Constructing and counting plane embeddings of biconnected partial 2-trees
(Draft)

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Abstract

We describe the structure of the graph induced by the union of minimal separators for any 2-tree. We give a formula for counting different plane embeddings of a given biconnected partial 2-tree and an algorithm constructing all such embeddings. We then solve the face independent vertex cover problems on biconnected partial 2-trees.

1 Introduction

1.1 Motivation

Partial 2-trees constitute a nontrivial class of planar graphs that includes outerplanar graphs. Since they inherit the definitional property of maximal such graphs, the 2-trees, which states that every minimal separator consists of end vertices of an edge, they admit efficient algorithms solving many inherently hard problems on general graphs ([7, 1]). We consider the problem of enumerating plane

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embeddings of planar graphs, restricted to partial 2-trees. (The problem for outerplanar graphs has been solved in [10].) Solution of the problem is facilitated by the fact that the union of minimal separators of any 2-tree has a very distinct structure. This fact implies a one-to-one correspondence between the frames of partial 2-trees (outerplanar subgraphs pivotal for plane embedding) and the frames of the imbedding 2-trees. This intermediate result complements the study of interior graphs of maximal outerplanar graphs in [4].

We consider also the notion of restricted coverings of faces of a plane graph by vertices. This notion has been introduced in [8] and investigated in [10]. We solve the problem of finding such a covering for biconnected partial 2-trees.

1.2 Definitions

We will deal with simple, loopless combinatorial graphs. An edge is *incident* with its *end vertices* which are mutually *adjacent*. A *simple path* between two vertices u and v is a sequence of edges such that each of their end vertices (other than u or v) is incident with exactly two neighboring edges. If $u = v$, we have a *simple cycle*. A graph is *connected* if there is a path between any two of its vertices. In a connected graph, a subset S of vertices is a *separator* if its removal disconnects the graph. A tree is a connected acyclic graph. A graph G is *outerplanar* if there is an embedding of G in the plane such that all vertices lie on the boundary of the infinite region of the plane (the outer face). Thus, an outerplanar graph has no subgraph homeomorphic to the complete bipartite graph $K_{2,3}$. The set of *boundary cycles* consists of the boundaries of the faces, regions of the plane in a plane embedding. We identify a plane embedding of a graph with the set of its boundary cycles. A subgraph of G *induced* by a subset S of the vertices of G consists of S and all edges of G with both end vertices in S.

A *2-tree* is either the complete graph on 3 vertices (the triangle K_3) or a graph with $n > 3$ vertices obtained from a 2-tree G on $n-1$ vertices by adding a new vertex adjacent exactly to both end vertices of an edge of G. (An alternative definition involves construction of a 2-tree by a sum of two smaller 2-trees that have an edge in common.) A *partial 2-tree* is a subgraph of a 2-tree (it can be imbedded in a 2-tree) with the same set of vertices. For emphasis, we will call 2-trees *full*. Also, we will distinguish between an *embedding* of a planar graph in the plane and an *imbedding* of a partial 2-tree in a full 2-tree.

It is well known that a graph is a partial 2-tree iff it contains no homeomorph of K_4. Moreover, every minimal separator of a full 2-tree consists of the end vertices of an edge [6]. We will use the following classification of edges in a full or partial 2-tree H. An edge $e = (a, b)$ is called *exterior* if $\{a, b\}$ is not a separator of H, otherwise it is called *interior*. An edge $e = (a, b)$ is called *strongly interior* if the graph $H - \{a, b\}$ has more than two connected components and *weakly interior* otherwise. A strongly interior edge $e = (a, b)$ is *terminal* iff all but at most one of the graphs $G_i = H - \bigoplus_{j \neq i} C_j$ are outerplanar. (Here, C_i's are the connected components of $H - \{a, b\}$.)

Lemma 1 *Every non-outerplanar 2-tree H has a terminal strongly interior edge.*

Proof: If a 2-tree H is not outerplanar, it has a strongly interior edge since it contains a homeomorph of $K_{2,3}$. Assume that there is no terminal such edge. Then, there is a strongly interior edge e that separates H into at least two non-outerplanar components: C, that has the maximum size over all strongly interior edges and the corresponding components, and D. The latter has a strongly interior edge f separating H into connected components, one of which is non-outerplanar and properly includes C, thus contradicting the definition of C as maximum size. (This argument should give an intuition about the name of the terminal strongly interior edge.) ■

2 Interior graphs of 2-trees

Hedetniemi *et al.* ([4]) defined the interior graph of a maximal outerplanar graph (*mop*, for short) as the union of its interior edges. They have completely characterized the interior graphs of mops and showed that such a graph is a connected union of mops and caterpillars. We will obtain a similar result for partial 2-trees.

Lemma 2 *Any tree is the interior graph of some 2-tree.*

Proof: (by induction on the number of vertices.) By inspection, the lemma is true for $n = 2$ and $n = 3$ vertices. For $n \geq 3$, consider a tree T with $n + 1$ vertices. Unless $T = K_{1,n}$ ('a star') we can split T into smaller trees T_1 and T_2 by removing an edge e, so that $|T_i|+1 \leq n$ $(i = 1, 2)$. By the inductive hypothesis, each of the trees T_i augmented by e is the interior graph of a 2-tree G_i $(i = 1, 2)$. A 2-tree G obtained from G_1 and G_2 by identifying the copies of e in each of them has T as its interior graph. $K_{1,n}$ is the interior graph of the 2-tree with a universal vertex and $n + 2$ remaining vertices inducing a path ("a wheel without an external edge"). ■

Theorem 1 *A connected partial 2-tree H is the interior graph of some 2-tree if and only if it has no induced cycles of length greater than 3.*

Proof: (sufficiency) Any such H has biconnected components that are either edges or 2-trees. A 2-tree H_i is the interior graph of a 2-tree G_i obtained from H_i by adding a triangle (a vertex adjacent to both end vertices of an edge) to each exterior edge. To every edge H_i, add add two triangles. Given an articulation point v of H, choose from each component H_i of $H - v$ an original edge e_i incident to v and connect the other end vertices of these edges by a path. Perform this operation for all articulation points of the augmented graph whose biconnected components have as the interior graphs the corresponding biconnected components of H.This results in a 2-tree that has H as its interior graph.

(necessity) Removing all vertices of degree 2 from a partial 2-tree results in a partial 2-tree and does not introduce any induced cycles. ∎

The necessity result of [4] that the interior graph of a mop is a connected union of mops and caterpillars is an immediate corollary of Theorem 1 (since caterpillars are the only acyclic interior graphs of mops). However, the outerplanarity constitutes a nontrivial hinder for sufficiency of this condition.

3 Planar embeddings of partial 2-trees

A *frame* in a 2-tree H is a maximal (with respect to subgraph inclusion) outerplanar subgraph of H that does not contain any strongly interior edge as an interior edge. Any frame is also a mop. Strongly interior edges of a 2-tree partition it into frames (if one allows multiple copies of those edges).

We will first prove that a partial 2-tree has the same plane embeddings (modulo embeddings of its frames) as any full 2-tree that imbedds it. Since the frames are outerplanar and the plane embeddings problem for outerplanar graphs has been solved ([10]), solving the problem for full 2-trees will imply the solution for partial 2-trees.

3.1 Imbeddings of partial 2-trees

Lemma 3 *A biconnected partial 2-tree G contains all exterior edges of any full 2-tree imbedding H with the same set of vertices.*

Proof: Removal of an exterior edge from a 2-tree introduces an articulation point. If there were an imbedding H of G missing an exterior edge, it would be separable and so would be any partial graph of H. This contradicts biconnectivity of G. ∎

Strongly interior edges of a 2-tree H partition H into maximal outerplanar components (frames) in the following sense: In every non-outerplanar 2-tree, there is a terminal strongly interior edge, say

$e = (a, b)$. Add the outerplanar graphs G_i (connected components C_i of $H - \{a, b\}$ augmented by $\{a, b\}$ and the adjacent edges) to the set of frames and remove the corresponding components C_i to obtain a 2-tree H'. Repeat the operation until only one edge remains.

Lemma 4 *Any 2-tree imbedding H of a biconnected partial 2-tree G contains the same set of strongly interior edges.*

Proof: The lemma follows from the uniqueness of the set of exterior edges: If (a, b) is a strongly interior edge in an imbedding H of G, then the removal of $\{a, b\}$ disconnects G into more than two components. Since $H - \{a, b\}$ consists of at least three connected components, G contains three disjoint paths between x and y. Moreover, there is no 2-tree imbedding of G in which any two of the three paths would be connected outside a, b, since this would implythe existence of a subgraph homeomorphic to K_4. Thus, $\{a, b\}$ is a separator in any 2-tree imbedding of G and the lemma holds by the characteristic property of 2-trees. ∎

3.2 Representation of plane embeddings of full 2-trees

We will now define a unique graph representing a full 2-tree G. This *associated graph* $D(G)$ is the intersection graph of triangles of G over the set of edges. Thus, the set of nodes of $D(G)$ is the set of triangles (maximal cliques) of G, and the set of edges of $D(G)$ represents the set of edges of G that are in at least two triangles. An example can be found in Fig. 3.2.

Figure 1: G and its associated graph $D(G)$

It can be easily verified that each node v of $D(G)$ is in at most three maximal cliques. Furthermore, there are at least two nodes that belong to exactly one maximal clique. We will call such nodes (and the corresponding triangles of G) *pendant*. Furthermore, there is at least one maximal clique in $D(G)$ with at most one non-pendant node. We will call such a clique *pendant*, as well.

We will now give an algorithm constructing a tree that represents a planar embedding of G.

Select a pendant node r from a pendant clique C as the *root* of $D(G)$. To represent a plane embedding of G, traverse $D(G)$ in a breadth-first fashion starting at r. This traversal determines a breadth-first tree T_r of G (Fig. 3.2b). When arriving at any node v of T_r during the traversal (crossing, in G, an edge of the triangle v), partition the children of v into two subsets $\mathrm{In}(v)$ and $\mathrm{Out}(v)$, depending on whether the corresponding triangles are inside or outside the triangle v in the particular drawing of the plane embedding of G. These nodes belong to at most two maximal cliques that also contain v. (The two cliques correspond to the other two edges of the triangle v of G.) Subdivide further the nodes in $\mathrm{In}(v)$ (respectively $\mathrm{Out}(v)$) into $\mathrm{In}_1(v)$ and $\mathrm{In}_2(v)$ (respectively $\mathrm{Out}_1(v)$ and $\mathrm{Out}_2(v)$) depending on to which maximal clique they belong. Order nodes in each of the four sets $\mathrm{In}_1(v)$, $\mathrm{In}_2(v)$, $\mathrm{Out}_1(v)$, $\mathrm{Out}_2(v)$ according to the relationship of the inclusion in the plane between the corresponding triangles of G. Each order defines a path in $D(G)$. T_r together with the root r and ordered subdivisions $\mathrm{In}_1(v)$, $\mathrm{In}_2(v)$, $\mathrm{Out}_1(v)$, $\mathrm{Out}_2(v)$ for every node v in $D(G)$ will be called an *in-graph* of $D(G)$ *rooted at* r (Fig. 3.2c). It will be denoted by $\vec{D}_r$.

Figure 2: $D(G)$, its breadth-first tree T_r and its in-graph $\vec{D}_r$

When drawing in-graphs as in Fig.3.2c, we will use open (respectively bold) circles to indicate nodes in Out-subsets (respectively In-subsets). The order in the subsets will be indicated by directed solid branches. The remaining branches will be undirected and dashed.

Lemma 5 *Every in-graph $\vec{D}_r$ defines a plane embedding of G.*

Proof. The embedding associated with $\vec{D}_r$ is obtained in the following manner (Fig. 3.2):

- Draw a triangle r.

- Traverse the nodes of T_r in any 'parent first' order. When arriving at a node v (could be r), draw in the nested fashion triangles corresponding to $\mathrm{In}_1(v)$ and $\mathrm{In}_2(v)$ (respectively $\mathrm{Out}_1(v)$ and $\mathrm{Out}_2(v)$) inside (respectively outside) the triangle v. The ordering of triangles is given by the directed paths (first node corresponding to the outermost triangle). It is always possible to place triangles without violating the planarity of the already embedded subgraph. ■

Figure 3: $\vec{D}_r$ and the corresponding embedding.

We will now investigate the existence of planar embeddings of a given 2-tree with a specified pendant triangle as the root face.

For a given embedding of a 2-tree G and a pendant triangle r of G that is a face in this embedding, there are exactly two in-graphs rooted in r representing that embedding. One of these represents a drawing of G with r being the exterior face ('the outermost triangle'). The other one will be called *standard* and represents the *standard drawing* of G, in which the rest of the graph is on the outside of r (and thus the set $\mathrm{In}(r)$ in the in-graph $\vec{D}_r$ is empty).

In any given plane embedding of a 2-tree G, there is a pendant triangle that is in the set of boundary cycles.

Observation 1 *For a given 2-tree G and a pendant clique C of $D(G)$, any plane embedding of G has one or two pendant triangles of C as faces*

Moreover, the in-graphs rooted at all pendant triangles of C include all possible plane embeddings of G.

Lemma 6 *Given a full 2-tree G and a pendant clique C of $D(G)$. Every plane embedding of G can be represented by an in-graph $\vec{D}_r$, where r is a pendant triangle of C.*

Proof. Let us fix a pendant clique C of $D(G)$. For every plane embedding of G, there is a pendant triangle r of C that is a face in the embedding. An in-graph $\vec{D}_r$ can be constructed in the obvious manner that reverses the procedure used in the proof of Lemma 5 and assumes that G is drawn in the standard way with the given set of faces. ■

For a maximal set S of pendant triangles of a pendant clique in $D(G)$, any two embeddings corresponding to standard in-graphs rooted at different pendant triangles of S are different unless both triangles are simultaneously faces in the embeddings. This property of embeddings can be translated into a property of in-graphs by defining a bijection ϕ among in-graphs.

Let r and s be pendant triangles of a pendant clique of $D(G)$, both faces in the given embedding of G. We define $\phi(\vec{D}_r)$ as $\vec{D}_s$ such that if $\text{Out}'(r) = \langle s, T_1, \ldots, T_i \rangle$, then $\text{Out}''(s) = \langle r, \bar{T}_i, \ldots, \bar{T}_1 \rangle$, where for each node T in $\text{Out}(r)$ of $\vec{D}_r$, the corresponding node $\bar{T}$ in $\text{Out}(s)$ of $\vec{D}_s$ has reversed the roles of In and Out: $\text{In}''(\bar{T}) = \text{Out}'(T)$ and $\text{Out}''(\bar{T}) = \text{In}'(T)$ ($'$ and $''$ denote functions in $\vec{D}_r$ and $\vec{D}_s$, respectively).

Lemma 7 *If $\vec{D}_r$ and $\vec{D}_s$ are two different standard in-graphs rooted at r and s, respectively, then they generate different sets of faces unless $\vec{D}_r = \phi(\vec{D}_s)$ (or $\vec{D}_s = \phi(\vec{D}_r)$).*

Proof: It can be verified by inspection, that if $\vec{D}_r = \phi(\vec{D}_s)$ then the set of faces in the respective embeddings are identical, since the mapping corresponds to a cyclic shift of the components of $G - \{a, b\}$, where (a, b) is the edge common for all triangles of C.

For the implication in the other direction, let us assume that two different embeddings of G are represented by standard in-graphs. Without loss of generality, we may assume that they are rooted at pendant triangles of the same pendant clique C of $D(G)$. The assumption of the in-graphs being in the standard form insures that two different in-graphs with the same pendant triangle as the root represent two different embeddings. There are two cases, one when the embeddings differ in the embedding of the non-triangular component of $G - C$, and the other, when the difference is in the faces involving the triangles of C. The former case can be treated by induction. In the latter, the mapping ϕ captures the only situation when the boundary cycles are identical. ■

3.3 Counting planar embeddings

For a planar graph G, let $\pi(G)$ be the number of plane embeddings on the sphere (*i.e.,* embeddings with different sets of boundary cycles). Let $\pi'(G)$ be the number of plane embeddings of G when the outer face containing a specified edge is distinguished. (It is easy to see that this number is independent of the particular edge chosen. When the graph G has at least 2 faces, then $\pi'(G) = 2\pi(G)$ since every edge is in exactly two faces.)

Lemma 8 *Let $e = (a, b)$ be an interior edge of a 2-tree H with l components C_i of $H - \{a, b\}$. Let $H_i = H - \bigoplus_{j \neq i} C_j$. Then*

$$\pi(H) = \frac{1}{2} l! \prod_{1 \leq i \leq l} \pi'(H_i)$$

Proof: We will use the idea of permuting the components H_i to produce all embeddings of H while avoiding duplication by omitting embeddings related in a manner similar to the mapping ϕ of the preceding subsection. The proof will follow by induction on l:

(i) $l = 2$. Assume that H is drawn with a vertical edge e separating C_1 'on the left of e' from C_2 'on the right' and consider given embeddings of H_1 and H_2. The same embedding of H can be found among plane drawings of H with C_1 and C_2 on one side of e. On the other hand, every embedding of H_1 and H_2 with the distinguished 'outer side' of e contributes multiplicatively a new set of boundary cycles. Thus, $\pi(H) = \pi'(H_1) \cdot \pi'(H_2) = \frac{1}{2} l! \prod \pi'(H_i)$.

(ii) $l > 2$. Let x be an end vertex of e. For each H_i, choose an arbitrary edge e_i incident with x. An embedding of H is uniquely given by the position of e_l in the ordering of e_i around x and the given embeddings of the H_i's ($1 \leq i < l$) fixing the 'outer side' of e. Assuming $\pi(H - C_l) = \frac{1}{2}(l-1)! \prod_{1 \leq i < l} \pi'(H_i)$, each embedding of H_l contributes multiplicatively to the number of different sets of boundary cycles and the above observation proves the desired formula, since there are l possible positions for e_l. ■

Note that the outerplanar case (solved in [10] as 2^{f-2}, where $f > 1$ is the number of interior faces of the graph) is a simple corollary of Lemma 8, since every separating edge of an outerplanar graph gives $l = 2$ and the absence of such an edge gives the base case of $\pi(H) = 1$. Since all mops of a given size have the same number of interior faces, the number of plane embeddings of a mop is completely determined by its size.

Lemma 9 *Given a biconnected partial 2-tree G, the number of embeddings on the sphere is the same for every 2-tree H imbedding G.*

Proof: By Lemma 3, any two 2-tree imbeddings of G differ at most on some subset of weakly interior edges. Yet, the sizes of the corresponding frames are identical. Since the frames of a 2-tree are maximal outerplanar, it follows by Lemma 8 that the number of planar embeddings of H is determined by the size of frames of H interacting through strongly interior edges of H. These are identical for all imbeddings of G. ■

From these lemmas follows immediately a formula counting the number of planar embeddings for a partial 2-tree with minimal separators that induce edges.

Theorem 2 *Let $\{a, b\}$ be a separator of a biconnected partial 2-tree G with l components C_i of $G - \{a, b\}$. Let $G_i = G - \bigoplus_{j \neq i} C_j$. Then*

(i) If (a, b) is an edge of G, then $\pi(G) = \frac{1}{2} l! \prod_{1 \leq i \leq l} \pi'(G_i)$.

(ii) Otherwise,

$$\pi(G) = \frac{1}{2}(l-1)! \prod_{1 \leq i \leq l} \pi'(G_i)$$

Proof: (i) Since it is almost identical with the proof of Lemma 8, we omit it.

(ii) Let $\{a, b\}$ be a minimal separator of G without the corresponding edge (as we assume that the number of connected components of $G - \{a, b\}$ is at least 3, (a, b) is an edge in any 2-tree imbedding H of G). We notice that in this case, the $l \geq 3$ components can be "permuted" in $\frac{1}{2}(l-1)!$ ways. (The following does not cover the case of $l = 2$. But then one can either find a strong separator in the above sense or G is outerplanar.) Each subgraph G_i of G will be defined as $G - \bigoplus_{j \neq i} C_j$ *augmented* by the edge (a, b). (In the previous case of the separator inducing an edge, the edge $e = (a, b)$ acts as an extra component.) ■

4 Independent covers

Let $G = \langle V, E \rangle$ be a biconnected planar graph embedded in the plane. A subset S of vertices is called a *face-independent vertex cover* (or *FIVC*, for short) of the faces (boundary cycles) of G if every face of G has exactly one vertex in S. A set W of faces of a graph H is called a *vertex-independent face cover* (or *VIFC*, for short) if every vertex of H is in exactly one face of W. A *VIFC* in H is simply a 2-factor of H which consists of facial cycles. In the geometric dual G^* of G, a *FIVC* of G corresponds to a set of faces of G^* which is a *VIFC* of the vertices of G^*. The problem of finding *FIVC* and *VIFC* are NP-complete in general, see [3, 2]. When restricted to outerplanar graphs, these problems are polynomially solvable, see [8].

4.1 Perfect FIVCs for Full 2-Trees

In this section, we describe a linear time algorithm that, given a 2-tree G, finds a plane embedding of G that admits a *FIVC*, or decides that no such embedding exists. In fact, the algorithm can be easily modified to yield a minimum cardinality such a cover, if it exists. The algorithm follows a similar approach as that of [9]. Namely, it processes in a bottom-up manner a breadth-first search spanning tree structure of the associated graph $D(G)$. To construct such a tree, we will use a standard embedding of G by fixing a pendant triangle r of an arbitrary pendant clique C of $D(G)$.

We will consider several cases of possible standard embeddings of G, depending on the number of pendant triangles in C and its parity (whether the number is even or odd). Let (a, b) be the edge of G common for all triangles of C and, for $1 \leq i \leq l$, let x_i be the third vertex of such a triangle. Let G_i be the components of $G - \{a, b\}$ augmented by (a, b) and the adjacent edges $(a, x_i), (b, x_i)$. Let G_1 denote r. If there is a non-triangular component, we denote it G_l. We will denote by G_l^a its component separated from x_1 by $\{a, x_l\}$ and by G_l^b the component separated from x_1 by $\{b, x_l\}$. The other triangles of C are subscripted according to the eventual relative position of the third vertex in an embedding that admits a *FIVC* (*cf.* Figure xx). In each case of the analysis below, the assumed position of G_l with respect to the vertices x_i (and, specifically, x_{l-1}) included in a *FIVC* of G will imply existence of a covering of G_l that possibly does not cover a face that includes the edge (a, b). (We will call such coverings *partial*.)

0. All components G_i are triangles. The obvious *FIVC* consists of one of the vertices a or b.

Otherwise, there is a non-triangular component G_l.

1. There is a *FIVC* of G that includes a or b.

    1. If there is a *FIVC* that includes a, it implies the existence of a *FIVS* of G_l^a that includes a and of a partial covering of G_l^b that does not cover exactly one face that includes the edge (a, b).
    2. If there is a *FIVC* that includes b, it implies the existence of a *FIVC* of G_l^b that includes b and of a partial covering of G_l^a that does not cover exactly one face that includes the edge (a, b).

If none of the vertices a or b are in the *FIVC*, any *FIVC* of G must include the third vertex, x_1, of the root triangle and, circularily following, every other vertex x_i $(1 < i < l)$. We will consider the two parity cases of l.

2. There is an even number of triangles G_i. Depending on the embedding of G_l we have:

   1. If G_l is embedded (a, b) and x_{l-1}, then there must exist a *FIVC* of G_l.
   2. If G_l is embedded elsewhere, then there must be a partial covering of G_l that does not cover either of its faces that include the edge (a, b).

3. There is an odd number of triangles G_i.

   Whatever is the position of G_l's embedding relative to the other triangles, it can be embedded so that its exterior face is covered by an x_i. The existence of a *FIVC* of G implies the existence of a partial covering of G_l omitting exactly one face that includes the edge (a, b).

The above analysis shows a need for a definition of several types of (partial) *FIVC*s of plane embeddings of a 2-tree. These are the following families of *FIVC*s:

$A(G)$ that include a.

$B(G)$ that include b.

$C(G)$ that do not include either a or b.

Partial covers of the following families do not include either a or b:

$X(G)$ cover all but one face that includes the edge (a, b).

$N(G)$ cover all but the faces that include the edge (a, b).

The case analysis provides an informal proof for the following theorem, that justifies a procedure for finding an embedding of G that admits a *FIVC*; in fact, a minimum cardinality such a cover. This procedure will follow in a bottom-up manner a breadth-first search spanning tree structure of the associated graph $D(G)$.

Theorem 3 *Let G be a given a 2-tree and r be a pendant triangle of a pendant clique C of G with l triangles. The family of FIVCs of G is $A(G) \cup B(G) \cup C(G)$ where, if C contains a non-triangular component G_l, the families of FIVCs can be found by solving the following recurrence relation:*

$$
\begin{aligned}
A(G) &= \{\alpha \cup \beta : \alpha \in A(G_l^a) \wedge \beta \in N(G_l^b)\} \\
B(G) &= \{\alpha \cup \beta : \alpha \in N(G_l^a) \wedge \beta \in B(G_l^b)\} \\
C(G) &= \{\ \alpha \cup \beta : \\
&\qquad \alpha = \{x_{2i-1} : 1 \leq i \leq \tfrac{l-1}{2}\} \wedge odd(l) \wedge \beta \in C(G_l) \vee \\
&\qquad \alpha = \{x_{l-1}\} \wedge \{x_{2i-1} : 1 \leq i \leq \tfrac{l+1}{2}\} \wedge odd(l) \wedge \beta \in N(G_l) \vee \\
&\qquad \alpha = \{x_{2i-1} : 1 \leq i \leq \tfrac{l}{2}\} \wedge even(l) \wedge \beta \in X(G_l)\} \\
X(G) &= \{\alpha \cup \beta : \alpha \in A(G_l^a) \wedge \beta \in X(G_l^b)\} \\
N(G) &= \{\alpha \cup \beta : \alpha \in A(G_l^a) \wedge \beta \in X(G_l^b)\}
\end{aligned}
$$

with the following initial conditions for a triangle $G = (a, b, c)$:

$$
\begin{aligned}
A(G) &= \{a\} \\
B(G) &= \{b\} \\
C(G) &= \{c\} \\
X(G) &= undefined \\
N(G) &= \emptyset
\end{aligned}
$$

■

Any node for which the above six covers have been determined is said to be *labeled*. Hence, all leaf nodes of $\vec{D}_r$ are initially the only labeled nodes. Assume that an unlabeled node v is chosen such that all its sons in $\vec{D}_r$ are labeled. What is $I(v)$? Clearly, if both $\mathrm{In}_1(v) = \emptyset$ and $\mathrm{In}_2(v) = \emptyset$, then $I(v)$ is undefined; it is impossible to cover the face corresponding to v. Suppose that the face corresponding to v is covered by a vertex from a triangle corresponding to a node from $\mathrm{In}_1(v)$. When looking for $I(v)$, we can then assume that $\mathrm{Out}_1(v) = \emptyset$ and $\mathrm{In}_2(v) = \emptyset$. If this is not the case, we can consider the embedding with $\overline{\mathrm{In}_v^1} = \mathrm{In}_1(v) \cup \mathrm{Out}_1(v)$, $\overline{\mathrm{Out}_v^1} = \emptyset$, $\overline{\mathrm{In}_v^2} = \emptyset$, $\overline{\mathrm{Out}_v^2} = \mathrm{Out}_2(v) \cup \mathrm{In}_2(v)$. Any $I(v)$ for the original embedding will also be a face-independent vertex cover for the new embedding with all but the exterior face and vice versa. We can in the following assume that either $\mathrm{In}_2(v) = \mathrm{Out}_1(v) = \emptyset$ or $\mathrm{In}_1(v) = \mathrm{Out}_2(v) = \emptyset$.

Suppose that $\mathrm{In}_1(v) = \mathrm{Out}_2(v) = \emptyset$. Let $\mathrm{Out}_1(v) = \{x_1, x_2, \ldots, x_k\}$

and $\mathrm{In}_2(v) = \{y_1, y_2, ..., y_l\}$. Assume that this ordering actually admits $I(v)$. Thus, $G_p^{x_1}$ in this embedding must be covered by either $I(x_1)$ or $B(x_1)$. If $G_p^{x_1}$ is covered by $I(x_1)$, then $G_p^{x_2}$ must be covered by either $\mathrm{In}(x_2)$ or $B(x_2)$. If $G_p^{x_1}$ is covered by $B(x_1)$, then $G_p^{x_2}$ must be covered by either $F(x_2)$ or $E(x_2)$. If $G_p^{x_2}$ is covered by $F(x_2)$, then $G_p^{x_3}$ must be covered by either $F(x_3)$ or $E(x_3)$. If $G_p^{x_2}$ is covered by $E(x_2)$, then $G_p^{x_3}$ must be covered by either $I(x_3)$ or $B(x_3)$. Note that $G_p^{x_k}$ must be covered by either $I(x_k)$ or $E(x_k)$. Hence, the covering sequence of $x_1, x_2, ..., x_k$ must be formed as a path in the following tree:

```
                          /----> I(x_3) :::::
         /-----> I(x_2)
        /                 \----> B(x_3) :::::
I(x_1)
        \                 /----> F(x_3) :::::
         \-----> B(x_2)
                          \----> E(x_3) :::::

                          /----> F(x_3) :::::
         /-----> F(x_2)
        /                 \----> E(x_3) :::::
B(x_1)
        \                 /----> I(x_3) :::::
         \-----> E(x_2)
                          \----> B(x_3) :::::
```

which ends in either $I(x_k)$ or $E(x_k)$.

Suppose that $I(x_i)$, $2 \leq i \leq k$, is preceeded by something else than $I(x_{i-1})$ (i.e. it is preceeded by $E(x_{i-1})$). But then we can place $I(x_i)$ in front without affecting $I(v)$. Hence, we can assume that $I(x_i)$ occurs only in the beginning of the covering sequence.

By turning $I(x_i)$ inside-out, we obtain a face-independent vertex cover of all but the face corresponding to x_i. Hence, $|I(x_i)| \geq |F(x_i)|$. By turning $F(x_i)$ inside-out, we obtain a face-independent vertex cover of all but the exterior face. Hence, $|I(x_i)| \leq |F(x_i)|$. In conclusion, $|I(x_i)| = |F(x_i)|$. Consequently, we can assume that no x_i, $1 \leq i \leq k$ is covered by $F(x_i)$. If it was, we could place x_i in front and cover it by $I(x_i)$.

In view of the above discussion, the covering sequence of $x_1, x_2, ..., x_k$ is a path in:

```
                      /----> I(x_3) :::::
       /-----> I(x_2)
      /               \----> B(x_3) :::::
I(x_1)
      \
       \-----> B(x_2)
                      \----> E(x_3) :::::




B(x_1)
      \
       \-----> E(x_2)
                      \----> B(x_3) :::::
```

which ends in either $I(x_k)$ or $E(x_k)$.

Given these restrictions, how can we determine $I(v)$. Suppose first that k is even. Consider a complete network with $x_1, x_2, ..., x_k$ as its vertices. Associate the cost

$$\min\{|I(x_i)| + |I(x_j)|, |B(x_i)| + |E(x_j)|, |E(x_i)| + |B(x_j)|\}$$

with every edge (x_i, x_j) of this network. We only need to solve the minimum cost perfect matching problem to find $I(v)$.

If k is odd, we have to take each x_i, $1 \leq i \leq k$ in turn, assume it is covered by $I(x_i)$, and find how to cover other subgraphs by solving the minimum cost perfect matching problem on the complete graph with $k - 1$ vertices.

Let us now look how to cover the embedding on the other side. $G_p^{y_1}$ must be covered by either $F(y_1)$ or $E(y_1)$.+ ....

```
                      /----> F(y_3) :::::
       /-----> F(y_2)
      /               \----> E(y_3) :::::
```

```
F(y_1)
      \              /----> I(y_3) :::::
       \-----> E(y_2)
                     \----> B(y_3) :::::

                     /----> I(y_3) :::::
       /-----> I(y_2)
      /              \----> B(y_3) :::::
E(y_1)
      \              /----> F(y_3) :::::
       \-----> B(y_2)
                     \----> E(y_3) :::::
```

which ends in either $I(x_k)$ or $E(x_k)$.

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