Modal Logic
Part IV – Truth!

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Modal Logic

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- Propositional classical logic plus the necessitation rule, axiom K and some combination of axioms D, T, B, 4 and 5.

$$\frac{\vdash A}{\vdash \Box A}\ \text{nec} \qquad\qquad \text{K} \quad \Box(A \to B) \to (\Box A \to \Box B)$$

$$\begin{array}{rl}
\text{D} & \Box A \to \Diamond A \\
\text{T} & \Box A \to A \\
\text{B} & A \to \Box\Diamond A \\
4 & \Box A \to \Box\Box A \\
5 & \Diamond A \to \Box\Diamond A
\end{array}$$

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$$\frac{\vdash A}{\vdash \Box A}\ \text{nec}$$

K $\Box(A \to B) \to (\Box A \to \Box B)$

D $\Box A \to \Diamond A$
T $\Box A \to A$
B $A \to \Box\Diamond A$
4 $\Box A \to \Box\Box A$
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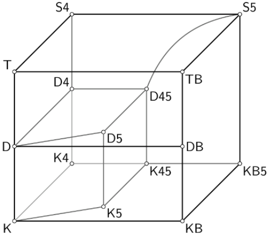

Kripke Semantics

- The usual semantics of possible worlds.

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & p \in V(w); \\
\mathcal{M}, w \Vdash \neg A & \text{iff} & \mathcal{M}, w \nVdash A; \\
\mathcal{M}, w \Vdash A \wedge B & \text{iff} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B; \\
\mathcal{M}, w \Vdash \Box A & \text{iff} & \text{for all } v, wRv \text{ implies } \mathcal{M}, v \Vdash A; \\
\mathcal{M}, w \Vdash \Diamond A & \text{iff} & \text{there exists } v \text{ s.t. } wRv \text{ and } \mathcal{M}, v \Vdash A.
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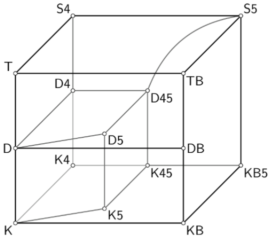
\end{array}
$$

- Frame conditions:

D : Seriality
T : Reflexivity
B : Symmetry
4 : Transitivity
5 : Euclidianness



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- Kurt Gödel (1932): there exists a formula that an *n*-valued matrix would designate as true, but which is not valid in IPL.
- Intuitively: Finite-valued tables: too static! They cannot capture the evolving proof-oriented nature of intuitionistic validity ☹

Bad news

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We are then in a sort of dead end...

- No need of possible worlds to explain the meaning of modalities.

Kearns Semantics

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- No need of possible worlds to explain the meaning of modalities.
- Non-deterministic matrix (Nmatrix) generalize truth tables [Avron].
- Let's try with IPL: negation of U $\rightsquigarrow \{\mathsf{U}, \mathsf{T}\}$
- $\neg\neg(\alpha \vee \neg\alpha) \rightsquigarrow \mathsf{U}$ ☹

Let's try S4

A	$\Box^{\mathsf{KT4}} A$	$\Diamond^{\mathsf{KT4}} A$
F	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$
f	$\{\mathsf{F}, \mathsf{f}\}$	$\{\mathsf{T}, \mathsf{t}\}$
t	$\{\mathsf{F}, \mathsf{f}\}$	$\{\mathsf{T}, \mathsf{t}\}$
T	$\{\mathsf{T}\}$	$\{\mathsf{T}\}$

$\alpha \to \beta$	F	f	t	T
F	T	T	T	T
f	t	T, t	T, t	T
t	f	f	T, t	T
T	F	f	t	T

- $p \to p$

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T	$\{\mathsf{T}\}$	$\{\mathsf{T}\}$

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t	f	f	T, t	T
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- $p \to p$
- $\Diamond(p \to p)$

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F	{F}	{F}
f	{F, f}	{T, t}
t	{F, f}	{T, t}
T	{T}	{T}

$\alpha \to \beta$	F	f	t	T
F	T	T	T	T
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T	F	f	t	T

- $p \to p$
- $\Diamond(p \to p)$
- $\Box(p \to p)$ (bad surprise...)

Soundness fails!

p	$p \to p$	$\Box(p \to p)$
F	T	T
f	T	T
f	t	F, f
t	T	T
t	t	F, f
T	T	T

Kearns Semantics

	p	$p \to p$	$\Box(p \to p)$	
	F	T	T	$\cdots$
	f	T	T	$\cdots$
$\times$	f	t	F, f	$\cdots$
	t	T	T	$\cdots$
$\times$	t	t	F, f	$\cdots$
	T	T	T	$\cdots$

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1. It requires to check all the tautologies.

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However, this is NOT a decision procedure!

1. It requires to check all the tautologies.
2. Some rows will be removed later (when do we stop?)

Grätz Procedure: Partial Valuations

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- NMatrices are not enough for the whole cube – the need of levels.
- Sound and complete decision procedure for KT and KT4.
- Only subformulas of the formula are evaluated.
- Certain values creates dependencies that must be satisfied.
- E.g., t below is not properly supported:

	p	$p \to p$	$\Box(p \to p)$
	F	T	T
	f	T	T
$\times$	f	t	F, f
	t	T	T
$\times$	t	t	F, f
	T	T	T

Ecumenism

The present [Kearnsean] semantic account is simpler than the standard [Kripkean] account [...] For I do not think there are such things as possible worlds, or even that they constitute a useful fiction." (Kearns)

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Both semantics are indeed very well related!

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Both semantics are indeed very well related!

Our relational model, on partial valuations, preserves the usual frame conditions in modal logics!

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- Kearns' level valuations for $9/15$ modal logics.

The idea

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- Kearns' level valuations and decision procedures for all the 15 logics.
- All such procedures are systematically constructed (and some of the previous ones are obtained as instances).
- The key point: meaning and classification of truth values.

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- The key point: meaning and classification of truth values.

This will be a symphony in 5 movements ☺

Outline

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How Many Truth Values?

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8!

How Many Truth Values?

The 8 values for K:

Truth-value	**Intuitive meaning**
$v(\alpha) = \mathsf{F}$	$\Diamond\neg\alpha \wedge \neg\alpha \wedge \Box\neg\alpha$
$v(\alpha) = \mathsf{f}$	$\Diamond\neg\alpha \wedge \neg\alpha \wedge \Diamond\alpha$
$v(\alpha) = \mathsf{f}_1$	$\Box\alpha \wedge \neg\alpha \wedge \Box\neg\alpha$
$v(\alpha) = \mathsf{f}_2$	$\Box\alpha \wedge \neg\alpha \wedge \Diamond\alpha$
$v(\alpha) = \mathsf{t}_2$	$\Diamond\neg\alpha \wedge \alpha \wedge \Box\neg\alpha$
$v(\alpha) = \mathsf{t}_1$	$\Box\alpha \wedge \alpha \wedge \Box\neg\alpha$
$v(\alpha) = \mathsf{t}$	$\Diamond\neg\alpha \wedge \alpha \wedge \Diamond\alpha$
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How Many Truth Values?

The non-designated values:

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How Many Truth Values?

The possible values:

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$v(\alpha) = \mathsf{f}$	$\Diamond\neg\alpha \wedge \neg\alpha \wedge \Diamond\alpha$
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How Many Truth Values?

The non-necessary values:

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$v(\alpha) = \mathsf{f}$	$\Diamond\neg\alpha \wedge \neg\alpha \wedge \Diamond\alpha$
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The non-possible values:

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The 8 values for K:

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Value function: $\mathsf{t}(\beta) = \Diamond\neg\beta \wedge \beta \wedge \Diamond\beta$

Outline

Distinguished Sets

Classification of the 8 values:

1. $\mathcal{D} = \{\mathsf{T}, \mathsf{t}, \mathsf{t}_1, \mathsf{t}_2\}$ (α is true)
2. $\mathcal{N} = \{\mathsf{T}, \mathsf{t}_1, \mathsf{f}_2, \mathsf{f}_1\}$ (α is necessary)
3. $\mathcal{I} = \{\mathsf{F}, \mathsf{f}_1, \mathsf{t}_2, \mathsf{t}_1\}$ ($\neg\alpha$ is necessary)
4. $\mathcal{P} = \{\mathsf{T}, \mathsf{t}, \mathsf{f}_2, \mathsf{f}\}$ (α is possible)
5. $\mathcal{PN} = \{\mathsf{F}, \mathsf{f}, \mathsf{t}, \mathsf{t}_2\}$ ($\neg\alpha$ is possible)

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5. $\mathcal{PN} = \{\mathsf{F}, \mathsf{f}, \mathsf{t}, \mathsf{t}_2\}$ ($\neg\alpha$ is possible)

For example, $\mathsf{t}_2(\alpha) = \Diamond\neg\alpha \wedge \alpha \wedge \Box\neg\alpha$, hence:

- $\mathsf{t}_2 \in \mathcal{PN}$ ($\neg\alpha$ is possible)
- $\mathsf{t}_2 \in \mathcal{D}$ (designated value, α is true "now")
- $\mathsf{t}_2 \in \mathcal{I}$ (α is impossible)

Do we need all the 8 values?

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- Not needed in logics characterising serial frames.
- Our approach: The modal characterisation of truth values yields conditions on those values.
- These conditions are systemically obtained for all the 15 logics.

Values per Family of Logics

Truth-value	Meaning
$v(\alpha) = \mathsf{F}$	$\Diamond\neg\alpha, \neg\alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{f}$	$\Diamond\neg\alpha, \neg\alpha, \Diamond\alpha$
$v(\alpha) = \mathsf{f}_1$	$\Box\alpha, \neg\alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{f}_2$	$\Box\alpha, \neg\alpha, \Diamond\alpha$
$v(\alpha) = \mathsf{t}_2$	$\Diamond\neg\alpha, \alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{t}_1$	$\Box\alpha, \alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{t}$	$\Diamond\neg\alpha, \alpha, \Diamond\alpha$
$v(\alpha) = \mathsf{T}$	$\Box\alpha, \alpha, \Diamond\alpha$

Distinguished sets

- $\mathcal{D} = \{\mathsf{T}, \mathsf{t}, \mathsf{t}_1, \mathsf{t}_2\}$
- $\mathcal{N} = \{\mathsf{T}, \mathsf{t}_1, \mathsf{f}_2, \mathsf{f}_1\}$
- $\mathcal{I} = \{\mathsf{F}, \mathsf{f}_1, \mathsf{t}_2, \mathsf{t}_1\}$
- $\mathcal{P} = \{\mathsf{T}, \mathsf{t}, \mathsf{f}_2, \mathsf{f}\}$
- $\mathcal{PN} = \{\mathsf{F}, \mathsf{f}, \mathsf{t}, \mathsf{t}_2\}$

Axiom		Condition	Rule
D	$\Box\alpha \to \Diamond\alpha$	$v(\alpha) \in \mathcal{N}$	$v(\alpha) \in \mathcal{P}$
		$v(\alpha) \in \mathcal{I}$	$v(\alpha) \in \mathcal{PN}$

Values allowed

- $\mathcal{V}(\mathsf{K}) = \{\mathsf{T}, \mathsf{t}, \mathsf{t}_1, \mathsf{t}_2, \mathsf{F}, \mathsf{f}, \mathsf{f}_1, \mathsf{f}_2\}$
- $\mathcal{V}(\mathsf{KD}) = \{\mathsf{T}, \mathsf{t}, \cancel{\mathsf{t}_1}, \mathsf{t}_2, \mathsf{F}, \mathsf{f}, \cancel{\mathsf{f}_1}, \mathsf{f}_2\}$

Values per Family of Logics

Truth-value	Meaning
$v(\alpha) = \mathsf{F}$	$\Diamond\neg\alpha, \neg\alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{f}$	$\Diamond\neg\alpha, \neg\alpha, \Diamond\alpha$
$v(\alpha) = \mathsf{f}_1$	$\Box\alpha, \neg\alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{f}_2$	$\Box\alpha, \neg\alpha, \Diamond\alpha$
$v(\alpha) = \mathsf{t}_2$	$\Diamond\neg\alpha, \alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{t}_1$	$\Box\alpha, \alpha, \Box\neg\alpha$
$v(\alpha) = \mathsf{t}$	$\Diamond\neg\alpha, \alpha, \Diamond\alpha$
$v(\alpha) = \mathsf{T}$	$\Box\alpha, \alpha, \Diamond\alpha$

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T	$\Box\alpha \to \alpha$	$v(\alpha) \in \mathcal{N}$	$v(\alpha) \in \mathcal{D}$
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- $\mathcal{V}(\mathsf{K}) = \{\mathsf{T}, \mathsf{t}, \mathsf{t}_1, \mathsf{t}_2, \mathsf{F}, \mathsf{f}, \mathsf{f}_1, \mathsf{f}_2\}$
- $\mathcal{V}(\mathsf{KD}) = \{\mathsf{T}, \mathsf{t}, \cancel{\mathsf{t}_1}, \mathsf{t}_2, \mathsf{F}, \mathsf{f}, \cancel{\mathsf{f}_1}, \mathsf{f}_2\}$
- $\mathcal{V}(\mathsf{KT}) = \{\mathsf{T}, \mathsf{t}, \cancel{\mathsf{t}_1}, \cancel{\mathsf{t}_2}, \mathsf{F}, \mathsf{f}, \cancel{\mathsf{f}_1}, \cancel{\mathsf{f}_2}\}$

Outline

The Matrices

$\alpha \to \beta$	F	f	f_1	f_2	t_2	t_1	t	T
F	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$
f	$\{t\}$	$\{T, t\}$	$\{t_1\}$	$\{T\}$	$\{t\}$	$\{T\}$	$\{T, t\}$	$\{T\}$
f_1	$\{t_2\}$	$\{t\}$	$\{t_1\}$	$\{T\}$	$\{t_2\}$	$\{t_1\}$	$\{t\}$	$\{T\}$
f_2	$\{t_2\}$	$\{t\}$	$\{t_1\}$	$\{T\}$	$\{t_2\}$	$\{t_1\}$	$\{t\}$	$\{T\}$
t_2	$\{f_2\}$	$\{f_2\}$	$\{f_2\}$	$\{f_2\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$
t_1	$\{F\}$	$\{f\}$	$\{f_1\}$	$\{f_2\}$	$\{t_2\}$	$\{t_1\}$	$\{t_2\}$	$\{T\}$
t	$\{f\}$	$\{f, f_2\}$	$\{f_2\}$	$\{f_2\}$	$\{t\}$	$\{T\}$	$\{T, t\}$	$\{T\}$
T	$\{F\}$	$\{f\}$	$\{f_1\}$	$\{f_2\}$	$\{t_2\}$	$\{t_1\}$	$\{t\}$	$\{T\}$

The Matrices

α	$\Box^{K}\alpha$	$\Box^{KB}\alpha$	$\Box^{K4}\alpha$	$\Box^{K5}\alpha$	$\Box^{K45}\alpha$
F	$\{F, f, f_2\}$	$\{F\}$	$\{F, f, f_2\}$	$\{F\}$	$\{F\}$
f	$\{F, f, f_2\}$	$\{F\}$	$\{F, f, f_2\}$	$\{F\}$	$\{F\}$
f_1	$\{t_1\}$	$\{t_1\}$	$\{t_1\}$	$\{t_1\}$	$\{t_1\}$
f_2	$\{T, t, t_2\}$	$\{t_2\}$	$\{T\}$	$\{T, t_2\}$	$\{T\}$
t_2	$\{F, f, f_2\}$	$\{F, f, f_2\}$	$\{F, f, f_2\}$	$\{F\}$	$\{F\}$
t_1	$\{t_1\}$	$\{t_1\}$	$\{t_1\}$	$\{t_1\}$	$\{t_1\}$
t	$\{F, f, f_2\}$	$\{F, f, f_2\}$	$\{F, f, f_2\}$	$\{F\}$	$\{F\}$
T	$\{T, t, t_2\}$	$\{T, t, t_2\}$	$\{T\}$	$\{T, t_2\}$	$\{T\}$

The Matrices

α	$\Box^{\mathsf{KT}}\alpha$	$\Box^{\mathsf{KTB}}\alpha$	$\Box^{\mathsf{KT4}}\alpha$	$\Box^{\mathsf{KTB45}}\alpha$
F	{F}	{F}	{F}	{F}
f	{F, f}	{F}	{F, f}	{F}
t	{F, f}	{F, f}	{F, f}	{F}
T	{T, t}	{T, t}	{T}	{T}

The Matrices

α	$\Box^{\mathsf{KD}}\alpha$	$\Box^{\mathsf{KDB}}\alpha$	$\Box^{\mathsf{KD4}}\alpha$	$\Box^{\mathsf{KD5}}\alpha$	$\Box^{\mathsf{KD45}}\alpha$
F	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$
f	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$
f_2	$\{\mathsf{T}, \mathsf{t}, \mathsf{t}_2\}$	$\{\mathsf{t}_2\}$	$\{\mathsf{T}\}$	$\{\mathsf{T}, \mathsf{t}_2\}$	$\{\mathsf{T}\}$
t_2	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$
t	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}, \mathsf{f}, \mathsf{f}_2\}$	$\{\mathsf{F}\}$	$\{\mathsf{F}\}$
T	$\{\mathsf{T}, \mathsf{t}, \mathsf{t}_2\}$	$\{\mathsf{T}, \mathsf{t}, \mathsf{t}_2\}$	$\{\mathsf{T}\}$	$\{\mathsf{T}, \mathsf{t}_2\}$	$\{\mathsf{T}\}$

Definition (Level valuation in $\mathcal{M}_\star$)

Let $Val(\mathcal{M}_\star)$ be the set of valuation functions in $\mathcal{M}_\star$.

$\mathcal{L}_0(\mathcal{M}_\star)$ Every $v \in Val(\mathcal{M}_\star)$ where, if $\exists\alpha.v(\alpha) \in \{\mathrm{t}_1, \mathrm{f}_1\}$, then $\forall\beta, v(\beta) \in \{\mathrm{t}_1, \mathrm{f}_1\}$.

$\{\mathrm{t}_1, \mathrm{f}_1\}$ are stable

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$\mathcal{L}_{k+1}(\mathcal{M}_\star)$ Every $v \in \mathcal{L}_k$ such that, for every formula α, if $\models^{\mathcal{L}_k} \alpha$, then $v(\alpha) \in \{\mathsf{T}, \mathsf{t}_1\}$

Theorems are stable

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The set of level valuations in $\mathcal{M}_\star$ is given by

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- Level valuations are total functions.

Soundness and completeness

Theorem (Soundness)

$\Gamma \vdash^{\star} \alpha \Rightarrow \Gamma \models^{\mathcal{L}(\mathcal{M}_{\star})} \alpha$.

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$\Gamma \vdash^{\star} \alpha \Rightarrow \Gamma \vDash^{\mathcal{L}(\mathcal{M}_{\star})} \alpha.$

- We use a Hilbert system (defining $\vdash^{\star}$) for the given logic $\mathcal{L}$, for instance, for S5:

$$
\begin{array}{l}
\Gamma, p \vdash^{S5} p \\
\Gamma \vdash^{S5} (p \to (q \to p)) \\
\Gamma \vdash^{S5} ((p \to (q \to r)) \to ((p \to q) \to (p \to r))) \\
\Gamma \vdash^{S5} ((\neg p) \to (\neg q)) \to (q \to p) \\
\Gamma \vdash^{S5} (\Box(p \to q)) \to (\Box p) \to \Box q \\
\Gamma \vdash^{S5} \Box p \to p \\
\Gamma \vdash^{S5} \Box p \to \Box\Box p \\
\Gamma \vdash^{S5} p \to \Box\Diamond p \\
\text{if } \Gamma \vdash^{S5} (p \to q) \text{ and } \Gamma \vdash^{S5} p \text{ then } \Gamma \vdash^{S5} q \\
\text{if } \vdash^{S5} p \text{ then } \vdash^{S5} \Box p
\end{array}
$$

- The base case (axioms) for every logic can be found in: `https://github.com/nmatrices`

Theorem (Soundness)

$\Gamma \vdash^{\star} \alpha \Rightarrow \Gamma \vDash^{\mathcal{L}(\mathcal{M}_{\star})} \alpha.$

Theorem (Completeness)

For every modal logic $\mathcal{L}$ and associated Nmatrix $\mathcal{M}$, $\Gamma \vDash^{\mathcal{L}(\mathcal{M}_{\star})} \alpha \Rightarrow \Gamma \vdash^{\star} \alpha$.

Soundness and completeness

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For every modal logic $\mathcal{L}$ and associated Nmatrix $\mathcal{M}$, $\Gamma \vDash^{\mathcal{L}(\mathcal{M}_{\star})} \alpha \Rightarrow \Gamma \vdash^{\star} \alpha$.

- Henkin construction where characteristic functions are obtained directly from the meaning of the truth values.

$$v^{\mathcal{L}}_{\Delta}(\alpha) = \iota \text{ iff } \Delta \vdash^{\mathcal{L}} \iota(\alpha)$$

For instance, for the family KT$\star$:

$$v^{\mathcal{L}}_{\Delta}(\alpha) = \begin{cases} \mathsf{F} & \text{iff } \Delta \vdash^{\mathcal{L}} \Box\neg\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \neg\alpha \wedge \Diamond\neg\alpha) \\ \mathsf{f} & \text{iff } \Delta \vdash^{\mathcal{L}} \neg\alpha \wedge \Diamond\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \Diamond\neg\alpha) \\ \mathsf{t} & \text{iff } \Delta \vdash^{\mathcal{L}} \alpha \wedge \Diamond\neg\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \Diamond\alpha) \\ \mathsf{T} & \text{iff } \Delta \vdash^{\mathcal{L}} \Box\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \alpha \wedge \Diamond\alpha) \end{cases}.$$

Lemma (Adequacy)

For every logic $\mathcal{L}$ and maximally consistent set Δ, $v^{\mathcal{L}}_{\Delta}$ is a level valuation.

Outline

The Meaning of Values – the mission!

Consider a valuation v s.t $v(\alpha) = \mathsf{f}_2$:

- Recall: $\mathsf{f}_2(\alpha) = \Box\alpha \wedge \neg\alpha \wedge \Diamond\alpha$.

The Meaning of Values – the mission!

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- Hence, $\Diamond\alpha$ needs to be true.

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- It imposes a dependency on v: $\exists v' \ldots \rightsquigarrow vRv'$.
- Such v' must satisfy some extra requirements (due to $\Box$):
  - By nec: if $v(\beta) \in \mathcal{N}$ and vRv' then $v'(\beta) \in \mathcal{D}$
  - In, e.g., K4: if $v(\beta) \in \mathcal{N}$ and vRv' then $v'(\beta) \in \mathcal{N}$.

The Meaning of Values – the mission!

Consider a valuation v s.t $v(\alpha) = \mathsf{f}_2$:

- Recall: $\mathsf{f}_2(\alpha) = \Box\alpha \wedge \neg\alpha \wedge \Diamond\alpha$.
- Hence, $\Diamond\alpha$ needs to be true.
- This requires the existence of a valuation v' s.t. $v'(\alpha) \in \mathcal{D}$, thus fulfilling the requirement $\Diamond\alpha$.
- It imposes a dependency on v: $\exists v'. \ldots \rightsquigarrow vRv'$.
- Such v' must satisfy some extra requirements (due to $\Box$):
  - By nec: if $v(\beta) \in \mathcal{N}$ and vRv' then $v'(\beta) \in \mathcal{D}$
  - In, e.g., K4: if $v(\beta) \in \mathcal{N}$ and vRv' then $v'(\beta) \in \mathcal{N}$.

We systematically build a relational model for partial valuations (and show it sound and complete).

Outline

Property		**Condition**	**Implies**
nec		$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{D}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \notin \mathcal{D}$

Property		Condition	Implies
nec		$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{D}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \notin \mathcal{D}$

- Notation $\iota \Rightarrow^R V$: if $v(\alpha) = \iota$ and vRv', then $v'(\alpha) \in V$.

Property		Condition	Implies
nec		$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{D}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \notin \mathcal{D}$

- Notation $\iota \Rightarrow^R V$: if $v(\alpha) = \iota$ and vRv', then $v'(\alpha) \in V$.

$\mathsf{T} \Rightarrow^R \mathsf{T}, \mathsf{t}, \mathsf{t}_1, \mathsf{t}_2$

$$\begin{array}{rl} & \alpha \ldots \beta \ldots \\ v: & \ldots \mathsf{T} \ldots \mathcal{P}, \mathcal{PN} \ldots \\ & \quad \Downarrow_R \quad \downarrow_\exists \\ v': & \ldots \mathcal{D} \ldots \mathcal{D}, \mathcal{D}^{\complement} \ldots \end{array}$$

From Pre-models to K-Models

Property		Condition	Implies
nec		$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{D}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \notin \mathcal{D}$

- Notation $\iota \Rightarrow^R V$: if $v(\alpha) = \iota$ and vRv', then $v'(\alpha) \in V$.

$\mathsf{T} \Rightarrow^R \mathsf{T}, \mathsf{t}, \mathsf{t}_1, \mathsf{t}_2$

$$\begin{array}{c} \alpha \ldots \beta \ldots \\ v : \ldots \mathsf{T} \ldots \mathcal{P}, \mathcal{PN} \ldots \\ \Downarrow_R \quad \downarrow_\exists \\ v' : \ldots \mathcal{D} \ldots \mathcal{D}, \mathcal{D}^{\complement} \ldots \end{array}$$

$\mathsf{F} \Rightarrow^R \mathsf{F}, \mathsf{f}, \mathsf{f}_1, \mathsf{f}_2$

$$\begin{array}{c} \alpha \ldots \ \beta \ldots \\ v : \ldots \mathsf{F} \ldots \ \mathcal{P}, \mathcal{PN} \ldots \\ \Downarrow_R \quad \downarrow_\exists \\ v' : \ldots \mathcal{D}^{\complement} \ldots \mathcal{D}, \mathcal{D}^{\complement} \ldots \end{array}$$

Property		Condition	Implies
4	$\Box\alpha \to \Box\Box\alpha$	$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{N}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \in \mathcal{I}$

From K-Models to K4$\star$ Models

Property		Condition	Implies
4	$\Box\alpha \to \Box\Box\alpha$	$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{N}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \in \mathcal{I}$

Distinguished sets

- $\mathcal{N} = \{\mathsf{T}, \mathsf{t}_1, \mathsf{f}_2, \mathsf{f}_1\}$
- $\mathcal{I} = \{\mathsf{F}, \mathsf{f}_1, \mathsf{t}_2, \mathsf{t}_1\}$

From K-Models to K4$\star$ Models

Property		Condition	Implies
4	$\Box\alpha \to \Box\Box\alpha$	$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{N}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \in \mathcal{I}$

Distinguished sets

- $\mathcal{N} = \{\mathsf{T}, \mathsf{t}_1, \mathsf{f}_2, \mathsf{f}_1\}$
- $\mathcal{I} = \{\mathsf{F}, \mathsf{f}_1, \mathsf{t}_2, \mathsf{t}_1\}$

K	K4
$\mathsf{T} \Rightarrow^R \mathsf{T}, \mathsf{t}, \mathsf{t}_1, \mathsf{t}_2$	$\mathsf{T} \Rightarrow^R \mathsf{T}, \mathsf{t}_1$
$\mathsf{t}_2 \Rightarrow^R \mathsf{F}, \mathsf{f}, \mathsf{f}_1, \mathsf{f}_2$	$\mathsf{t}_2 \Rightarrow^R \mathsf{F}, \mathsf{f}_1$

The Whole Picture (The Recipe)

Property		Condition	Implies
nec		$w(\alpha) \in \mathcal{N}$ and wRw'	$w'(\alpha) \in \mathcal{D}$
		$w(\alpha) \in \mathcal{I}$ and wRw'	$w'(\alpha) \notin \mathcal{D}$
t	$\Box\alpha \to \alpha$	$w(\alpha) \in \mathcal{N}$	$w(\alpha) \in \mathcal{D}$
		$w(\alpha) \in \mathcal{I}$	$w(\alpha) \notin \mathcal{D}$
d	$\Box\alpha \to \Diamond\alpha$	$w(\alpha) \in \mathcal{N}$	$w(\alpha) \in \mathcal{P}$
		$w(\alpha) \in \mathcal{I}$	$w(\alpha) \in \mathcal{PN}$
b	$\alpha \to \Box\Diamond\alpha$	$w(\alpha) \in \mathcal{D}$ and wRw'	$w'(\alpha) \in \mathcal{P}$
		$w(\alpha) \notin \mathcal{D}$ and wRw'	$w'(\alpha) \in \mathcal{PN}$
4	$\Box\alpha \to \Box\Box\alpha$	$w(\alpha) \in \mathcal{N}$ and wRw'	$w'(\alpha) \in \mathcal{N}$
		$w(\alpha) \in \mathcal{I}$ and wRw'	$w'(\alpha) \in \mathcal{I}$
5	$\Diamond\alpha \to \Box\Diamond\alpha$	$w(\alpha) \in \mathcal{P}$ and wRw'	$w'(\alpha) \in \mathcal{P}$
		$w(\alpha) \in \mathcal{PN}$ and wRw'	$w'(\alpha) \in \mathcal{PN}$
	$\Box\Box\alpha \to \Box\Box\Box\alpha$	$w(\alpha), w'(\alpha) \in \mathcal{N}$, wRw' and (wRw'' or $w'Rw''$)	$w''(\alpha) \in \mathcal{N}$
		$w(\alpha), w'(\alpha) \in \mathcal{I}$, wRw' and (wRw'' or $w'Rw''$)	$w''(\alpha) \in \mathcal{I}$

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2}
\end{array}$$

(a) K

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{f}, \mathbf{f_2} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{f}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{t}, \mathbf{t_2} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{t}, \mathbf{t_2} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(b) KB

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t_1} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f_1} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t_1} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f_1}
\end{array}$$

(c) K4

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(d) K5

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(e) K45

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(f) KB45

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_2} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_2} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_2}
\end{array}$$

(g) KD

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{f}, \mathbf{f_2} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{f}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{t}, \mathbf{t_2} \\
\mathbf{f} & \Rightarrow^R & \mathbf{F}, \mathbf{t}, \mathbf{f}, \mathbf{t_2} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(h) KDB

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(i) KD4

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(j) KD5

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T} \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(k) KD45

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{F}, \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{T}, \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(l) KT

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{f} \\
\mathbf{f} & \Rightarrow^R & \mathbf{F}, \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(m) KTB

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{F}, \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{T}, \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(n) KT4

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(o) KTB45

The Whole Picture (The Dishes)

$$\begin{array}{lcl}\mathbf{T} & \Rightarrow^R & \mathbf{T},\mathbf{t},\mathbf{t}_1,\mathbf{t}_2\\ \mathbf{t}_1 & \Rightarrow^R & \bullet\\ \mathbf{t}_2 & \Rightarrow^R & \mathbf{F},\mathbf{f},\mathbf{f}_1,\mathbf{f}_2\\ \mathbf{f}_2 & \Rightarrow^R & \mathbf{T},\mathbf{t},\mathbf{t}_1,\mathbf{t}_2\\ \mathbf{f}_1 & \Rightarrow^R & \bullet\\ \mathbf{F} & \Rightarrow^R & \mathbf{F},\mathbf{f},\mathbf{f}_1,\mathbf{f}_2\end{array}$$

(a) K

$$\begin{array}{lcl}\mathbf{T} & \Rightarrow^R & \mathbf{T},\mathbf{t}\\ \mathbf{t} & \Rightarrow^R & \mathbf{T},\mathbf{t},\mathbf{f},\mathbf{f}_2\\ \mathbf{t}_1 & \Rightarrow^R & \bullet\\ \mathbf{t}_2 & \Rightarrow^R & \mathbf{f},\mathbf{f}_2\\ \mathbf{f}_2 & \Rightarrow^R & \mathbf{t},\mathbf{t}_2\\ \mathbf{f}_1 & \Rightarrow^R & \bullet\\ \mathbf{f} & \Rightarrow^R & \mathbf{F},\mathbf{f},\mathbf{t},\mathbf{t}_2\\ \mathbf{F} & \Rightarrow^R & \mathbf{F},\mathbf{f}\end{array}$$

(b) KB

$$\begin{array}{lcl}\mathbf{T} & \Rightarrow^R & \mathbf{T},\mathbf{t}_1\\ \mathbf{t}_1 & \Rightarrow^R & \bullet\\ \mathbf{t}_2 & \Rightarrow^R & \mathbf{F},\mathbf{f}_1\\ \mathbf{f}_2 & \Rightarrow^R & \mathbf{T},\mathbf{t}_1\\ \mathbf{f}_1 & \Rightarrow^R & \bullet\\ \mathbf{F} & \Rightarrow^R & \mathbf{F},\mathbf{f}_1\end{array}$$

(c) K4

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(d) K5

$$\begin{array}{lcl}\mathbf{T} & \Rightarrow^R & \mathbf{T}\\ \mathbf{t} & \Rightarrow^R & \mathbf{t},\mathbf{f}\\ \mathbf{t}_1 & \Rightarrow^R & \bullet\\ \mathbf{t}_2 & \Rightarrow^R & \mathbf{F}\\ \mathbf{f}_2 & \Rightarrow^R & \mathbf{T}\\ \mathbf{f}_1 & \Rightarrow^R & \bullet\\ \mathbf{f} & \Rightarrow^R & \mathbf{t},\mathbf{f}\\ \mathbf{F} & \Rightarrow^R & \mathbf{F}\end{array}$$

(e) K45

$$\begin{array}{lcl}\mathbf{T} & \Rightarrow^R & \mathbf{T}\\ \mathbf{t} & \Rightarrow^R & \mathbf{t},\mathbf{f}\\ \mathbf{t}_1 & \Rightarrow^R & \bullet\\ \mathbf{f}_1 & \Rightarrow^R & \bullet\\ \mathbf{f} & \Rightarrow^R & \mathbf{t},\mathbf{f}\\ \mathbf{F} & \Rightarrow^R & \mathbf{F}\end{array}$$

(f) KB45

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(g) KD

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(k) KD45

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(l) KT

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(m) KTB

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(n) KT4

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(o) KTB45

- According to the logic $\mathcal{L}$, the relation induced by $\Rightarrow^R$ is serial/reflexive/symmetric/transitive/Euclidian.

Outline

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Theorem (Analyticity (Procedure))

Every partial level-valuation can be extended to a level-valuation.

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Theorem (Analyticity (Procedure))

Every partial level-valuation can be extended to a level-valuation.

Theorem (Soundness)

For every $\mathcal{L}$, $\Gamma \vdash^{\mathcal{L}} \alpha \Rightarrow \Gamma \models^{\mathcal{L}} \alpha$.

Completeness of Partial Valuations

- We show that level valuations restricted to a (closed) domain are good partial valuations.

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For every level-valuation v and every set closed under subformulas Λ, $v \downarrow_\Lambda$ is a partial level-valuation.

Theorem (Completeness)

For every $\mathcal{L}$, $\Gamma \vDash^{\mathcal{L}} \alpha \Rightarrow \Gamma \vdash^{\mathcal{L}} \alpha$.

Outline

Concluding Remarks

Our Nmatrices were computed (and refined) with the aid of a Rocq procedure (impossible by hand!). E.g, axiom **K** in KTB45:

Info Command Prooftree Interrupt Restart Help

```
= ([! "p0"; ! "p1"; ! "p0" ~> ! "p1"; [] ! "p0"; [] ! "p1"; []
    (! "p0" ~> ! "p1"); [] ! "p0" ~> [] ! "p1";
    [] (! "p0" ~> ! "p1") ~> ([] ! "p0" ~> [] ! "p1")];
  [(12; [0; 0; 7; 0; 0; 7; 7; 7]); (24; [0; 1; 7; 0; 0; 7; 7; 7]);
   (36; [0; 6; 7; 0; 0; 7; 7; 7]); (47; [0; 7; 7; 0; 7; 7; 7; 7]);
   (59; [1; 0; 6; 0; 0; 0; 7; 7]); (71; [1; 1; 7; 0; 0; 7; 7; 7]);
   (83; [1; 1; 6; 0; 0; 0; 7; 7]); (95; [1; 6; 7; 0; 0; 7; 7; 7]);
   (107; [1; 6; 6; 0; 0; 0; 7; 7]); (118; [1; 7; 7; 0; 7; 7; 7; 7]);
   (130; [6; 0; 1; 0; 0; 0; 7; 7]); (142; [6; 1; 1; 0; 0; 0; 7; 7]);
   (154; [6; 6; 7; 0; 0; 7; 7; 7]); (166; [6; 6; 6; 0; 0; 0; 7; 7]);
   (177; [6; 7; 7; 0; 7; 7; 7; 7]); (188; [7; 0; 0; 7; 0; 0; 0; 7]);
   (199; [7; 1; 1; 7; 0; 0; 0; 7]); (210; [7; 6; 6; 7; 0; 0; 0; 7]);
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: pair (list LF) (list (pair nat (list nat)))
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       [(12; [0; 0; 7; 0; 0; 7; 7; 7]); (24; [0; 1; 7; 0; 0; 7; 7; 7]);
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Future work

Full mechanization of our proofs. Partial results for S5.

Concluding Remarks

Showing analyticity correct for KD requires checking more than 15M cases!

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Maude to the rescue:

```
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t]: (2 <- 1),
[A : f2, C : f2, NEW : t2]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t2]: (2 <- 1),
[A : f2, C : f2, NEW : T]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
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[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t]: (2 <- 1),
[A : f2, C : f2, NEW : t2]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t2]: (2 <- 1),
[A : f2, C : f2, NEW : T]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t2]: (2 <- 1),
[A : f2, C : f2, NEW : t]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t2]: (2 <- 1),
[A : f2, C : f2, NEW : t2]: (2 <- 1)))
```

- Automated theorem provers based on RN-matrices: Satisfiability Modulo Theories (SMT) problems – ask Carlos!!!
- Counter-models (Rocq tool)

Some reflections

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Applications of modal logic

- Hardware and Software Verification – Linear Temporal Logic (LTL) and Computation Tree Logic (CTL)
- Distributed Systems and Cryptography – Epistemic Logic: $K_i P =$ "Agent i knows P;" Common Knowledge $\rightsquigarrow$ agreement protocols.
- Game Theory and Multi-Agent Systems – Autonomous agents must make decisions based on what they think other players will do.
- Legal Tech and Compliance – Deontic Logic: $OP =$ "P is obligatory."
- AI and KR – Description Logics: The Semantic Web need a way to organize vast amounts of data conceptually.
- etc ☺

End of Part IV and this tutorial!



Thanks Zena, Marco and Pierre-Louis for the wonderful event!

Thanks Ana, Chandradeep and Stephanie for all the support and brilliant work!

End of Part IV and this tutorial!

Thank you all for the fantastic time we shared in Eugene!!

References

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