Modal Logic: Proof Theory, Semantics and Applications

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1 Substructural Games

1.1 Linear Logic [1]

Linear logic is a refinement of classical and intuitionistic logic, which emphasizes the role of formulas as **resources**, instead of emphasizing truth, as in classical logic, or proof, as in intuitionistic logic.

There are two different versions of each connective in linear logic: an additive version and a multiplicative version of conjunction and of disjunction.

On one hand, for the conjunction, we now have &, the additive and, and $\otimes$, the multiplicative and, with the following right introduction rules:

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \& B}\ \&_R \qquad \frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \otimes B}\ \otimes_R\ .$$

On the other hand, for the disjunction, we now have $\oplus$, the additive or, and $\parr$, the multiplicative or, with the following right introduction rules:

$$\frac{\Gamma \vdash \Delta, A_i}{\Gamma \vdash \Delta, A_1 \oplus A_2}\ \oplus_{R_i} \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \parr B}\ \parr_R.$$

This duplication of the connectives provides a clearer understanding of the conflict between classical and intuitionistic logic. For example, there are two readings of the excluded middle: the additive $(A \oplus \neg A)$ and the multiplicative $(A \parr \neg A)$. While the first is not provable and corresponds to the intuitionistic reading of disjunction, the second is trivially provable and simply corresponds to the tautology $A \to A$, which is perfectly acceptable in intuitionistic logic too.

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Once contraction and weakening are eliminated, formulas no longer behave as immutable truth values. For example, if we have a proof of $A \multimap B$ and a proof of A, we must consume them to obtain a proof of B. Thus, linear logic formulas behave like resources that can only be used once.

1.2 Modalities

To recover the full expressive power of intuitionistic and classical logic, we add two dual modalities (the exponentials), ! and ?, that allow us to arbitrarily duplicate or discard resources.

In particular, ! allows contraction and weakening to be applied to the formula $!B$ in the left-hand sequent context, while ? allows contraction and weakening to be applied to the formula $?B$ on the right-hand sequent context. This leads to the full propositional linear logic.

An online tool for building proofs in linear logic: Interactive linear logic prover.

1.3 Playing with substructural modalities

We will look at the game for the affine intuitionistic multiplicative additive LL ($\mathcal{C}$) (see slides [4], page 18 for a sequent system for $\mathcal{C}$). Formulas in this logic refer to resources that can be built from the atomic propositions, the connectives $\otimes$, $\&$, $\oplus$, $\multimap$, and the units 0 and 1. As before, we have the Proponent and the Opponent. The game states are multisets of sequents of the form $\Gamma \vdash F$. The Proponent acts as the scheduler, selecting a sequent S from the current state, a principal formula in S, and a matching rule instance.

The game proceeds in one of two ways based on the rule applied.

$$\begin{array}{lllll} (1) & G \cup \{S\} & \rightsquigarrow & G \cup \{S'\} \\ (2) & G \cup \{S\} & \rightsquigarrow & G \cup \{S_1\} \cup \{S_2\} \end{array}$$

While the person who gets the choice is similar to the games seen before, there is a difference between the additive and the multiplicative conjunction. With $A \& B$, P needs to play only the side O chooses, but they need to have the resources to play any of the sides. With $A \otimes B$, P needs to play both subgames and win.

Next, an early decision to pick an $\oplus$ on the left might be problematic, as O might be able to coerce the game into a state where the resources are no longer enough to satisfy the right.

For an example, see the slides [4], page 21.

1.4 Subexponentials

Instead of allowing arbitrary duplication using the exponential modalities (see below), subexponentials let us control the number of copies of a resource we can work with.

$$\frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\ ! \qquad \frac{!A_1, \ldots, !A_n \vdash A}{!A_1, \ldots, !\,A_n \vdash !A}\ !_R$$

Subexponential modalities involve labeling the exponential modalities with resource counts. We will use them to denote resource costs of a formula.

$$\frac{\Gamma, A \vdash C}{\Gamma, !^a A \vdash C}\ !^a{}_L \qquad \frac{!^{a_1} A_1, \ldots, !^{a_n} A_n \vdash A}{!^{a_1} A_1, \ldots, !^{a_n} A_n \vdash !^a A}\ !^a{}_R, \text{ provided } a \preceq a_i$$

We call this new system with costs $\in \mathbb{R}^+$ the system $\mathcal{C}(\mathbb{R}^+)$, and we extend our game to the game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ by tweaking the states slightly.

Instead of multisets of sequents, we now have tuples (H, b) where H is a finite multiset of $\mathbb{R}^+$ valued sequents and $b \in \mathbb{R}$ is a budget. The game now proceeds in the following way instead:

$$\begin{array}{llcl} (1) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S'\}, b') \\ (2) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S^1\} \cup \{S^2\}, b) \end{array}$$

If the rule is not $!_L$, then the game proceeds as before, with budget b. For $!_L$ with premise S and principal formula $!^a A$, the game proceeds to the game state $(G \cup \{S\}, b—a)$. To win the game, the Proponent needs to have a non-negative final budget.

For an example, see page 26 of the slides [4].

To preserve the budget information in the sequent rules, we add labels to the sequents. We call this the system $\mathcal{C}^l(\mathbb{R}^+)$. A $\mathcal{C}^l(\mathbb{R}^+)$-proof is built from labeled sequents of the form $b : \Gamma \vdash A$, where $\Gamma \vdash A$ is a $\mathcal{C}(\mathbb{R}^+)$ sequent and $b \in \mathbb{R}^+$.

Theorem 1.1. *Given a $\mathcal{C}(\mathbb{R}^+)$-proof Ξ of a sequent S, there exists a smallest budget with $\mathtt{cost}(\Xi)$ that suffices to win the game $\mathcal{G}_{\mathcal{C}}(\mathbb{R}^+)$ on S when following the strategy corresponding to Ξ.*

Definition 1.1 (Spectrum)**.** $\mathtt{spec}(S) := \{\mathtt{cost}(\Xi) \mid \Xi$ *is a* $\mathcal{C}(\mathbb{R}^+)$*-proof of* $S\}$.

Theorem 1.2. *If $\vdash_{\mathcal{C}(\mathbb{R}^+)} \Gamma \vdash A$, then $\mathtt{spec}(\Gamma \vdash A)$ has a least element. In other words, there is a smallest b such that $\vdash_{\mathcal{C}^l(\mathbb{R}^+)} \Gamma \vdash_b A$.*

For the sequent rules and an example, see pages 28 and 29 of the slides [4].

2 Truth Table Semantics[2, 3]

Classical propositional logic has a finite truth table: each connective is a function on finitely many truth values, and validity can be directly decided by checking all rows. Through lectures we have built up a rich theory of modal logic by other means, such as the modal cube, Kripke models, and the frame conditions corresponding to **D**, **T**, **B**, **4** and **5** (the corresponding theory). It is natural to ask next if modalities have the same truth table treatment.

The answer is no, and the rest of this section explains why, and then we explore at a high level how something table-like could be recovered.

2.1 Three values ($\mathsf{T}, \mathsf{F}, \mathsf{U}$) are not enough

The obstruction already shows up in intuitionistic propositional logic, which has a similar flavor: we can only assert what we can prove. With three values, true (T), false (F), and undetermined (U), intuitionistic logic faces the following dilemma. It's natural to define negation of the first two, $\neg\mathsf{T} = \mathsf{F}$ and $\neg\mathsf{F} = \mathsf{T}$, but what should $\neg\mathsf{U}$ be? Every choice is not ideal.

- $\neg\mathsf{U} = \mathsf{F}$. This is very tempting, since lacking a proof of a statement does not guarantee a proof of its impossibility. However, a formula that is not a theorem in intuitionistic logic gets designated as True on every row:
$$\neg\alpha \vee \neg\neg\alpha.$$
- $\neg\mathsf{U} = \mathsf{T}$. Similarly, the dual problem exists, and other non-theorem formulas become valid.
- $\neg\mathsf{U} = \mathsf{U}$. There are also formulas that should be valid but are no longer designated, for instance, $\neg(\alpha \wedge \neg\alpha)$.

The takeaway of the discussion is that a fixed three-valued table is too static: it cannot track the *evolving, proof-oriented* character of intuitionistic validity.

2.2 The negative results

Kurt Gödel (1932) showed that for every finite-valued matrix, there is a formula the matrix designates as true, but which is not valid in intuitionistic propositional logic (IPL). This concludes no finite table is sound and complete for IPL.

The failure transfers to modal logic. Since IPL is faithfully embedded in $\mathsf{S4}$, $\mathsf{S4}$ itself has no finite-valued table either. Dugundji extended this result to the whole modal cube: none of these modal logics can be captured by a finite matrix.

2.3 Non-determinism and levels

Avron's non-deterministic matrices. Although we may seem to be in a sort of dead end, there are remedies done by relaxing some requirements and designs. Avron's *non-deterministic matrices* (Nmatrices) generalize classical truth tables by allowing a connective to return a *set* of values rather than one. A valuation then picks some value from that set. For instance, we can let $\neg \mathsf{U} \rightsquigarrow \{\mathsf{U}, \mathsf{T}\}$ in IPL, so that the negation of an undetermined formula is free to be either undetermined or true. This relaxation is not enough, though: the intuitionistic theorem $\neg\neg(\alpha \vee \neg\alpha)$ can still take the value U, which is not designated, and hence the table is unsound.

The modal table. The same issue appears in modal logic. In S4, we have four values:

$$\mathbf{T}: \text{ necessarily true} \quad \mathbf{t}: \text{ true but not necessary} \quad \mathbf{f}: \text{ false but possible} \quad \mathbf{F}: \text{ necessarily false.}$$

The values $\mathbf{t}$ and $\mathbf{T}$ are *designated*. The modalities and implication are given in Table 1.

A	$\Box^{\mathsf{S4}}A$	$\Diamond^{\mathsf{S4}}A$
$\mathbf{F}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
$\mathbf{f}$	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{T}, \mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{T}, \mathbf{t}\}$
$\mathbf{T}$	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$

$\alpha \to \beta$	$\mathbf{F}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{T}$
$\mathbf{F}$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{T}, \mathbf{t}$	$\mathbf{T}, \mathbf{t}$	$\mathbf{T}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{T}, \mathbf{t}$	$\mathbf{T}$
$\mathbf{T}$	$\mathbf{F}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{T}$

Table 1: Nmatrix for S4: the modalities $\Box^{\mathsf{S4}}$ and $\Diamond^{\mathsf{S4}}$ (left), and implication (right).

Nmatrices are not sufficient. Non-determinism alone does not give a sound semantics in modal logic. Take the tautology $p \to p$. By necessitation rule, $\Box(p \to p)$ should be a theorem, so it should be designated. However, Table 2 shows the undesired result.

p	$p \to p$	$\Box(p \to p)$
$\mathbf{F}$	$\mathbf{T}$	$\mathbf{T}$
$\mathbf{f}$	$\mathbf{T}$	$\mathbf{T}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{F}, \mathbf{f}$
$\mathbf{t}$	$\mathbf{T}$	$\mathbf{T}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{F}, \mathbf{f}$
$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$

Table 2: Evaluating $\Box(p \to p)$ in the S4 Nmatrix.

The $\mathbf{f}$ and $\mathbf{t}$ rows have two values because $p \to p$ itself is non-deterministic there, taking value $\mathbf{T}$ or $\mathbf{t}$. Both are designated, so $p \to p$ should be true throughout. But $\mathbf{t}$ is true *without being necessary*, and $\Box^{\mathsf{S4}}\mathbf{t} = \{\mathbf{F}, \mathbf{f}\}$, so a valuation that gives the tautology the value $\mathbf{t}$ makes $\Box(p \to p)$ non-designated in these two red rows and breaks soundness.

Kearns' level valuations. Kearns' solution is to throw away the *bad* valuations that give a tautology a designated but non-necessary value. Since $p \to p$ is a tautology, a *good* valuation must give $\Box(p \to p)$ a designated value, which forces $p \to p$ onto $\mathbf{T}$ and removes the red rows. This *enforces* the necessitation rule.

The price of such change is that this is *not* a decision procedure. Selecting only the good valuations requires testing all tautologies, of which there are infinitely many, with no point at which the process is guaranteed to stop.

2.4 A uniform construction

Before the work[2], the state of art was uneven. Kearns' level valuations only covered 9 of the 15 logics as semantics, and Grätz turned just two of them, KT and S4, into a decision procedure. The construction presented in the lecture[5] extends this to all 15 logics of the cube in one systematic and modular way, recovering the earlier cases as instances. It rests on a uniform set of eight truth values, and the meaning of those values is what drives the whole development.

The work presented here extends the decision procedure to all 15 logics of the cube, constructed in a systematic and uniform way, with the earlier results recovered as special cases. The guiding idea is to fix what each truth value means and classify them correctly.

The construction was framed as a symphony in five movements, which there was no time to play. The development itself, together with the matrices and the soundness and completeness results, is deferred to the slides[5] and the paper[2, 3].

References

[1] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–101, 1987.

[2] Renato Leme, Carlos Olarte, Elaine Pimentel, and Marcelo Esteban Coniglio. The modal cube revisited: Semantics without worlds. In Gian Luca Pozzato and Tarmo Uustalu, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, pages 181–200, Cham, 2026. Springer Nature Switzerland.

[3] Renato R. Leme, Carlos Olarte, and Elaine Pimentel. Efficient decision procedures for rnmatrix semantics, 2026.

[4] Elaine Pimentel. Modal logic, part IV: Substructural games!, 2026.

[5] Elaine Pimentel. Modal logic, part IV: Truth!, 2026.