# Modal Logic
# Part IV – Substructural games!


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OPLSS Summer School
Oregon, USA

UCL, UK

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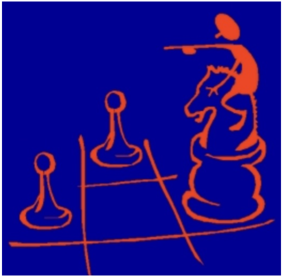

# Recap of Lecture III

- Valitity games
- Provability games
- Modal games

# In this lecture: substructural games!

- Playing with linear logic
- How to stay within your budget

# Outline

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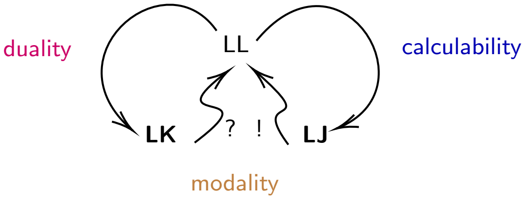
Linear logic

Modalities

Playing with substructural modalities

# Linear logic LL [Girard]



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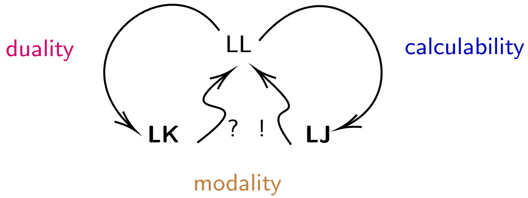


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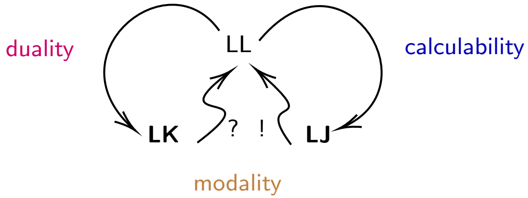


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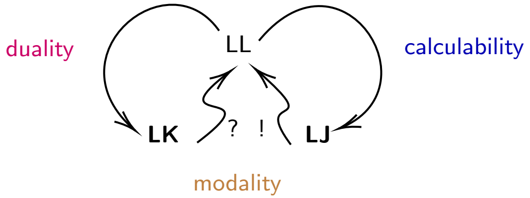


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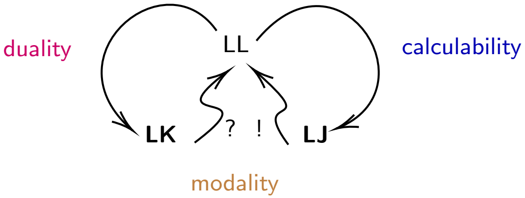


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- … is resource conscious: specification of states.
- … captures several notions of dualities.

# Classical and intuitionistic logics

- In classical logic:

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \wedge R_a \qquad \frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B} \wedge R_m$$

  are equivalent in the presence of the structural rules contraction and weakening.

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$$\dfrac{\dfrac{\Gamma_1 \vdash \Delta_1, A}{\Gamma \vdash \Delta, A}\, weak \qquad \dfrac{\Gamma_2 \vdash \Delta_2, B}{\Gamma \vdash \Delta, B}\, weak}{\Gamma \vdash \Delta, A \wedge B} \wedge R_a \qquad \dfrac{\dfrac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma, \Gamma \vdash \Delta, \Delta, A \wedge B} \wedge R_m}{\Gamma \vdash \Delta, A \wedge B}\, contr$$

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- Differences:
  - $\&_R$ is invertible, $\otimes_R$ is not;
  - sharing versus splitting resources.

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  - additive = not provable, corresponds to the intuitionistic reading of disjunction;
  - multiplicative = trivially provable and simply corresponds to the tautology $(A \Rightarrow A)$ that is perfectly acceptable in intuitionistic logic too.

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What is wrong with this picture?

# Outline

# Jean-Yves Girard on Modal Logic – Thank you Finnton Wentworth!!

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"If we take the best – or rather less bad – modal logic S4, one is stricken by the fact that the additional operations such as the necessity $\Box A$ bring nothing new: typically, the erasure of all symbols $\Box, \Diamond$ preserves proofs. The situation can be summarised by the question :

*What is the necessity of necessity?*

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"Linear logic yielded another answer, through the exponentials $!A$, $?A$ which are in charge of perenniality ; those are indeed modalities, even if the symbols $\Box, \Diamond$ have been relinquished to avoid the infamous company of S5 and its likes."

Jean-Yves Girard, 2007. "Truth, modality and intersubjectivity," available at https://girard.perso.math.cnrs.fr/truth.pdf.

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$$\frac{\Gamma, !\,A, !\,A \vdash \Delta}{\Gamma, !\,A \vdash \Delta}\ \mathrm{C}_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !\,A \vdash \Delta}\ \mathrm{W}_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !\,A \vdash \Delta}\ der_L$$

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$$\frac{!\,\Gamma \vdash ?\Delta, A}{!\,\Gamma \vdash ?\Delta, !\,A}\ !_R \qquad \frac{!\,\Gamma, A \vdash ?\Delta}{!\,\Gamma, ?A \vdash ?\Delta}\ ?_L$$

They behave classically!

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## Sequent system LL

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}\ \otimes_R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \otimes_L \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta}\ \&_R$$

$$\frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_1 \,\&\, A_2 \vdash \Delta}\ \&_{L_i} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ \multimap_R \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \vdash \Delta_1, \Delta_2}\ \multimap_L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \parr_R \qquad \frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \parr B \vdash \Delta_1, \Delta_2}\ \parr_L \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_1 \oplus A_2, \Delta}\ \oplus_{R_i}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\ \oplus_L \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{C}_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{W}_L$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathrm{der}_L \qquad \frac{\Gamma \vdash \Delta, ?A, ?A}{\Gamma \vdash \Delta, ?A}\ \mathsf{C}_R \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, ?A}\ \mathsf{W}_R \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, ?A}\ \mathrm{der}_R$$

$$\frac{!\Gamma \vdash ?\Delta, A}{!\Gamma \vdash ?\Delta, !A}\ !_R \qquad \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}\ ?_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ \bot_R \qquad \frac{}{\bot \vdash}\ \bot_L$$

$$\frac{}{\vdash 1}\ 1_R \qquad \frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}\ 1_L \qquad \frac{}{\Gamma \vdash \top, \Delta}\ \top_R \qquad \frac{}{\Gamma, 0 \vdash \Delta}\ 0_L \qquad \frac{}{p \vdash p}\ \mathit{init}$$

# One sided!

$$\frac{A,\Gamma_1 \quad B,\Gamma_2}{A\otimes B,\Gamma_1,\Gamma_2}\otimes \qquad \frac{A,\Gamma \quad B,\Gamma}{A\,\&\,B,\Gamma}\,\& \qquad \frac{A_i,\Gamma}{A_1\oplus A_2,\Gamma}\oplus_i \qquad \frac{A,B,\Gamma}{A\parr B,\Gamma}\parr$$

$$\frac{\Gamma,?A,?A}{\Gamma,?A}\,\mathsf{C} \qquad \frac{\Gamma}{\Gamma,?A}\,\mathsf{W} \qquad \frac{\Gamma,A}{\Gamma,?A}\,\mathsf{der} \qquad \frac{?\Gamma,A}{?\Gamma,!A}\,\mathsf{prom}$$

$$\frac{\Gamma}{\bot,\Gamma}\bot \qquad \frac{}{1}\,1 \qquad \frac{}{\top,\Gamma}\top$$

$$\frac{}{p^\perp,p}\,\mathit{init}$$

# Structural and "modal" modalities

$$4 : \ \Box A \to \Box\Box A \qquad\qquad \mathsf{T} : \ \Box A \to A$$

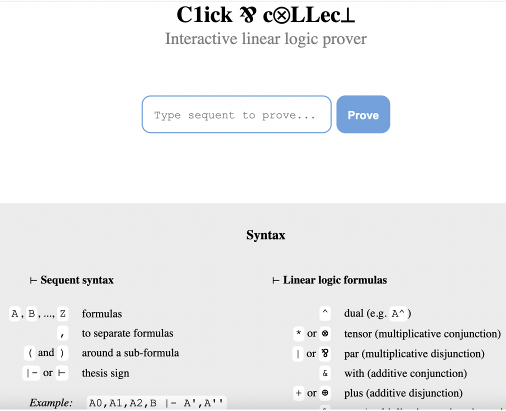
$$\frac{F, \Diamond \Gamma}{\Box F, \Diamond \Gamma, \Delta}\ 4 \qquad\qquad \frac{F, \Gamma}{\Diamond F, \Gamma}\ \mathsf{T}$$

# Structural and “modal” modalities

$$4 : \ !A \multimap !!A \qquad\qquad T : \ !A \multimap A$$

$$\frac{F, ?\Gamma}{!F, ?\Gamma}\ \text{prom} \qquad\qquad \frac{F, \Gamma}{?F, \Gamma}\ \text{der}$$

# Hands on!

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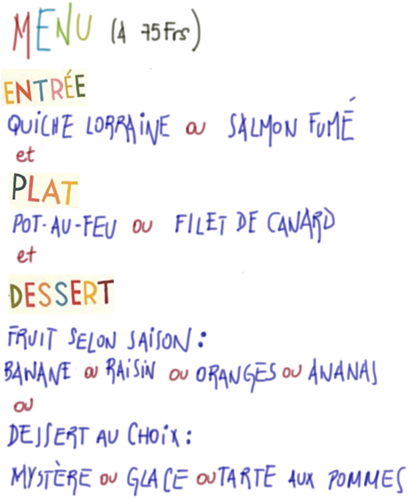
# Outline

# Game semantics & LL

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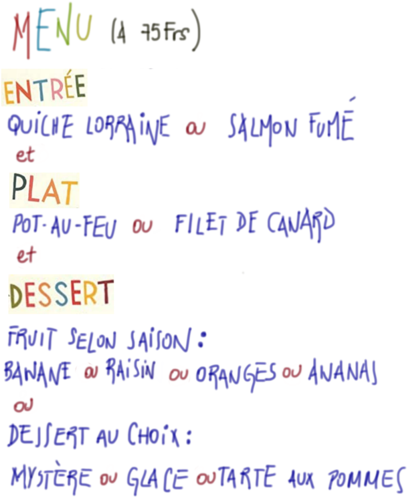

MENU (A 75Frs)

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QUICHE LORRAINE ou SALMON FUMÉ

et

PLAT

POT-AU-FEU ou FILET DE CANARD

et

DESSERT

FRUIT SELON SAISON :

BANANE ou RAISIN ou ORANGES ou ANANAS

ou

DESSERT AU CHOIX :

MYSTÈRE ou GLACE ou TARTE AUX POMMES

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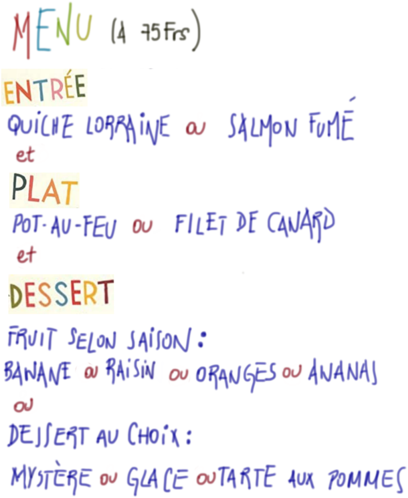
- Gentzen's sequent calculus is a (the?) natural starting point for connecting inference and resource consciousness.

## Game semantics & LL



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- To breathe life into the resource metaphor, we need dynamics

  $\Longrightarrow$ game semantics for substructural sequent calculi.

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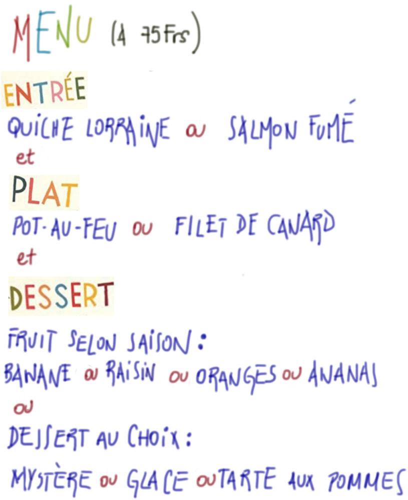


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- To breathe life into the resource metaphor, we need dynamics

  $\Longrightarrow$ game semantics for substructural sequent calculi.
- Our view of game semantics: a playground for illuminating specific intuitions underlying certain proof systems.

# Affine intuitionistic multiplicative additive LL ($\mathcal{C}$)

Sequent System for $\mathcal{C}$

$$\frac{\Delta_1 \vdash A \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}\ \otimes_R \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ \&_R \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap_R$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes_L \qquad \frac{\Delta_1 \vdash A \quad \Delta_2, B \vdash C}{\Delta_1, \Delta_2, A \multimap B \vdash C}\ \multimap_L \qquad \frac{\Gamma, A_i \vdash B}{\Gamma, A_1 \,\&\, A_2 \vdash B}\ \&_{L_i}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus_L \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2}\ \oplus_{R_i}$$

$$\frac{}{\Gamma, p \vdash p}\ I \qquad \frac{}{\Gamma \vdash 1}\ 1_R \qquad \frac{}{\Gamma, 0 \vdash A}\ 0_L$$

# The game for $\mathcal{C}$ [Fermüller,Lang'17]

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- P chooses a sequent $S$ among the current game state, a principal formula in $S$ and a matching rule instance $r$.
- P acts as the scheduler of the game.

# Multiplicative vs Additive

Both are (right) branching rules:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \mathbin{\&} B}\ \&_R \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes_R$$

However, the intended meaning is different:

- $A \mathbin{\&} B$: P must be prepared to play either $A$ or $B$ (O choice) but only one game is actually played.
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## Branching structure:

Both definitions (a single or a parallel game) are equivalent: the existence of winning strategies for P remains the same.

However, semantically, they provide different viewpoints of the connectives.

# The game for $\mathcal{C}$

$$\dfrac{}{p, q \oplus r \vdash (p \otimes q) \oplus (p \otimes r)} \oplus_{R_1}$$

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$$\dfrac{\dfrac{\overset{☺}{p \vdash p} \qquad q \oplus r \vdash q}{p, q \oplus r \vdash (p \otimes q)}\ \otimes_R}{p, q \oplus r \vdash (p \otimes q) \oplus (p \otimes r)}\ \oplus_{R_1}$$

# The game for $\mathcal{C}$

$$\dfrac{\dfrac{\overset{\ddot\smile}{p \vdash p} \qquad \dfrac{\overset{\ddot\smile}{q \vdash q}}{q \oplus r \vdash q}\,\oplus_L}{p, q \oplus r \vdash (p \otimes q)}\,\otimes_R}{p, q \oplus r \vdash (p \otimes q) \oplus (p \otimes r)}\,\oplus_{R_1}$$

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$$

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$$\dfrac{p, q \vdash (p \otimes q) \oplus (p \otimes r) \qquad \dfrac{\dfrac{\overset{\text{☺}}{p \vdash p} \qquad \overset{\text{☺}}{r \vdash r}}{p, r \vdash p \otimes r}\otimes_R}{p, r \vdash (p \otimes q) \oplus (p \otimes r)}\oplus_{R_2}}{p, q \oplus r \vdash (p \otimes q) \oplus (p \otimes r)}\oplus_L$$

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$$
\dfrac{
\dfrac{\dfrac{\smile}{p \vdash p} \quad \dfrac{\smile}{q \vdash q}}{p, q \vdash (p \otimes q)}\otimes_R
}{p, q \vdash (p \otimes q) \oplus (p \otimes r)}\oplus_{R_1}
\qquad
\dfrac{
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}{p, r \vdash (p \otimes q) \oplus (p \otimes r)}\oplus_{R_2}
$$

$$
\dfrac{p, q \vdash (p \otimes q) \oplus (p \otimes r) \qquad p, r \vdash (p \otimes q) \oplus (p \otimes r)}{p, q \oplus r \vdash (p \otimes q) \oplus (p \otimes r)}\oplus_L
$$

# Lafont's menu revisited

MENU (A 75 Frs)

ENTRÉE

QUICHE LORRAINE ou SALMON FUMÉ

et

PLAT

POT-AU-FEU ou FILET DE CANARD

et

DESSERT

FRUIT SELON SAISON :

BANANE ou RAISIN ou ORANGES ou ANANAS

ou

DESSERT AU CHOIX :

MYSTÈRE ou GLACE ou TARTE AUX POMMES

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# Subexponentials [Danos,Joinet,Schellinx'93]

Exponentials in ILL:

$$\frac{\Gamma, A \vdash C}{\Gamma, !\, A \vdash C}\ !_L \qquad \frac{!\, A_1, \ldots, !\, A_n \vdash A}{!\, A_1, \ldots, !\, A_n \vdash !\, A}\ !_R$$

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Sub-exponentials in ILL:

$$\frac{\Gamma, A \vdash C}{\Gamma, !^a A \vdash C}\ {!^a}_L \qquad \frac{!^{a_1} A_1, \ldots, !^{a_n} A_n \vdash A}{!^{a_1} A_1, \ldots, !^{a_n} A_n \vdash !^a A}\ {!^a}_R, \text{ provided } a \preceq a_i$$

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Then:

$!^a A \not\equiv !^b A$ for any $a \neq b$.

# Assumptions plus cost – system $\mathcal{C}(\mathbb{R}^+)$

Augment assumptions with costs, where assumptions are formulas occurring negatively on sequents.

$$\frac{\Gamma, !^a A, A \vdash C}{\Gamma, !^a A \vdash C} \; !^a{}_L, \;\; a \in \mathbb{R}^+$$

# The game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ [Lang,Olarte,Pimentel,Fermüller'19]

- States: tuples $(H, b)$, where $H$ is a finite multiset of $\mathbb{R}^+$-valued sequents and $b \in \mathbb{R}$ is a budget.
- Rounds: the successor state is determined according to rules that fit one of the two following schemes:

$$
\begin{array}{llcl}
(1) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S'\}, b') \\
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- Depending on the $r$, the round proceeds as follows:
  1. If the rule $r$ is not $!_L$, then the game proceeds as before, with budget $b$.
  2. Budget decrease: $!_L$ with premise $S'$ and principal formula $!^a A$, then the game proceeds in the game state $(G \cup \{S'\}, b - a)$.
  3. To win the game: non negative final budget.

## Example

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7. $(\{G, w, b \vdash b\} \cup \{G, b \vdash b\}, 0)$ (P plays $\otimes_R$, parallelism)

# Labelled system $\mathcal{C}^{\ell}(\mathbb{R}^+)$

- Weak adequacy: information about the budget $b$ is lost in the proof theoretic representation.
- In other words, the game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ is more expressive than the calculus $\mathcal{C}(\mathbb{R}^+)$.

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- Weak adequacy: information about the budget $b$ is lost in the proof theoretic representation.
- In other words, the game $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ is more expressive than the calculus $\mathcal{C}(\mathbb{R}^+)$.
- To overcome this mismatch: a labelled extension of $\mathcal{C}(\mathbb{R}^+)$.
- A $\mathcal{C}^\ell(\mathbb{R}^+)$-proof is build from labelled sequents

$$b : \Gamma \vdash A$$

where $\Gamma \vdash A$ is a $\mathcal{C}(\mathbb{R}^+)$ sequent and $b \in \mathbb{R}^+$.

# Sequent rules for $\mathcal{C}^\ell(\mathbb{R}^+)$

**Labelled sequent system for $\mathcal{C}^\ell(\mathbb{R}^+)$**

$$\frac{a : !\Gamma, \Delta_1 \vdash A \quad b : !\Gamma, \Delta_2 \vdash B}{a + b : !\Gamma, \Delta_1, \Delta_2 \vdash A \otimes B} \otimes_R \qquad \frac{a : \Gamma \vdash A \quad b : \Gamma \vdash B}{\max\{a, b\} : \Gamma \vdash A \,\&\, B} \&_R$$

$$\frac{c : \Gamma, !^a A, A \vdash C}{a + c : \Gamma, !^a A \vdash C} \,!_L$$

$$\frac{}{b : \Gamma, p \vdash p} \; I \; b \geq 0 \qquad \frac{}{b : \Gamma \vdash 1} \; 1_R \; b \geq 0 \qquad \frac{}{b : \Gamma, 0 \vdash A} \; 0_L \; b \geq 0$$

# Example

A printer costs \$500 and produces copies for \$0.1. I want to make 2 copies. Which is the budget needed?

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$$\cfrac{\cfrac{\cfrac{}{0 : C, C \vdash C \otimes C}\ \otimes, \mathit{init}}{0.20 : !^{0.1} C \vdash C \otimes C}\ !^{0.10}}{500.20 : !^{500}(!^{0.1} C) \vdash C \otimes C}\ !^{500}$$

$$\cfrac{\cfrac{\cfrac{}{0 : C, C \vdash C \otimes C}\ \otimes, \mathit{init}}{0.20 : !^{0.1} C, !^{0.1} C \vdash C \otimes C}\ !^{0.10}}{1,000.20 : !^{500}(!^{0.1} C) \vdash C \otimes C}\ !^{500}$$

Reasonable strategy:
Buy one printer,
make 2 copies out of it.

Stupid strategy:
Buy two printers,
make one copy in each.

# Results

**Theorem**

*Given a $\mathcal{C}(\mathbb{R}^+)$-proof $\Xi$ of a sequent $S$, there exists a smallest budget with $\mathit{cost}(\Xi)$ that suffices to win the game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ on $S$ when following the strategy corresponding to $\Xi$.*

**Spectrum**

$\mathtt{spec}(S :) = \{\mathtt{cost}(\Xi) \mid \Xi \text{ is an } \mathcal{C}(\mathbb{R}^+)\text{-proof of } S\}$.

**Theorem**

*If $\vdash_{\mathcal{C}(\mathbb{R}^+)} \Gamma \vdash A$, then $\mathit{spec}(\Gamma \vdash A)$ has a least element. In other words, there is a smallest $b$ such that $\vdash_{\mathcal{C}^{\ell}(\mathbb{R}^+)} \Gamma \vdash_b A$.*

# Generalizing the notion of cost
## Time units

$$\mathcal{K}_t = \langle \mathbb{R}^+_\infty, \geq_{\mathbb{R}^+_\infty}, +_\mathbb{R}, 0 \rangle$$

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$\mathbb{R}^+_\infty$ is the completion of $\mathbb{R}^+$ with $\infty$.

$\geq_{\mathbb{R}^+_\infty}$ elects the "fastest" path when there are different ways of going from $s_x$ to $s_y$.

## Generalizing the notion of cost

### Time units

$$\mathcal{K}_t = \langle \mathbb{R}^+_\infty, \geq_{\mathbb{R}^+_\infty}, +_\mathbb{R}, 0 \rangle$$

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Amount of computational resources (e.g., RAM) available to process a series of tasks.

Idea: to know what is the least amount of resources $b$ s.t. some jobs $\Gamma$ can be all of them executed, sequentially if needed.

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$$\mathcal{K}_{c/p} = \langle \{\mathtt{pub}, \mathtt{conf}\}, \mathtt{pub} < \mathtt{conf}, \mathsf{min}, \mathtt{conf} \rangle$$

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$$\mathcal{K}_{c/p} = \langle \{\mathtt{pub}, \mathtt{conf}\}, \mathtt{pub} < \mathtt{conf}, \min, \mathtt{conf} \rangle$$

$!^{\mathrm{pub}} F$ = *public* information.

$!^{\mathrm{conf}} F$ = *secret* information.

$\mathtt{conf} : \Gamma, !^{\mathrm{pub}} F \not\vdash G$ since $\mathtt{conf} \not< \mathtt{pub}$.

# Generalizing the notion of cost

## Time units

$$\mathcal{K}_t = \langle \mathbb{R}^+_\infty, \geq_{\mathbb{R}^+_\infty}, +_\mathbb{R}, 0 \rangle$$

## Least amount of resources to execute a task

$$\mathcal{K}_{\mathsf{max}} = \langle \mathbb{R}^+_\infty, \geq_{\mathbb{R}^+_\infty}, \mathsf{max}, 0 \rangle$$

## Protected resources

$$\mathcal{K}_{c/p} = \langle \{\mathbf{pub}, \mathbf{conf}\}, \mathbf{pub} < \mathbf{conf}, \mathsf{min}, \mathbf{conf} \rangle$$

## Probabilistic choice

$$\mathcal{K}_p = \langle [0, 1], \leq, \times, 1 \rangle$$

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$P +_\alpha Q = (!^\alpha P) \,\&\, (!^{1-\alpha} Q).$

# References


1. Notes on game semantics by Pierre-Louis Curien, 2019.
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