

G3cp. This sequent calculus (see page 12, [3]) was developed by Gentzen for **classical** propositional logic and has some interesting meta-theoretical properties, such as having all its rules invertible. This means that we may apply any rule in any order, and if there is one leaf that is not an instance of the initial axiom, then the sequent is not provable. Thus, the system gives us counter-models for free.

Example 2.1. *If we try to prove $p \to q, q \vdash p$, which is not a classical propositional axiom, we can build the following partial derivation:*

$$\dfrac{q \vdash p, p \qquad q, q \vdash p}{p \to q, q \vdash p}\; L\to$$

Notice that there is no inference rule that we can apply to prove either $q \vdash p, p$ or $q, q \vdash p$, so we can conclude that the initial sequent is not provable. Besides that, it is immediate that there is valuation v such that $v(q) = T$ and $v(p) = F$, and $p \to q, q \not\models p$.

2.2 Intuitionistic Propositional Logic

G3ip. This sequent calculus (see page 13, [3]) was developed by Gentzen for **intuitionistic** propositional logic, but unlike the analogous system for classical propositional logic, it has some subtleties: the right introduction rules for disjunction and the left introduction rule for implication are not invertible. Also, since we are working in an intuitionistic framework, we don't have a direct way to extract counter-models from a proof search that failed.

mLJ. This sequent calculus (see page 13, [3]) was also developed for **intuitionistic** propositional logic. The system handles sequents $\Gamma \vdash \Delta$, with multiple formulas on the right-hand side, and the right introduction rule for implication is not invertible. Also, we don't have a direct way to extract counter-models from a proof search that failed.

2.2.1 Relational (Kripke) Model

As mentioned in subsection 1.1.1, we define the Kripke model for intuitionistic logic as a triple $\mathcal{M} = \langle W, \leq, V \rangle$, with the accessibility relation restricted to be a partial order $\leq$. The satisfiability relation $\mathcal{M}, w \vDash A$ is defined in Table 1.

Theorem 2.1 (**Monotonicity**). *If $\mathcal{M}, w \vDash A$ and $w \leq v$, then $\mathcal{M}, v \vDash A$.*

Theorem 2.2. *If $A \to B$ is a tautology, then the following rules are admissible in G3cp and G3ip:*

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta}$$

This theorem is useful in the process of transforming semantics into syntax.

Example 2.2. *Recall the satisfiability rule:*

$$\mathcal{M}, w \vDash A \wedge B \qquad \textit{iff} \qquad \mathcal{M}, w \vDash A \textit{ and } \mathcal{M}, w \vDash B.$$

We can discharge the information about the model and have the following:

$$w : A \wedge B \qquad \leftrightarrow \qquad w : A \wedge w : B.$$

We split the equivalence in two implications. For the $\leftarrow$, we have:

$$w : A \wedge B \qquad \leftarrow \qquad w : A \wedge w : B.$$

By the first admissible rule of Theorem 2.2,

$$\frac{\Gamma \vdash w : A \wedge w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta}\ .$$

Finally, we split the premise in two different premises and recover the right introduction rule for conjunction.

$$\frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta}.$$

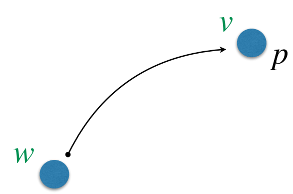
The inverse implication is analogous.

This process allows us to build a deductive system equivalent to G3ip and mLJ, which can provide a counter-model when the proof search does not succeed.

Example 2.3. *If we try to prove the law of excluded middle, $p \vee \neg p$, which is obviously not an intuitionistic theorem, we can build the following partial derivation:*

$$\cfrac{\cfrac{wRv, v : p \vdash w : p}{\vdash w : p, w : \neg p}\ \neg R}{\vdash w : p \vee \neg p}\ \vee R$$

Since $v \neq w$, there is no inference rule we can apply to prove $wRv, v : p \vdash w : p$. Thus, the sequent is not provable. A counter-model would be:



2.3 Classical Modal Logic (K)

In subsection 1.1.1, we defined the Kripke model for **classical modal logic (K)** as a triple $\mathcal{M} = \langle W, R, V \rangle$, where $W \neq \emptyset$, $R \subseteq W \times W$ and $V : At \to \mathcal{P}(W)$. The satisfiability relation $\mathcal{M}, w \vDash A$ is defined in Table 1.

To obtain the deductive system for classical modal logic, we just add the following two inference rules to G3cp:

$$\frac{\Gamma, wRv \vdash v : A, \Delta}{\Gamma \vdash w : \Box A, \Delta}\ \Box R \qquad \frac{\Gamma, wRv, v : A \vdash \Delta}{\Gamma, wRv, w : \Box A \vdash \Delta}\ \Box L.$$

Theorem 2.3. *The following inference rule is admissible in **K**:*

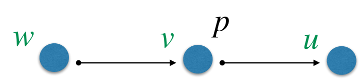
$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A, \Delta}\ k.$$

Given these, we can now look for the proofs of valid sequents or, otherwise, find counter-models.

Example 2.4. *If we try to prove $\vdash w : \Box p \to \Box\Box p$, we can build the following partial derivation:*

$$\dfrac{\dfrac{\dfrac{wRv, vRu, v : p \vdash u : p}{wRv, vRu, w : \Box p \vdash u : p}\ \Box L}{w : \Box p \vdash w : \Box\Box p}\ \Box R}{\vdash w : \Box p \to \Box\Box p}\ \to R$$

Since $v \neq u$, there is no inference rule we can apply to prove $wRv, vRu, v : p \vdash u : p$. Thus, the sequent is not provable. A counter-model would be:



3 Extending K via the ✨Beautiful✨ Correspondence Theory

Modal logic **K** is the weakest modal logic with a single axiom K on top of classical propositional logic. Stronger modal logic can arise by adding further axioms. Beautifully, common properties of interest on the accessibility relation can be precisely captured by an axiom about modalities, as shown in Table 2.

Principles about modalities	**Properties on the accessibility relation**
$T : \Box A \to A$	$\forall w.\, wRw$ (Reflexivity)
$4 : \Box A \to \Box\Box A$	$\forall w, v, u.\, wRv \wedge vRu \to wRu$ (Transitivity)
$D : \Box A \to \Diamond A$	$\forall w. \exists v.\, wRv$ (Seriality)
$B : A \to \Box\Diamond A$	$\forall w, v.\, wRv \to vRw$ (Symmetry)
$5 : \Diamond A \to \Box\Diamond A$	$\forall w, v, u.\, wRv \wedge wRu \to vRu$ (Euclideaness)

Table 2: Correspondence between modal axioms and frame conditions

Theorem 3.1. *Given a model $\mathcal{M} = \langle W, R, V\rangle$, the axiom*

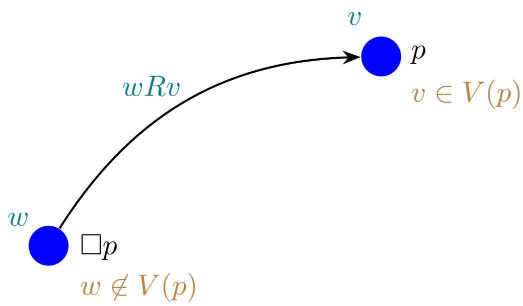
$$T : \Box A \to A$$

is valid if and only if R is reflexive.

Proof. ($\Leftarrow$) Suppose R is reflexive and $\mathcal{M}, w \vDash \Box A$. Since wRw, then it must be the case that $\mathcal{M}, w \vDash A$.

($\Rightarrow$) Suppose R is not reflexive for contraposition. To show T is not valid, we can construct a model as counterexample with such R, as shown in the graph below. Every world v with wRv satisfies $\mathcal{M}, v \vDash p$, so $\mathcal{M}, w \vDash \Box p$. However, it's clear by construction that $\mathcal{M}, w \nvDash p$. Hence,

$$\mathcal{M}, w \nvDash \Box p \to p.$$



$\square$

Just as in section 2, axiom T can be internalized into the following rule, which extends the labeled system for **K** to one for **T**.

$$\frac{\Gamma, xRx \vdash \Delta}{\Gamma \vdash \Delta}\ \textsf{Ref}$$

With the rule, T becomes derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{wRw,\ w : A \vdash w : A}\ \textsf{init}}{wRw,\ w : \Box A \vdash w : A}\ \Box L}{wRw \vdash w : \Box A \to A}\ \to R}{\vdash w : \Box A \to A}\ \textsf{Ref}$$

Similarly, axiom 4 (transitivity) and axiom D (seriality) can be internalized into rules and combined with **K** freely. The cube in figure 1 shows the resulting modal logics. Some combinations do collapse, such as T implies D.

$$\frac{\Gamma, wRv, vRu, wRu \vdash \Delta}{\Gamma, wRv, vRu \vdash \Delta}\ \textsf{trans} \qquad \frac{\Gamma, wRv \vdash \Delta}{\Gamma \vdash \Delta}\ \textsf{Ser}(v\ \textsf{fresh})$$

4 Sequent Calculi for Modal Logics

We can represent the rules about $\Box$ as inference rules.

💖 $\mathsf{k} : \Box(A \to B) \to \Box A \to \Box B$

$$\frac{\Gamma \vdash A}{\Gamma', \Box\Gamma \vdash \Box A, \Delta}\ \textsf{k}$$

- 💖 T : $\Box A \to A$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\ \mathsf{t}$$

- 💖 4 : $\Box A \to \Box\Box A$

$$\frac{\Box\Gamma \vdash A}{\Gamma', \Box\Gamma \vdash \Box A, \Delta}\ 4$$

- 💖 D : $\Box A \to \neg\Box\neg A$

$$\frac{\Gamma \vdash \bot}{\Gamma', \Box\Gamma \vdash \Delta}\ \mathsf{d}$$

4.1 The boundary of sequent calculi

Consider, however, S5 from Figure 1. We get S5 by adding the axioms T, 5, and 4. This corresponds to adding the following inference rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\ \mathsf{T} \qquad \frac{\Box\Gamma \vdash A, \Box\Delta}{\Gamma', \Box\Gamma \vdash \Box A, \Box\Delta, \Delta'}\ 4\ 5$$

This is sound and complete. However, it is not cut-free. For example, the sequent $p \vdash \Box\Diamond p$ is not cut-free derivable in S5. This can be solved through nested systems, hypersequents, and labeled systems, but this is non-trivial.

5 Bonus: Why Intuitionism

5.1 Direct proof vs. Contraposition vs. Contradiction

Going classical is not always necessary. The three ways to prove $p \to q$ differ on the kind of knowledge revealed. A *direct* proof of $p \to q$ assumes p and deduces q through some intermediate conclusions, exposing knowledge foreseen from the p worlds. A proof $\neg q \to \neg p$ by *contraposition* does the same for the worlds where q fails. A proof of $p \wedge \neg q \to \bot$ by contradiction explores starting from worlds containing hypotheses that hold nowhere, so the proof provides knowledge about a nonexistent, contradictory land. [2]

5.2 Brouwer on the Law of Excluded Middle

Brouwer [1] rejected the law of excluded middle, arguing that it is not constructively valid for the reals. Using the digits of π, he constructed a real number that cannot be shown to be negative, zero, or positive, since doing so would require deciding an undecidable property of the expansion.

References

[1] L.E.J. Brouwer. Über die bedeutung des satzes vom ausgeschlossenen dritten in der mathematik, insbesondere in der funktionentheorie. *Journal für die reine und angewandte Mathematik*, 1925(154):1–7, 1925.

[2] Terry Tao (https://mathoverflow.net/users/766/terry tao). Reductio ad absurdum or the contrapositive? MathOverflow. URL:https://mathoverflow.net/q/12427 (version: 2021-03-18).

[3] Elaine Pimentel. Modal logic - part ii. OPLSS 2026. https://www.cs.uoregon.edu/research/summerschool/summer26/_lectures/Pimentel_Lecture2.pdf.