Modal Logic
Part III – Games!

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Recap of Lecture II

- Kripke semantics.
- Proof theory for modal logics.
- Extensions: the beautiful modal cube!

In this lecture: games!

- Valitity games
- Provability games
- Modal games

Outline

Outline

Satisfiability games

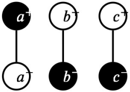
From satisfiability to validity

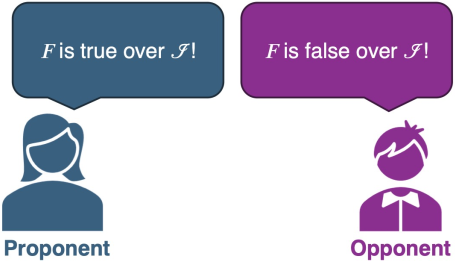
Modal games

Group polarisation and modalities

Hintikka's semantic games

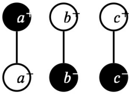
Formula $F = \left[a^- \vee \left(b^+ \wedge c^-\right)\right] \vee \left[a^+ \wedge b^-\right]$
Model $\mathscr{I} = \left\{a^+, b^-, c^-\right\}$





Hintikka's semantic games

Formula $F = \left[a^- \lor \left(b^+ \land c^-\right)\right] \lor \left[a^+ \land b^-\right]$
Model $\mathcal{I} = \left\{a^+, b^-, c^-\right\}$



$\lor$

at $F_1 \lor F_2$,
P chooses
between
F_1 and F_2

$\land$

at $F_1 \land F_2$,
O chooses
between
F_1 and F_2

$a^\pm$

at $a^\pm$,
P wins iff
$\mathcal{I} \vDash a^\pm$

$a^\pm$

at $a^\pm$,
O wins iff
$\mathcal{I} \nvDash a^\pm$

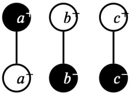
F is true over $\mathcal{I}$!

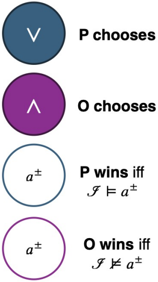
F is false over $\mathcal{I}$!

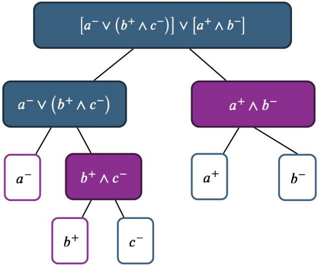
Hintikka's semantic games

Formula $F = \left[a^- \vee \left(b^+ \wedge c^-\right)\right] \vee \left[a^+ \wedge b^-\right]$

Model $\mathscr{I} = \left\{a^+, b^-, c^-\right\}$







Hintikka's semantic games

Formula $F = \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge b^-\right]$
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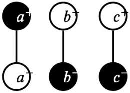
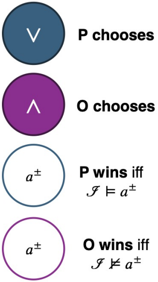
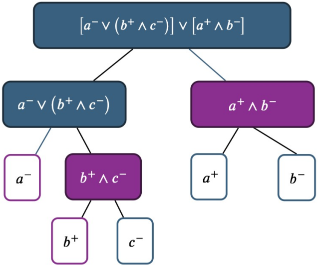




P has a winning strategy:



Hintikka's semantic games

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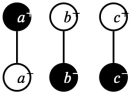
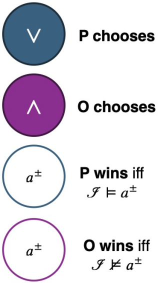
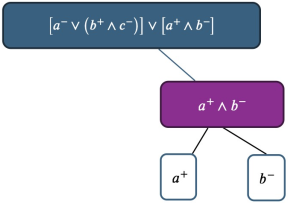




P has a winning strategy:



P has a winning strategy in $\mathrm{S}^{\mathrm{CL}}_{\mathscr{I}}(F)$ iff $\mathscr{I} \models F$.

Outline

From semantic games to provability games

Can **P** play the game such that she wins in every model?

The provability game

Formula $F = \left[a^- \vee \left(b^+ \wedge c^-\right)\right] \vee \left[a^+ \wedge \left(b^- \vee c^+\right)\right]$

$\vee$ **P chooses**

$\wedge$ **O chooses**

$a^\pm$ **P wins** in $a^\pm$ iff for every model $\mathscr{I}$, $\mathscr{I} \models a^\pm$

$a^\pm$ **O wins** otherwise

B **P can create backup copies:** at F, continue with $F \parallel F$

The provability game

Formula $F = \left[a^- \vee \left(b^+ \wedge c^-\right)\right] \vee \left[a^+ \wedge \left(b^- \vee c^+\right)\right]$

$\vee$ **P chooses**

$\wedge$ **O chooses**

Win **P wins** in $a_1^\pm \parallel \ldots \parallel a_n^\pm$ iff for every model $\mathscr{I}$, there is some i with $\mathscr{I} \vDash a_i^\pm$

Win **O wins** otherwise

B **P can create backup copies:** at $F_1 \parallel \ldots \parallel F_i \parallel \ldots \parallel F_n$, continue with $F_1 \parallel \ldots \parallel F_i \parallel F_i \parallel \ldots \parallel F_n$

S **P is the scheduler**

The provability game

$$[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

The provability game

$$\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

$$\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right] \parallel \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

The provability game

$$[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

$$\underline{\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]} \parallel \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

The provability game

$$[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

$$\underline{\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]} \parallel \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

$$a^- \vee (b^+ \wedge c^-) \parallel [a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

The provability game

$$[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

$$\underline{\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]} \parallel \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

$$a^- \vee (b^+ \wedge c^-) \parallel \underline{[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]}$$

The provability game

$$[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

$$\underline{\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]} \parallel \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

$$a^- \vee (b^+ \wedge c^-) \parallel \underline{[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]}$$

$$a^- \vee (b^+ \wedge c^-) \parallel \underline{a^+ \wedge (b^- \vee c^+)}$$

The provability game

$$[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

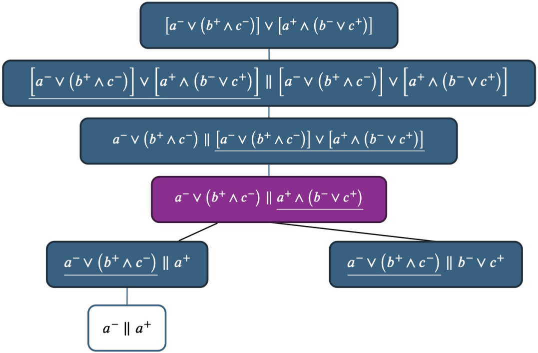
$$\underline{\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]} \parallel \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

$$a^- \vee (b^+ \wedge c^-) \parallel \underline{[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]}$$

$$a^- \vee (b^+ \wedge c^-) \parallel \underline{a^+ \wedge (b^- \vee c^+)}$$

$$\underline{a^- \vee (b^+ \wedge c^-)} \parallel a^+ \qquad \underline{a^- \vee (b^+ \wedge c^-)} \parallel b^- \vee c^+$$

The provability game



The provability game

$$[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$$

$$\underline{\left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]} \parallel \left[a^- \vee (b^+ \wedge c^-)\right] \vee \left[a^+ \wedge (b^- \vee c^+)\right]$$

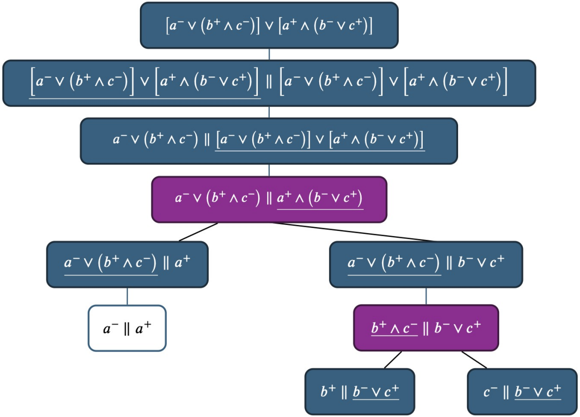
$$a^- \vee (b^+ \wedge c^-) \parallel \underline{[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]}$$

$$a^- \vee (b^+ \wedge c^-) \parallel \underline{a^+ \wedge (b^- \vee c^+)}$$

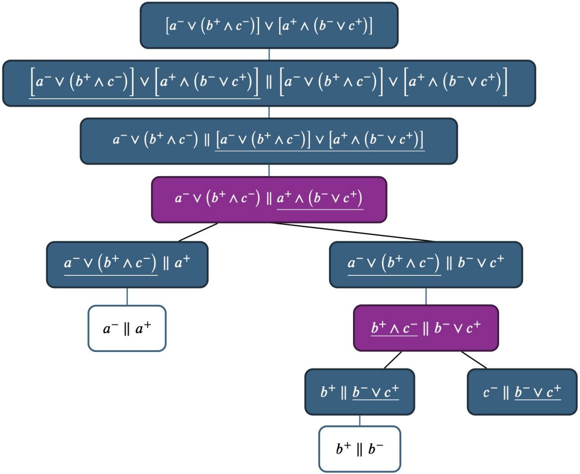
$$\underline{a^- \vee (b^+ \wedge c^-)} \parallel a^+ \qquad \underline{a^- \vee (b^+ \wedge c^-)} \parallel b^- \vee c^+$$

$$a^- \parallel a^+ \qquad \underline{b^+ \wedge c^-} \parallel b^- \vee c^+$$

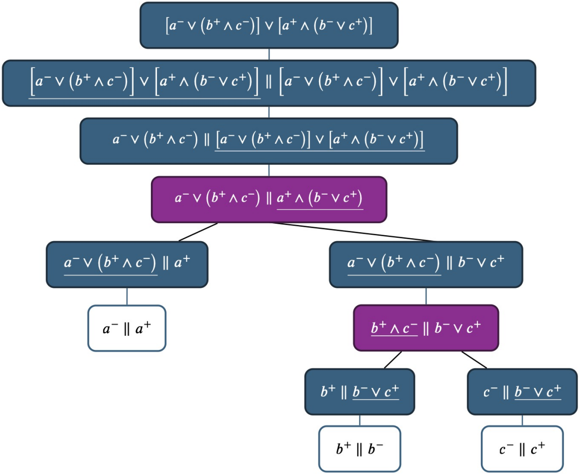
The provability game



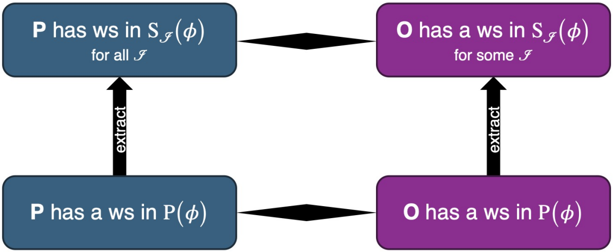
The provability game



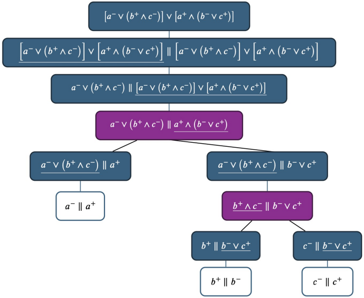
The provability game

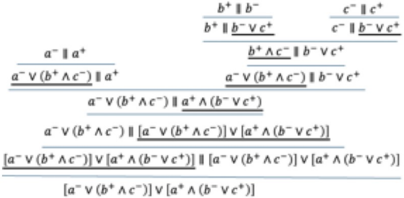


Adequacy


P has ws in $S_{\mathscr{I}}(\phi)$
for all $\mathscr{I}$
O has a ws in $S_{\mathscr{I}}(\phi)$
for some $\mathscr{I}$
extract
extract
P has a ws in $P(\phi)$
O has a ws in $P(\phi)$

From probability games to proofs





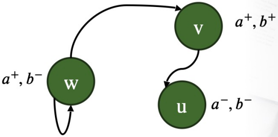
Proof theory!

$\vee$	$\wedge$	B	Win
at $F_1 \vee F_2$, **P chooses** between F_1 and F_2	at $F_1 \wedge F_2$, **O chooses** between F_1 and F_2	**P can create backup copies:** at $\mathscr{D} \parallel F$, continue with $\mathscr{D} \parallel F \parallel F$	**P wins** in $a_1^{\pm} \parallel \ldots \parallel a_n^{\pm} \parallel b^{+} \parallel b^{-}$
⬇	⬇	⬇	⬇
$\dfrac{\mathscr{D} \parallel F_1}{\mathscr{D} \parallel F_1 \vee F_2}(\vee)_1$ $\dfrac{\mathscr{D} \parallel F_2}{\mathscr{D} \parallel F_1 \vee F_2}(\vee)_2$	$\dfrac{\mathscr{D} \parallel F_1 \quad \mathscr{D} \parallel F_2}{\mathscr{D} \parallel F_1 \wedge F_2}(\wedge)$	$\dfrac{\mathscr{D} \parallel F \parallel F}{\mathscr{D} \parallel F}(c)$	$a_1^{\pm} \parallel \ldots \parallel a_n^{\pm} \parallel b^{+} \parallel b^{-}$ (Axiom)

There is a proof of F iff F is valid in propositional classical logic!

Outline

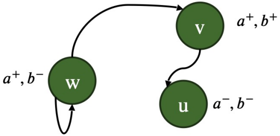
How about modalities?



$\Box F$ … F is…

… necessary
… known
… obligatory
…

Playing with modalities



Formula $F = \Diamond(a^+ \wedge b^-)$ + model + world

$\Diamond$ — at $\mathrm{w}\colon \Diamond F$,
P chooses an wRv
to continue with $\mathrm{v}\colon F$

$\Box$ — at $\mathrm{w}\colon \Box F$,
O chooses an wRv
to continue with $\mathrm{v}\colon F$

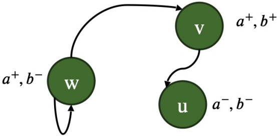
$\vee$ — **P chooses**

$\wedge$ — **O chooses**

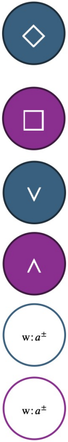
$\mathrm{w}\colon a^{\pm}$ — **P wins** iff
$\mathcal{M}, \mathrm{w} \vDash a^{\pm}$

$\mathrm{w}\colon a^{\pm}$ — **O wins** iff
$\mathcal{M}, \mathrm{w} \nvDash a^{\pm}$

Playing with modalities



Formula $F = \Diamond(a^+ \wedge b^-)$ + model + world



at $\mathrm{w}: \Diamond F$,
P chooses an wRv
to continue with $\mathrm{v}: F$

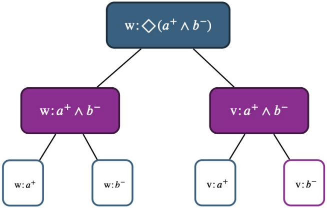
at $\mathrm{w}: \Box F$,
O chooses an wRv
to continue with $\mathrm{v}: F$

P chooses

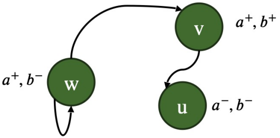
O chooses

P wins iff
$\mathscr{M}, \mathrm{w} \vDash a^{\pm}$

O wins iff
$\mathscr{M}, \mathrm{w} \nvDash a^{\pm}$



Playing with modalities



Formula $F = \Diamond(a^+ \wedge b^-)$ + model + world

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O chooses an wRv
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$\vee$ **P chooses**

$\wedge$ **O chooses**

$\mathrm{w}: a^{\pm}$ **P wins** iff
$\mathscr{M}, \mathrm{w} \vDash a^{\pm}$

$\mathrm{w}: a^{\pm}$ **O wins** iff
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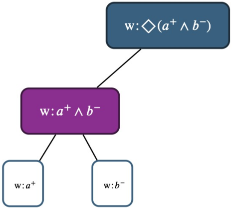
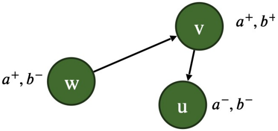
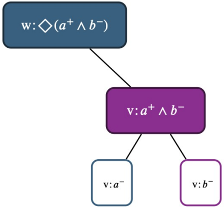


Playing with modalities



Formula $F = \Diamond(a^+ \wedge b^-)$ + model + world

- $\Diamond$: at $\mathrm{w}: \Diamond F$, **P chooses** an wRv to continue with $\mathrm{v}: F$
- $\Box$: at $\mathrm{w}: \Box F$, **O chooses** an wRv to continue with $\mathrm{v}: F$
- $\vee$: **P chooses**
- $\wedge$: **O chooses**
- $\mathrm{w}: a^{\pm}$: **P wins** iff $\mathcal{M}, \mathrm{w} \vDash a^{\pm}$
- $\mathrm{w}: a^{\pm}$: **O wins** iff $\mathcal{M}, \mathrm{w} \nvDash a^{\pm}$



Playing with modalities

$$\frac{\mathscr{D} \parallel i : F_1}{\mathscr{D} \parallel i : F_1 \vee F_2}(\vee)_1 \qquad \frac{\mathscr{D} \parallel i : F_1 \quad \mathscr{D} \parallel i : F_2}{\mathscr{D} \parallel i : F_1 \wedge F_2}(\wedge) \qquad \frac{\mathscr{D} \parallel i : F \parallel i : F}{\mathscr{D} \parallel i : F}(c)$$

$a_1^{\pm} \parallel \dots \parallel a_n^{\pm}$ (Axiom)
… winning*

$$\frac{\mathscr{D} \parallel i : F_2}{\mathscr{D} \parallel i : F_1 \vee F_2}(\vee)_2$$

$$\frac{\mathscr{D} \parallel j : R^{+}(i,j) \wedge G}{\mathscr{D} \parallel i : \Diamond G}(\Diamond) \qquad \frac{\mathscr{D} \parallel j : R^{-}(i,j) \vee G}{\mathscr{D} \parallel i : \Box G}(\Box)$$

… j is not in $\mathscr{D}$, G

*P-time checkable

$\Diamond$

at $i : \Diamond F$,
P chooses a j to continue with $j : R^{+}(i,j) \wedge F$

$\Box$

at $i : \Box F$,
O chooses a j to continue with $j : R^{-}(i,j) \vee F$

There is a proof of $i : F$ iff* F is valid in H.

*i does not appear in F

Outline

Social network logic [Pedersen'24]

Social network: a graph where

- nodes = agents or groups of agents;
- arrows = social relation (such as friendship) or a communication channel.

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Social network: a graph where

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- arrows = social relation (such as friendship) or a communication channel.

Social network logic: $\{$*social network analysis*$\} \cap \{$*formal logic*$\}$

- use logical methods in the explicit study of social networks;
- broaden our understanding of specific social phenomena.

Group polarisation

- Individuals in a group adopt more extreme positions after group discussions;
- balance: stable configurations of connections between agents;
- modal logic: models the evolution of opinions, preferences, or alliances.

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- modal logic: models the evolution of opinions, preferences, or alliances.

Positive-Negative Logic (PNL) [Xiong'17, Xiong,Ågotnes'19]:

- two distinct relations to model networks;
- agents can be related positively or negatively, but not both.

PNL Logic

Grammar:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid \qquad\qquad \mid \oplus F \mid \ominus F \mid$$

- $\ominus F$ holds for agent a whenever F holds for some a's enemy;
- $\oplus F$ holds for agent a whenever F holds for some a's friend;

PNL Logic [Pedersen,Smets,Ågotnes'21]

Grammar:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^{+}(i,j) \mid R^{-}(i,j) \mid \diamondplus F \mid \diamondminus F \mid [A]F$$

- $\diamondminus F$ holds for agent a whenever F holds for some a's enemy;
- $\diamondplus F$ holds for agent a whenever F holds for some a's friend;
- $[A]F$ holds whenever F holds for all agents.

PNL Logic [Pedersen,Smets,Ågotnes'21]

Grammar:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^+(i,j) \mid R^-(i,j) \mid \oplus F \mid \ominus F \mid [A]F$$

- $\ominus F$ holds for agent a whenever F holds for some a's enemy;
- $\oplus F$ holds for agent a whenever F holds for some a's friend;
- $[A]F$ holds whenever F holds for all agents.

Model $\mathcal{M} = \langle \mathrm{A}, \mathrm{R}^+, \mathrm{R}^-, \mathrm{V}, \mathrm{g} \rangle$:

PNL Logic [Pedersen,Smets,Ågotnes'21]

Grammar:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^+(i,j) \mid R^-(i,j) \mid \oplus F \mid \ominus F \mid [A]F$$

- $\ominus F$ holds for agent a whenever F holds for some a's enemy;
- $\oplus F$ holds for agent a whenever F holds for some a's friend;
- $[A]F$ holds whenever F holds for all agents.

Model $\mathcal{M} = \langle \mathrm{A}, \mathrm{R}^+, \mathrm{R}^-, \mathrm{V}, \mathrm{g} \rangle$:

- A is a set (of agents);
- $\mathrm{g} : N \to \mathrm{A}$ is the denotation function – gives names to agents;
- $\mathrm{R}^+, \mathrm{R}^- \subseteq \mathrm{A} \times \mathrm{A}$;
- $\mathrm{V} : \mathrm{At} \to \mathcal{P}(\mathrm{A})$.

PNL Logic [Pedersen,Smets,Ågotnes'21]

Grammar:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^+(i,j) \mid R^-(i,j) \mid \oplus F \mid \ominus F \mid [A]F$$

- $\ominus F$ holds for agent a whenever F holds for some a's enemy;
- $\oplus F$ holds for agent a whenever F holds for some a's friend;
- $[A]F$ holds whenever F holds for all agents.

Model $\mathcal{M} = \langle \mathrm{A}, \mathrm{R}^+, \mathrm{R}^-, \mathrm{V}, \mathrm{g} \rangle$:

- $(\mathrm{a}, \mathrm{b}) \in \mathrm{R}^+$: agent a agrees with (is a friend of) agent b;
- $(\mathrm{a}, \mathrm{b}) \in \mathrm{R}^-$: agent a disagrees with (is an enemy of) agent b.

PNL Logic [Pedersen,Smets,Ågotnes'21]

Grammar:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^+(i,j) \mid R^-(i,j) \mid \oplus F \mid \ominus F \mid [A]F$$

- $\ominus F$ holds for agent a whenever F holds for some a's enemy;
- $\oplus F$ holds for agent a whenever F holds for some a's friend;
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- g is surjective (every agent has a name);
- R^+ is reflexive;
- R^+ and R^- are symmetric;
- these relations do not overlap.

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- R^+ and R^- are symmetric; Optional axiom: collectively connectedness
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PNL Logic [Pedersen,Smets,Ågotnes'21]

Grammar:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^+(i,j) \mid R^-(i,j) \mid \diamond^+ F \mid \diamond^- F \mid [A]F$$

- $\diamond^- F$ holds for agent a whenever F holds for some a's enemy;
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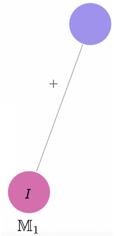
Let $\mathrm{a} = \mathrm{g}(i)$. Then:

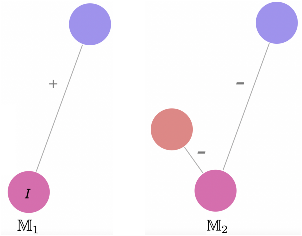
$$\begin{array}{ll} \mathcal{M}, \mathrm{a} \Vdash \diamond^{\pm} F & \text{iff there is } j \in N \text{ such that } \mathcal{M}, \mathrm{g}(j) \Vdash R^{\pm}(i,j) \text{ and } \mathcal{M}, \mathrm{g}(j) \Vdash F \\ \mathcal{M}, \mathrm{a} \Vdash [A]F & \text{iff } \mathcal{M}, \mathrm{g}(j) \Vdash F, \text{ for all } j \in N. \end{array}$$

Agreement and disagreement relations



- $\mathcal{M}_1$: I have a friend who does not believe in p: $\mathcal{M}_1, I \Vdash \oplus \neg p$;
-

Agreement and disagreement relations



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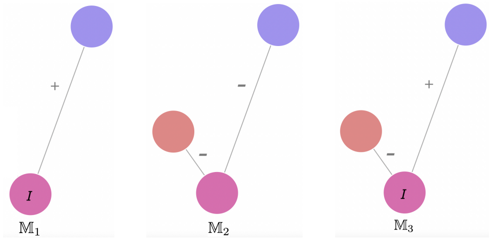

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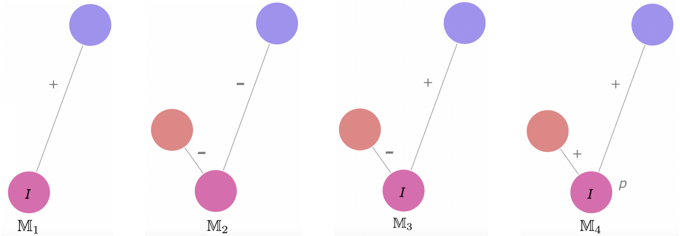
- $\mathcal{M}_3$: *I* have an enemy: $\mathcal{M}_3, I \Vdash \ominus \top$;

Agreement and disagreement relations



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- $\mathcal{M}_3$: I have an enemy: $\mathcal{M}_3, I \Vdash \ominus\, \top$;
- $\mathcal{M}_4$: Everybody has a friend where p: $\mathcal{M}_4, a \Vdash [A] \oplus p$ for any agent a.

The need for nominals

Extending Hintikka dialogue games to modal logic is straightforward:

- current roles of the players;
- current formula F;
- current world w in the model.

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- accessibility relation within the object language.

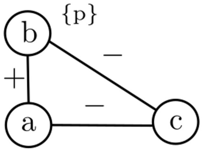
From now on:

- internalise the nominals: identify an agent a with its nominal i.

Semantic Game (truth in a model)[Freiman, Olarte, Pimentel, Fermüller'24]

Satisfiability

- played over a PNL model;



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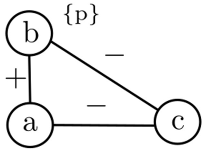

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  - I am the proponent $\mathrm{P}, i : F$
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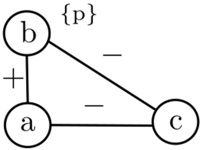

P: Proponent

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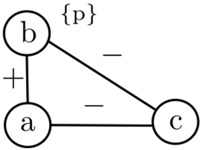
- My and Your choices and strategies.



P: Proponent   O: Opponent

Semantic Game

Conjunction $\mathrm{P}, i : F \wedge G$

Semantic Game

Conjunction $\mathrm{P}, i : F \wedge G$

$$
\begin{tikzcd}
 & \text{YOU} \arrow[dl] \arrow[dr] & \\
P, i : F & & P, i : G
\end{tikzcd}
$$

Semantic Game

Conjunction $\quad O, i : F \wedge G$

Semantic Game

Conjunction $\quad \mathrm{O}, i : F \wedge G$

$$
\begin{array}{c}
\text{ME} \\
\downarrow \\
O, i : F
\end{array}
$$

Semantic Game

Conjunction $\mathrm{O}, i : F \wedge G$

$$\begin{array}{c} \text{ME} \\ \big\downarrow \\ O, i : G \end{array}$$

Semantic Game

Diamond $\mathsf{P}, i : \Diamond^{\pm} F$

Semantic Game

 Diamond $\mathsf{P}, i :\Diamond^{\pm} F$

No successor $\rightsquigarrow$ You win

Semantic Game

Diamond $\quad \mathrm{P}, i : \Diamond^{\pm} F$

$$\mathrm{P}, i : \Diamond^{\pm} F \;\xrightarrow{\text{ME}}\; P, j : F$$

Semantic Game

Diamond $\quad O, i : \Diamond^{\pm} F$

Semantic Game

Diamond $\quad \mathrm{O}, i : \Diamond^{\pm} F$

No successor $\rightsquigarrow$ I win

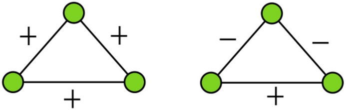
Semantic Game

Diamond $\quad O, i : \Diamond^{\pm} F$

YOU

$$P, j_1 : F \qquad \ldots \qquad P, j_n : F$$

An example: Balance (stable configurations)



$$((\Diamond_{+}\Diamond_{+}\, p \vee \Diamond_{-}\Diamond_{-}\, p) \rightarrow \Diamond_{+}\, p) \wedge$$
$$((\Diamond_{+}\Diamond_{-}\, p \vee \Diamond_{-}\Diamond_{+}\, p) \rightarrow \Diamond_{-}\, p) \qquad \textbf{(4B)}$$

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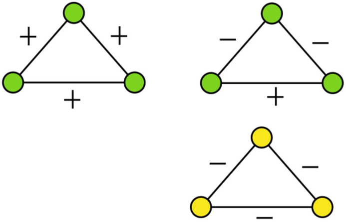
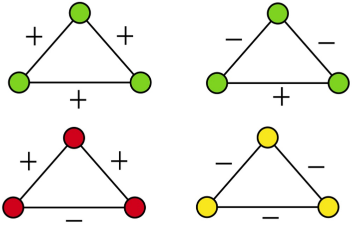


$$((\langle + \rangle\langle + \rangle\, p \vee \langle - \rangle\langle - \rangle\, p) \rightarrow \langle + \rangle\, p) \wedge$$
$$((\langle + \rangle\langle - \rangle\, p \vee \langle - \rangle\langle + \rangle\, p) \rightarrow \langle - \rangle\, p) \qquad \textbf{(4B)}$$

$$((\langle + \rangle\langle + \rangle\, p) \rightarrow \langle + \rangle\, p) \wedge$$
$$((\langle + \rangle\langle - \rangle\, p \vee \langle - \rangle\langle + \rangle\, p) \rightarrow \langle - \rangle\, p) \qquad \textbf{(4W)}$$

An example: Balance (stable configurations)

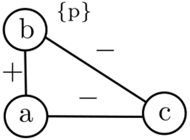


$$((\langle + \rangle\langle + \rangle\, p \vee \langle - \rangle\langle - \rangle\, p) \rightarrow \langle + \rangle\, p) \wedge$$
$$((\langle + \rangle\langle - \rangle\, p \vee \langle - \rangle\langle + \rangle\, p) \rightarrow \langle - \rangle\, p) \qquad \textbf{(4B)}$$

$$((\langle + \rangle\langle + \rangle\, p) \rightarrow \langle + \rangle\, p) \wedge$$
$$((\langle + \rangle\langle - \rangle\, p \vee \langle - \rangle\langle + \rangle\, p) \rightarrow \langle - \rangle\, p) \qquad \textbf{(4W)}$$

Dynamics: playing the game!

```
[✓,Y] : P @ a : (¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p) ∧ ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
├── [✓,I] : P @ a : ¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p
│   └── [✓,I] : P @ a : ⊕ p
│       └── [✓,I] : P @ b : p
└── [✓,I] : P @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
    └── [✓,I] : P @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p)
        └── [✓,Y] : O @ a : ⊕ ⊖ p ∨ ⊖ ⊕ p
            ├── [✓,Y] : O @ a : ⊕ ⊖ p
            │   ├── [✓,Y] : O @ a : ⊖ p
            │   │   └── [✓,Y] : O @ c : p
            │   └── [✓,Y] : O @ b : ⊖ p
            │       └── [✓,Y] : O @ c : p
            └── [✓,Y] : O @ a : ⊖ ⊕ p
                └── [✓,Y] : O @ c : ⊕ p
                    └── [✓,Y] : O @ c : p
```



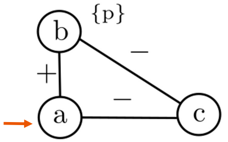
$$((\oplus\oplus\, p \vee \ominus\ominus\, p) \to \oplus\, p) \wedge ((\oplus\ominus\, p \vee \ominus\oplus\, p) \to \ominus\, p) \qquad \textbf{(4B)}$$

Dynamics: playing the game!

[✓,Y] : P @ a : $(\neg(\langle+\rangle\langle+\rangle p \vee \langle-\rangle\langle-\rangle p) \vee \langle+\rangle p) \wedge \neg(\langle+\rangle\langle-\rangle p \vee \langle-\rangle\langle+\rangle p) \vee \langle-\rangle p$

├── [✓,I] : P @ a : $\neg(\langle+\rangle\langle+\rangle p \vee \langle-\rangle\langle-\rangle p) \vee \langle+\rangle p$

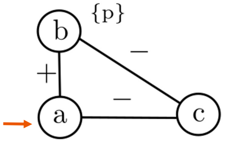
└── [✓,I] : P @ a : $\neg(\langle+\rangle\langle-\rangle p \vee \langle-\rangle\langle+\rangle p) \vee \langle-\rangle p$



$$((\langle+\rangle\langle+\rangle p \vee \langle-\rangle\langle-\rangle p) \to \langle+\rangle p) \wedge ((\langle+\rangle\langle-\rangle p \vee \langle-\rangle\langle+\rangle p) \to \langle-\rangle p) \quad \textbf{(4B)}$$

Dynamics: playing the game!

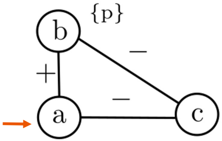
— [✓,I] : P @ a : $\neg\, (\Diamond^{+} \Diamond^{-}\, p \lor \Diamond^{-} \Diamond^{+}\, p) \lor \Diamond^{-}\, p$
└── [✓,I] : P @ a : $\neg\, (\Diamond^{+} \Diamond^{-}\, p \lor \Diamond^{-} \Diamond^{+}\, p)$



$$((\Diamond^{+}\Diamond^{+}\, p \lor \Diamond^{-}\Diamond^{-}\, p) \to \Diamond^{+}\, p) \land$$
$$((\Diamond^{+}\Diamond^{-}\, p \lor \Diamond^{-}\Diamond^{+}\, p) \to \Diamond^{-}\, p) \qquad \textbf{(4B)}$$

Dynamics: playing the game!

$$\text{└── [✓,I] : P @ a : } \neg(\Diamond^{+}\Diamond^{-} p \vee \Diamond^{-}\Diamond^{+} p)$$
$$\quad\text{└── [✓,Y] : O @ a : } \Diamond^{+}\Diamond^{-} p \vee \Diamond^{-}\Diamond^{+} p$$



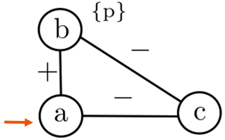
$$((\Diamond^{+}\Diamond^{+} p \vee \Diamond^{-}\Diamond^{-} p) \to \Diamond^{+} p) \wedge$$
$$((\Diamond^{+}\Diamond^{-} p \vee \Diamond^{-}\Diamond^{+} p) \to \Diamond^{-} p) \qquad \textbf{(4B)}$$

Dynamics: playing the game!

└── [✓,Y] : 0 @ a : $\Diamond^+ \Diamond^- p \vee \Diamond^- \Diamond^+ p$

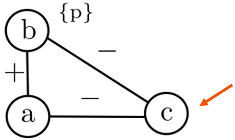
  ├── [✓,Y] : 0 @ a : $\Diamond^+ \Diamond^- p$

  └── [✓,Y] : 0 @ a : $\Diamond^- \Diamond^+ p$



$$((\Diamond^+\Diamond^+ p \vee \Diamond^-\Diamond^- p) \rightarrow \Diamond^+ p) \wedge$$
$$'(\Diamond^+\Diamond^- p \vee \Diamond^-\Diamond^+ p) \rightarrow \Diamond^- p) \qquad \textbf{(4B)}$$

Dynamics: playing the game!




```
└── [✓,Y] : 0 @ a : ◈− ◈+ p
    └── [✓,Y] : 0 @ c : ◈+ p
```

$$((\Diamond^{+}\Diamond^{+}\, p \vee \Diamond^{-}\Diamond^{-}\, p) \rightarrow \Diamond^{+}\, p) \wedge$$
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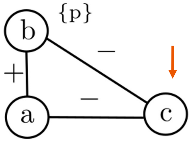
Dynamics: playing the game!

I win!




```
└── [✓,Y] : 0 @ c : ⟨+⟩ p
    └── [✓,Y] : 0 @ c : p
```

$$((\langle + \rangle\langle + \rangle\, p \vee \langle - \rangle\langle - \rangle\, p) \rightarrow \langle + \rangle\, p) \wedge$$
$$((\langle + \rangle\langle - \rangle\, p \vee \langle - \rangle\langle + \rangle\, p) \rightarrow \langle - \rangle\, p) \qquad \textbf{(4B)}$$

Semantic Game $\mathrm{G}_{\mathcal{M}}(g)$

Strategy for Me $=$ subtree σ of the associated game tree such that:

1. $g \in \sigma$,
2. if $h \in \sigma$ is a node labelled Y, then all children of h are in σ,
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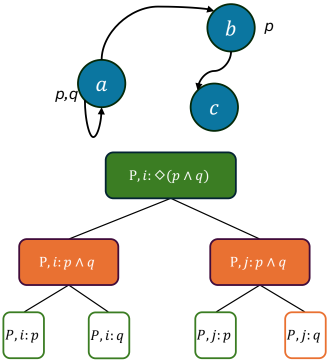

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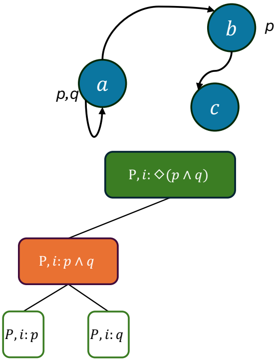

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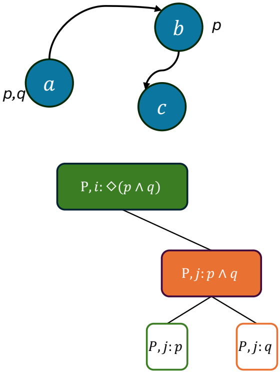

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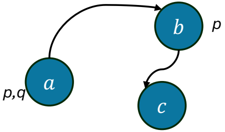
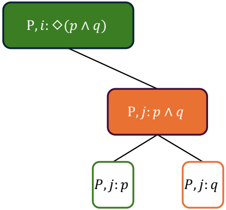
The strategy σ is called winning if all leaves in the tree σ are labelled I.





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Adequacy of semantic games:

- I have a winning strategy for $\mathsf{G}_{\mathcal{M}}(\mathsf{P}, i : F)$ iff $\mathcal{M}, \mathsf{a} \models F$.
- You have a winning strategy for $\mathsf{G}_{\mathcal{M}}(\mathsf{P}, i : F)$ iff $\mathcal{M}, \mathsf{a} \not\models F$.

What about validity?

There is no uniform strategy for winning $P, i : p \vee \neg p$.

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Validity/disjunctive game:

- Multiset of game states.
- The ability to duplicate states.
- The model is relevant 'only' in elementary states.

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$$
\begin{array}{c}
P, i : p \vee \neg p \\
\downarrow \\
\{P, i : p \vee \neg p \; ; P, i : p \vee \neg p\} \\
\downarrow \\
\{P, i : p \; ; P, i : \neg p\} \\
\downarrow \\
\{P, i : p \; ; O, i : p\}
\end{array}
$$

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Corollary. F is **PNL**-valid iff I have a winning strategy in $\mathrm{DG}(\mathrm{P}, i : [A]F)$.

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$$\mathsf{O}, i_1 : F_1; \ldots; \mathsf{O}, i_n : F_n; \quad \mathsf{P}, j_1 : G_1; \ldots; \mathsf{P}, j_m : G_m$$

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Important:

Formal relationship between elementary game states and logical axioms.

Some rules

$$\frac{\Gamma \longrightarrow R^{\pm}(i,j), \Delta \qquad \Gamma \longrightarrow j : F, \Delta}{\Gamma \longrightarrow i : \Diamond^{\pm} F, \Delta} \; (R_{\Diamond^{\pm}}) \qquad \frac{\Gamma, j : F \longrightarrow \Delta}{\Gamma, i : [A]F \longrightarrow \Delta} \; (L_{[A]})$$

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Collective connectedness:

$$(\boxplus p \rightarrow [A]p) \vee (\boxminus p \rightarrow [A]p)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\longrightarrow j : p, R^{+}(i,j), R^{-}(i,j)}\ cc \qquad \dfrac{}{\longrightarrow j : p, j : \neg p, R^{+}(i,j)}\ R_{\neg}, init}{\longrightarrow j : p, i : \diamondminus \neg p, R^{+}(i,j)}\ R_{\diamondminus} \qquad \dfrac{}{\longrightarrow j : p, j : \neg p, i : \diamondminus \neg p}\ R_{\neg}, init}{\longrightarrow j : p, i : \diamondplus \neg p, i : \diamondminus \neg p}\ R_{\diamondplus}}{\longrightarrow i : [A]p, i : \diamondplus \neg p, i : \diamondminus \neg p}\ R_{[A]}}{\longrightarrow i : (\boxplus p \rightarrow [A]p) \vee (\boxminus p \rightarrow [A]p)}\ R_{\vee}, R_{w}, R_{\neg}, L_{\neg}$$

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- Games for constructive and intuitionistic modal logics.
- Correspondence between winning innocent strategies in games played on Hyland-Ong arenas and proofs in modal constructive logics.

End of Part III

Thank you!!!

Obrigada!!!

Gracias!!!

☺

References

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