Modal Logic: Proof Theory, Semantics and Applications

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1 Hintikka's Semantic Game

Hintikka's semantic games are a framework in philosophical logic that defines the truth value (or satisfiability) of a logical formula F in a model $\mathcal{F}$ through a game.

There are two players, the **proponent** (**P**), who tries to prove the formula **true**, and the **opponent** (**O**), who tries to prove the formula **false**, with the following rules:

- 💎 at $F_1 \vee F_2$, **P chooses** between F_1 and F_2 (since for a disjunction to be true, only one of the statements must be true);
- 💎 at $F_1 \wedge F_2$, **O chooses** between F_1 and F_2 (since for a conjunction to be false, only one of the statements must be false);
- 💎 at a^+ (respectively, a^-) ,
  - 🏆 **P wins** if $\mathcal{F} \models a^+$ (respectively, $\mathcal{F} \models a^-$) , that is, if a is true (respectively, false) in the model $\mathcal{F}$;
  - 😭 **O wins** if $\mathcal{F} \not\models a^+$ (respectively, $\mathcal{F} \not\models a^-$) , that is, if a is false, (respectively, true) in the model $\mathcal{F}$.

The formula is satisfiable F in the model $\mathcal{F}$, $\mathcal{F} \models F$, if and only if **P** has at least one winning strategy.

Example 1.1. *Consider the formula* $F = \left[a^- \vee (b^+ \wedge c^-)\right] \vee [a^+ \wedge b^-]$ *within the model* $\mathcal{F} = \{a^+, b^-, c^-\}$. *We want to show that* $\mathcal{F} \models F$.

The game proceeds in the following way:

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- 💎 *Take* $F = \big[a^- \vee (b + \wedge c-)\big] \vee [a^+ \wedge b^-]$. *The main connective is a disjunction, so* ***P chooses*** *between* F_1 *and* F_2*, and tries to prove it true.*
  - 💎 $F_1 = a^- \vee (b^+ \wedge c^-)$. *The main connective is a disjunction, so* ***P chooses*** *between* F_{11} *and* F_{12}*, and tries to prove it true.*
    - 😭 $F_{11} = a^-$. *Since* $\mathcal{F} \not\models a^-$, ***O wins****.*
    - 💎 $F_{12} = b^+ \wedge c^-$. *The main connective is a conjunction, so* ***O chooses*** *between* F_{121} *and* F_{122}*, and tries to prove it false.*
      - 😭 $F_{121} = b^+$. *Since* $\mathcal{F} \not\models b^+$, ***O wins****.*
      - 🏆 $F_{122} = c^-$. *Since* $\mathcal{F} \models c^-$, ***P wins****.*
  - 💎 $F_2 = a^+ \wedge b^-$. *The main connective is a conjunction, so* ***O chooses*** *between* F_{21} *and* F_{22}*, and tries to prove it false.*
    - 🏆 $F_{21} = a^+$. *Since* $\mathcal{F} \models a^+$, ***P wins****.*
    - 🏆 $F_{22} = b^-$. *Since* $\mathcal{F} \models b^-$, ***P wins****.*

P has a winning strategy $F \to F_2 \to F_{21} \,\&\, F_{22} \to$🏆*, so we conclude that* $\mathcal{F} \models F$*.*

2 The Provability Game

To show that a formula F is valid, that is, satisfiable in every model $\mathcal{F}$, $\models F$, we extend the game from section 1, by adding two additional rules:

- 💎 at $F_1 \,||\, ... \,||\, F_i \,||\, ... \,||\, F_n$, **P can create backup copies** and continue with $F_1 \,||\, ... \,||\, F_i \,||\, F_i \,||\, ... \,||\, F_n$;
- 💎 **P is the scheduler**, so P is the one who chooses which formula will be played.

Example 2.1. *Consider the formula* $F = \big[a^- \vee (b^+ \wedge c^-)\big] \vee [a^+ \wedge (b^- \vee c^+)]$. *We want to show that* $\models F$*.*

The game proceeds in the following way:

- 💎 *Take* $F = \big[a^- \vee (b^+ \wedge c^-)\big] \vee [a^+ \wedge (b^- \vee c^+)]$. ***P creates a backup copy of F****.*
- 💎 $\big[a^- \vee (b^+ \wedge c^-)\big] \vee [a^+ \wedge (b^- \vee c^+)] \,||\, \big[a^- \vee (b^+ \wedge c^-)\big] \vee [a^+ \wedge (b^- \vee c^+)]$. *There are two formulas, so* ***P decides to focus*** *on the first formula.*
- 💎 $\underline{\big[a^- \vee (b^+ \wedge c^-)\big] \vee [a^+ \wedge (b^- \vee c^+)]} \,||\, \big[a^- \vee (b^+ \wedge c^-)\big] \vee [a^+ \wedge (b^- \vee c^+)]$. *The main connective of the first formula is a disjunction, so* ***P chooses*** *between* $a^- \vee (b^+ \wedge c^-)$ *and* $a^+ \wedge (b^- \vee c^+)$*, and tries to prove it true. We present only the first case for simplicity.*

- $\underline{a^- \vee (b^+ \wedge c^-)} \,||\, [a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]$. ***P decides to focus*** *on the second formula.*
- $a^- \vee (b^+ \wedge c^-) \,||\, \underline{[a^- \vee (b^+ \wedge c^-)] \vee [a^+ \wedge (b^- \vee c^+)]}$. *The main connective of the second formula is a disjunction, so* ***P chooses*** *between* $a^- \vee (b^+ \wedge c^-)$ *and* $a^+ \wedge (b^- \vee c^+)$, *and tries to prove it true. We present only the second case for simplicity.*
  - $a^- \vee (b^+ \wedge c^-) \,||\, \underline{a^+ \wedge (b^- \vee c^+)}$. *The main connective of the second formula is a conjunction, so* ***O chooses*** *between* a^+ *and* $b^- \vee c^+$, *and tries to prove it false.*
    - $a^- \vee (b^+ \wedge c^-) \,||\, \underline{a^+}$. ***P decides to focus*** *on the first formula.*
      - $\underline{a^- \vee (b^+ \wedge c^-)} \,||\, a^+$. *The main connective of the first formula is a disjunction, so* ***P chooses*** *between* a^- *and* $b^+ \wedge c^-$, *and tries to prove it true. We present only the first for simplicity.*
        - $a^- \,||\, a^+$. *Since for every model* $\mathcal{F}$, *we have either* $\mathcal{F} \models a^-$ *or* $\mathcal{F} \models a^+$, ***P wins.***
    - $a^- \vee (b^+ \wedge c^-) \,||\, \underline{b^- \vee c^+}$. ***P decides to focus*** *on the first formula.*
      - $\underline{a^- \vee (b^+ \wedge c^-)} \,||\, b^- \vee c^+$. *The main connective of the first formula is a disjunction, so* ***P chooses*** *between* a^- *and* $b^+ \wedge c^-$, *and tries to prove it true. We present only the second case for simplicity.*
      - $\underline{b^+ \wedge c^-} \,||\, b^- \vee c^+$. *The main connective of the first formula is a conjunction, so* ***O chooses*** *between* b^+ *and* c^-, *and tries to prove it false.*
        - $\underline{b^+} \,||\, b^- \vee c^+$. ***P decides to focus*** *on the second formula.*
          - $b^+ \,||\, \underline{b^- \vee c^+}$. *The main connective of the second formula is a disjunction, so* ***P chooses*** *between* b^- *and* c^+, *and tries to prove it true. We present only the first for simplicity.*
            - $b^+ \,||\, b^-$. *Since for every model* $\mathcal{F}$, *we have either* $\mathcal{F} \models b^+$ *or* $\mathcal{F} \models b^-$, ***P wins.***
        - $\underline{c^-} \,||\, b^- \vee c^+$. ***P decides to focus*** *on the second formula.*
          - $c^- \,||\, \underline{b^- \vee c^+}$. *The main connective of the second formula is a disjunction, so* ***P chooses*** *between* b^- *and* c^+, *and tries to prove it true. We present only the second for simplicity.*
            - $c^- \,||\, c^+$. *Since for every model* $\mathcal{F}$, *we have either* $\mathcal{F} \models c^-$ *or* $\mathcal{F} \models c^+$, ***P wins.***

P has a winning strategy, regardless of the model. So, we conclude that $\models F$.

3 Games with Modalities[2]

3.1 Adding Modalities

We can also extend the game from Section 1, by tweaking the winning rules and adding two additional rules for modalities. The game is now played at a world w of the model $\mathcal{F}$, with the propositional rules played within the current world:

- 💎 at $F_1 \vee F_2$, **P chooses** between F_1 and F_2 (since for a disjunction to be true, only one of the statements must be true);
- 💎 at $F_1 \wedge F_2$, **O chooses** between F_1 and F_2 (since for a conjunction to be false, only one of the statements must be false);
- 💎 at $\Diamond F$, **P chooses** a successor v of the current world w (wRv), and the game continues with F at v (if w has no successor, **O wins**, since the diamond cannot be satisfied);
- 💎 at $\Box F$, **O chooses** a successor v of the current world w (wRv), and the game continues with F at v (if w has no successor, **P wins**, since the box holds vacuously);
- 💎 at $a^{\pm}$,
  - 🏆 **P wins** if $\mathcal{F}, w \models a^{\pm}$, that is, if $a^{\pm}$ holds at the current world w;
  - 😭 **O wins** if $\mathcal{F}, w \not\models a^{\pm}$.

Example 3.1. *Consider the formula $F = \Diamond(a^+ \wedge b^-)$ and the model $\mathcal{F}$ with worlds w, v, and u, accessibility relation wRw, wRv, and vRu, and valuations $w = \{a^+, b^-\}$, $v = \{a^+, b^+\}$ and $u = \{a^-, b^-\}$, as shown in Figure 1. We play the game at the world w and want to show that $\mathcal{F}, w \models F$.*

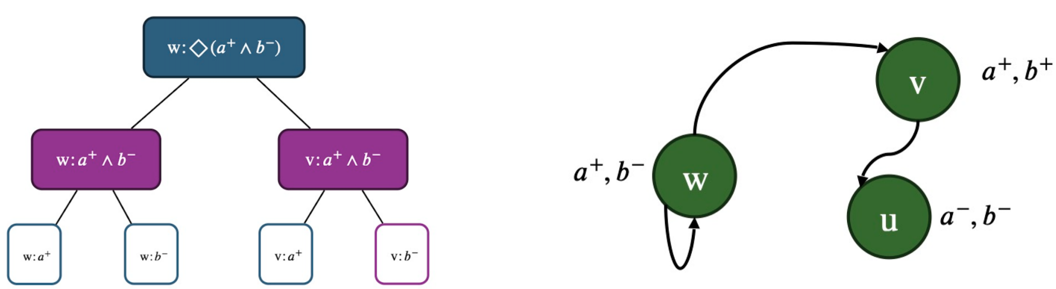


Figure 1: The game tree (left) and the world model (right)

The game proceeds in the following way:

- 💎 *Take $F = \Diamond(a^+ \wedge b^-)$ at w. The main connective is a diamond, so **P chooses** a successor of w, either v or w itself, and tries to prove $a^+ \wedge b^-$ true.*
  - 💎 *$a^+ \wedge b^-$ at v. The main connective is a conjunction, so **O chooses** between a^+ and b^-, and tries to prove it false.*
    - 🏆 *a^+ at v. Since $\mathcal{F}, v \models a^+$, **P wins**.*
    - 😭 *b^- at v. Since $\mathcal{F}, v \not\models b^-$, **O wins**.*
  - 💎 *$a^+ \wedge b^-$ at w. The main connective is a conjunction, so **O chooses** between a^+ and b^-, and tries to prove it false.*
    - 🏆 *a^+ at w. Since $\mathcal{F}, w \models a^+$, **P wins**.*
    - 🏆 *b^- at w. Since $\mathcal{F}, w \models b^-$, **P wins**.*

P has a winning strategy *by staying at w, $F \to (a^+ \wedge b^-) \to a^+ \,\&\, b^- \to$🏆, so we conclude that $\mathcal{F}, w \models F$.*

3.2 Positive-negative Logic (PNL)

"I am my own friend, I love myself." — E. Pimentel

Let us now see some different modalities. Consider a social network with agents who are friends or enemies based on whether they agree or disagree with each other. We will see how tools from formal logic let us answer questions about the nature of such systems [1].

Consider the following syntax:

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^+(i,j) \mid R^-(i,j) \mid \diamondplus\, F \mid \diamondminus\, F \mid [A]F$$

The new bits deal with the relationships between the agents and their agreement/disagreement on atomic propositions.

- 🙉 $\diamondplus\, F$ holds for agent a whenever F holds for some a's friend;
- 🙈 $\diamondminus\, F$ holds for agent a whenever F holds for some a's enemy;
- 🙊 $[A]F$ holds whenever F holds *for all agents*.

We will talk about $R^{\pm}(i,j)$ in the syntax later.

For our model, we will use $\mathcal{M} = \langle \mathsf{A}, \mathsf{R}^+, \mathsf{R}^-, \mathsf{V}, \mathsf{g} \rangle$, where

- A is a set (of agents);
- $\mathsf{g} : N \to \mathsf{A}$ is the denotation function – gives names to agents;
- $\mathsf{R}^+, \mathsf{R}^- \subseteq \mathsf{A} \times \mathsf{A}$;
- $\mathsf{V} : \mathsf{At} \to \mathcal{P}(\mathsf{A})$.

Membership in R^+ and R^- is defined as follows:

- $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^+$: agent a agrees with (is a friend of) agent b;
- $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^-$: agent a disagrees with (is an enemy of) agent b.

These satisfy the following properties:

- g is surjective (every agent has a name);
- R^+ is reflexive;
- R^+ and R^- are symmetric;
- these relations do not overlap

We also have the optional axiom: collective connectedness, which is discussed later.

We can now formally define what it means for $\Diamond^{+} F$, $\Diamond^{-} F$, and $[A]F$ to hold. Let $\mathsf{a} = \mathsf{g}(i)$,

$$
\begin{array}{ll}
\mathcal{M}, \mathsf{a} \Vdash \Diamond^{+} F & \text{iff there is } j \in N \text{ such that } \mathcal{M}, \mathsf{g}(j) \Vdash R^+(i,j) \text{ and } \mathcal{M}, \mathsf{g}(j) \Vdash F \\
\mathcal{M}, \mathsf{a} \Vdash \Diamond^{-} F & \text{iff there is } j \in N \text{ such that } \mathcal{M}, \mathsf{g}(j) \Vdash R^-(i,j) \text{ and } \mathcal{M}, \mathsf{g}(j) \Vdash F \\
\mathcal{M}, \mathsf{a} \Vdash [A]F & \text{iff } \mathcal{M}, \mathsf{g}(j) \Vdash F, \text{ for all } j \in N.
\end{array}
$$

Some sentences that we can talk about in this system include:

- $\mathcal{M}_1$: I have a friend who does not believe in p: $\mathcal{M}_1, I \Vdash \Diamond^{+} \neg p$;
- $\mathcal{M}_2$: I have an enemy that does not believe in p: $\mathcal{M}_2, I \Vdash \Diamond^{-} \neg p$;
- $\mathcal{M}_3$: I have an enemy: $\mathcal{M}_3, I \Vdash \Diamond^{-} \top$;
- $\mathcal{M}_4$: Everybody has a friend where p: $\mathcal{M}_4, \mathsf{a} \Vdash [A] \Diamond^{+} p$ for any agent a.

Just like the other modal logics, we can extend this system with new inference rules/axioms. This also has predictable effects on the nature of $R^{\pm}$.

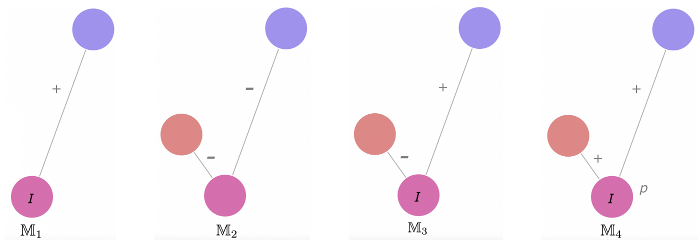
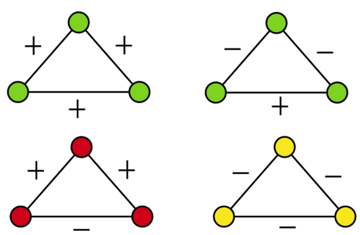


Figure 2: Example sentences in **PNL**

3.2.1 Balance

- 🦒 We have the rule $4B$, also called strong balance. This enforces the 'enemy of my enemy is my friend' structure in the models.
- 🦒 We have the rule $4W$, also called weak balance. This relaxes strong balance by letting three mutual enemies exist.



$$((\Diamond^{+}\Diamond^{+}\, p \vee \Diamond^{-}\Diamond^{-}\, p) \rightarrow \Diamond^{+}\, p) \wedge ((\Diamond^{+}\Diamond^{-}\, p \vee \Diamond^{-}\Diamond^{+}\, p) \rightarrow \Diamond^{-}\, p) \quad \textbf{(4B)}$$

$$((\Diamond^{+}\Diamond^{+}\, p) \rightarrow \Diamond^{+}\, p) \wedge ((\Diamond^{+}\Diamond^{-}\, p \vee \Diamond^{-}\Diamond^{+}\, p) \rightarrow \Diamond^{-}\, p) \quad \textbf{(4W)}$$

3.2.2 Extending Hintikka games

To extend the games to this model, we need to track a couple of extra things, namely:

- current roles of the players;
- current formula F; and
- the current world w in the model.

This is fairly straightforward, but there are a couple of drawbacks compared to propositional logic. The game tree becomes dependent on the relational structure of the model. This is unlike propositional logic, where the truth information is only necessary at the end to determine whether the statement is provable or not. This also means that the trees are not uniform across models.

There are solutions, however, namely hybrid logics. We will internalize the nominals for the agents in the syntax through the $R^{\pm}(i, j)$ constructs.

We play the game over a model. We have the two game states $\mathbf{P}, i : F$ when I am the proponent and $\mathbf{O}, i : F$ when I am the opponent. i is the nominal of the agent. The game goes as follows:

- at $\mathbf{P}, i : F_1 \wedge F_2$, **YOU choose** between F_1 and F_2 (since for a conjunction to be false, only one of the statements must be false);
- at $\mathbf{O}, i : F_1 \wedge F_2$, **I choose** between F_1 and F_2 (this time I want to prove the statement false, while YOU want it to be true);
- at $\mathbf{P}, i : \Diamond^{\pm} F$, **I choose** the potential friend/enemy with nominal j, and **YOU choose** between $\mathbf{P}, \_ : \mathsf{R}^{\pm}(i, j)$ and $\mathbf{P}, j : F$ (since either of these being false would make $\Diamond^{\pm} F$ false at i).
- at $\mathbf{O}, i : \Diamond^{\pm} F$, **YOU choose** the j, and **I choose** between $\mathbf{O}, \_ : \mathsf{R}^{\pm}(i, j)$ and $\mathbf{O}, j : F$.

Example 3.2. *See slides, page 30.*

3.2.3 Winning strategy

Let g be the formula. A strategy for Me is a subtree σ of the associated game tree $G_{\mathcal{M}}(g)$ such that:

1. $g \in \sigma$,
2. if $h \in \sigma$ is a node labelled Y, then all children of h are in σ,
3. if $h \in \sigma$ is a node labelled I, then exactly one child of h is in σ.

where the node labels are based on who has the choice.

σ is called a winning strategy if all leaves in the tree σ are labelled I. In fact, I have a winning strategy for $G_{\mathcal{M}}(\mathbf{P}, i : F)$ iff $\mathcal{M}, a \Vdash F$ and You have a winning strategy for $G_{\mathcal{M}}(\mathbf{P}, i : F)$ iff $\mathcal{M}, a \nVdash F$.

3.2.4 Validity

We can once again duplicate the game state as necessary and reach the elementary states in each game. Now we play the validity game $\mathbf{DG}(\mathbf{P}, i : F)$ over all **PNL**-models. I win (and you lose) if for every **PNL**-model there is an elementary game state where I win the semantics game.

Once again, we have the adequacy result: I have a winning strategy in $\mathbf{DG}(D)$ iff for every **PNL**-model $\mathcal{M}$, there is some $g \in D$ such that I have a winning strategy in $\mathbf{G}_{\mathcal{M}}(g)$. As a corollary, we have that F is **PNL**-valid iff I have a *winning strategy* in $\mathbf{DG}(\mathbf{P}, i : [A]F)$.

See the slides (page 35) for some of the related inference rules (including collective connectedness).

References

[1] Mina Young Pedersen, Sonja Smets, and Thomas Ågotnes. Modal logics and group polarization. *Journal of Logic and Computation*, 31(8):2240–2269, 2021.

[2] Elaine Pimentel, Carlos Olarte, Timo Lang, Robert Freiman, and Christian G. Fermüller. Playing with Modalities. In Jörg Endrullis and Sylvain Schmitz, editors, *33rd EACSL Annual Conference on Computer Science Logic (CSL 2025)*, volume 326 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 4:1–4:20, Dagstuhl, Germany, 2025. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.