Minimum Self-Repairing Graphs

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Abstract

A graph is self-repairing if it is 2-connected and such that the removal of any single vertex results in no increase in distance between any pair of remaining vertices of the graph. We completely characterize the class of minimum self-repairing graphs, which have the fewest edges for a given number of vertices.

1 Preliminaries

In this paper, we investigate the class of graphs which we call *self-repairing*. These are graphs for which the removal of a single vertex results in no increase in distance between any pair of remaining vertices. Study of such graphs is motivated by issues of reliability of communication networks. A site failure in a network with topology described by such a graph would not increase communication time (see, for instance, [3, 4]). We will show that the minimum number of edges in a self-repairing graph with n vertices is $2n-4$ and determine the class of all such minimum size self-repairing graphs.

By a *graph* we will understand a simple undirected graph $G = \langle V, E\rangle$, consisting of a set V of vertices and a set E of edges, each edge connecting a pair of vertices (v_1, v_2). For a given vertex v, let $i_G(v)$ represent the set

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of edges *incident* to v in graph G, *i.e.*, those for which v is a member of the edge pair. Two vertices are *adjacent* if they have a common incident edge. Similarly, two edges are adjacent if they have a common incident vertex. For a given vertex v of graph G, let $N_G(v)$ denote the *open neighborhood* of v, being the set of vertices adjacent to v. Let $N_G[v]$ denote the *closed neighborhood* of v, being $N_G(v) \cup \{v\}$.

For a given pair of vertices, u and v, in G, let the distance between them, $dist_G(u, v)$ be the length of a shortest path from u to v in G. Two adjacent vertices are at a distance 1 and are said to be *neighbors*. A graph is *connected* if there exists a path between every pair of vertices. A graph is *2-connected* if for every vertex v of G, the graph $G' = \langle V - v, E - i(v)\rangle$ is connected. In other words, removing any vertex of G does not disconnect the graph. Obviously, a self-repairing graph must be 2-connected. Various properties of k-connected graphs, for k =2 and 3, have been defined and studied previously [5, 6, 2].

In the following, we will refer by name to the *three-cycle* (cycle of three vertices), the four-cycle (cycle of four vertices) and the *cube graph* (the 8 vertex, 12 edge skeleton of the cube) and the complete bipartite graph $K_{2,k}$ having $k + 2$ vertices, two of which are adjacent to k other vertices and have no other adjacencies (see Figure 1).

We first establish a property of the neighborhoods of vertices in aself-repairing graph.

Theorem 1: A graph G is self-repairing if and only if, for every vertex v and for each pair of vertices $u, w \in N_G(v)$, $N_{G'}[u] \cap N_{G'}[w] \neq \emptyset$, where $G' = \langle V - \{v\}, E - i_G(v)\rangle$.

Proof: The necessity of the condition follows easily for any two vertices $u, w \in N_G(v)$, for if $N_{G'}[u] \cap N_{G'}[w] = \emptyset$ then $dist_{G'}(u, w) > dist_G(u, w) = 2$.

For sufficiency consider a pair of vertices, x and y in G', such that there exists a shortest path in G between x and y that includes v. Then we must show that there exists a path in G' of the same length. The shortest path through v in G contains edges (u, v) and (v, w), for some vertices $u, w \in N_G(v)$, where there are zero or more vertices between u and x and between y and w. By our assumption as to the non-empty intersection of neighborhoods of u and w in G', there exist vertex v' and edges (u, v') and (v', w) in G' (u and w are not adjacent because the (x, y) path has the shortest length). Thus, $dist_G(x, y) = dist_{G'}(x, y)$. ∎

We can restate Theorem 1 as the following corollary in terms of the maximum possible length of a minimum cycle involving a pair of neighbors of any vertex in a self-repairing graph.

Corollary: G is a self-repairing graph if and only if, for each vertex v in G and for each pair of vertices $u, w \in N_G(v)$, vertices u, v, and w are members of a cycle of length at most 4 in G, *i.e.*, a three- or a four-cycle. ∎

2 Main result

We want to characterize self-repairing graphs with minimum number of edges. We first define the class of twin graphs, which are self-repairing .

Two vertices x and y in a graph G are *twins* if and only if $N_G(x) = N_G(y)$. Based upon this notion of twin vertices, we define *twin graphs* recursively, as follows: (i) the four-cycle is a twin graph; (ii) if G is a twin graph, then the graph G' constructed by connecting a new vertex by two edges to a pair of twins in G is a twin graph.

Note that, when a new vertex is connected to twins x and y, vertices x and y remain twins in G'. In fact, once a pair of twin vertices have degree higher than 2, the two vertices must remain unique twins of each other in any twin graph constructed from this smaller graph.

Fact: In a twin graph, vertices of degree greater than 2 occur in uniquely defined pairs of twins. ∎

Theorem 2: A twin graph G is self-repairing.

Proof: Let G be a twin graph. Each vertex has degree 2 or higher. Let a vertex v of degree 2 be removed from G to form G'. In G, v is connected to twin vertices x and y, which, by definition, share a non-empty neighborhood of vertices in G'.

Now, consider v having degree greater than 2. Such a vertex has a unique twin vertex w (by the Fact above). As such, every neighbor of v in G is also a neighbor of w. Thus, if v is removed to form graph G', the intersections of neighborhoods (in G') of all pairs of vertices in $N_G(v)$ include w.

It follows that G is self-repairing, by Theorem 1. ∎

Twin graphs with n vertices have $2n - 4$ edges since the four-cycle has $2n-4$ edges and 2 edges are added for each new vertex connected to the graph. We will show that, together with the cube graph, twin graphs constitute exactly the class of minimum size self-repairing graphs.

In our proofs of the tight lower bound on the number of edges in a self-repairing graph, we will use the notion of *level graph* of a given graph G with respect to (*wrt.*) a fixed vertex x. Vertices of G are arranged in levels, depending on their distance from the vertex x. Given a vertex x in a graph G, we assign a *level* to each vertex by defining *level*$(x) = 0$ and *level*$(y) = i > 0$ to every vertex y at distance i from x (see Figure 2(a)).

Lemma 1: In a level graph of a self-repairing graph, any vertex at level $i > 1$ is adjacent to at least two vertices at level $i - 1$.

Proof: Consider a vertex v at level $i > 1$. It must have a neighbor x at level $i - 1$, which in turn has a neighbor y at level $i - 2$. Since v, x and y are in a four-cycle, v must have another neighbor at level $i - 1$. ∎

With the fact that self-repairing graphs are 2-connected, Lemma 1 gives us the following lemma.

Lemma 2: In a level graph of a self-repairing graph, each level $i > 0$, except for the maximum level, contains at least two vertices. ∎

We want to prove that there are no self-repairing graphs of n vertices with fewer than $2n - 4$ edges. We will proceed by assuming to the contrary and proving that such a graph cannot have a vertex of degree 2. This in turn will be used to show that no such graph exists.

Lemma 3: A self-repairing graph G with n vertices and fewer than $2n - 4$ edges has no degree 2 vertices.

Proof: Assume that there is a degree 2 vertex x in G. We will consider the levels defined *wrt.* x. By Lemma 1, every vertex at level $i > 1$ is adjacent to at least 2 vertices at level $i - 1$, which totals at least $2(n - 3)$ edges. The vertex x and its neighbors induce at least 2 edges. Thus G has at least $2n - 6 + 2 = 2n - 4$ edges, a contradiction. ∎

Not having any degree 2 vertices, a self-repairing graph G with n vertices and fewer than $2n-4$ edges must have some degree 3 vertices. We will show that this cannot happen. Thus, there are no such graphs.

Lemma 4: In a level graph of a self-repairing graph G with n vertices and fewer than $2n-4$ edges, where levels are defined *wrt.* a degree 3 vertex, every vertex at level $i > 1$ has exactly two neighbors at level $i-1$ and no neighbors at level i.

Proof: By Lemma 3, such a graph G does not have a degree 2 vertex and thus must have a degree 3 vertex x. We will consider the levels defined *wrt.* x. Lemma 1 requires that there are at least 2 level $i-1$ neighbors of a level $i > 1$ vertex; thus the $n-4$ vertices of $G - N[x]$ account for at least $2n-8$ edges. This is also the maximum number of edges incident with vertices at levels $i > 1$ if G is to have fewer than $2n-4$ edges, since at least 3 edges are induced by $N[x]$. Thus every vertex at level $i > 1$ has exactly two neighbors at level $i-1$ and no neighbors at level i ∎

Theorem 3: The minimum number of edges in a self-repairing graph of n vertices is $2n-4$.

Proof: In a self-repairing graph G with n vertices and fewer than $2n-4$ edges, every vertex at level $i > 1$ has exactly two neighbors at level $i-1$ and no neighbors at level i (by Lemma 4). This implies that vertices on the maximum level defined *wrt.* x have degree 2. This contradicts the result of Lemma 3, which states that there are no degree 2 vertices in G. Thus, there are no self-repairing graphs with n vertices and fewer than $2n-4$ edges. ∎

We have seen that twin graphs are self-repairing and reach the lower bound on the number of edges. We will show that, except for the cube graph, there are no other minimum self-repairing graphs. Here again, we will consider level graphs of minimum self-repairing graphs defined *wrt.* a vertex of degree 2 or 3.

Lemma 5: A minimum self-repairing graph with more than 4 vertices and having a degree 2 vertex is a twin graph.

Proof: Assume to the contrary and consider a smallest minimum self-repairing graph G, with a degree 2 vertex x, that is not a twin graph. In the level graph of G defined *wrt.* x, every vertex at level $i > 1$ is adjacent to exactly 2 vertices at level $i-1$ and has no neighbors at level i. By Lemma 1, the number of level $i-1$ neighbors for such a vertex is at least 2 and x is incident with 2 edges for a total of $2(n-3)+2=2n-4$ edges. The two neighbors of x have identical neighborhoods (x and all level 2 vertices) and thus are twins. Therefore, x can be removed from G resulting in a minimum self-repairing graph G'. G' has a degree 2 vertex (as discussed in the proof of Theorem 3). Thus, G' is a twin graph and so is G,which contradicts our assumption. ∎

Lemma 6: A minimum self-repairing graph without degree 2 vertices has no three-cycle involving a degree 3 vertex.

Proof: Assume to the contrary that three vertices induce a cycle. Consider the levels defined *wrt.* a degree 3 vertex x from this cycle. The remaining $n-4$ vertices account for at least $2n-8$ edges which together with the 4 edges induced by x and its neighbors gives $2n-4$ edges. Thus, the vertices at the maximum level must have degree 2, a contradiction. ∎

Lemma 7: The only minimum self-repairing graph with no degree 2 vertices is the cube graph.

Proof: Such a graph G with n vertices has $2n-4$ edges and thus at least 8 degree 3 vertices. Consider the level graph of G defined *wrt.* a vertex v of degree 3. By an edge counting argument similar to that in the proof of Lemma 6, only one vertex at any of the levels $i>1$ can have 3 neighbors at level $i-1$, or only two vertices at the same level $i>1$ can be neighbors. By the absence of degree 2 vertices, the above conditions could apply only to the maximum level, k. In fact, vertices at level k must satisfy one of these two conditions, by the same edge counting argument.

If there were two adjacent vertices of degree 3 at level k, they would have to be adjacent to four different level $k-1$ vertices (by Lemma 6). This and the absence of other edges between vertices at the same level (Lemma 4) violates the requirement that the vertices of any two adjacent edges in a self-repairing graph belong to a four-cycle.

Assume therefore that the maximum level k consists of exactly one degree 3 vertex u. The level graph of G *wrt.* to u has the same connectivity properties as the level graph of G *wrt.* to v. Thus, every vertex at level i, $0<i<k-1$, (*wrt.* v) has exactly two neighbors at level $i+1$. This implies that all intermediate levels have 3 vertices and the same pattern of adjacencies with vertices at the neighboring levels. Graphs with vertices at three (or

more) intermediate levels violate the self-repairing property. Suppose there exist three consecutive levels, $i-1, i, i+1$, with three vertices each. A vertex x at level $i-1$ is connected to two vertices at level i, which have only one common neighbor at level $i+1$. Neither of the other two vertices is in a four-cycle with x. Thus, $k \leq 3$. For $k = 2$ we have $K_{2,3}$, a twin graph, and for $k = 3$ we have the cube graph. ■

Lemmas 6 and 7 imply our main result.

Theorem 4: A minimum self-repairing graph is either a twin graph or the cube graph. ■

3 Conclusions

We have presented a constructive characterization of the class of minimum self-repairing graphs. With one exception, the class is identical with the class of twin graphs, which are thus useful as graphs underlying communication networks immune to certain element failures ([3, 4]). It is easy to see that twin graphs can be recognized by an efficient algorithm. Such a linear time algorithm is based on iterated removal of degree 2 vertices adjacent to vertices of identical neighborhoods. There is an additional advantage of the twin graph recognition algorithm, pertaining to the graph as the topology of a self-repairing communication network, [4]. Since the recognition process reverses a possible iterative construction process, if a self-repairing communication protocol (*i.e.,* routing tables) was not established during a graph's construction, it is possible to create correct entries during the recognition process.

There is a fairly obvious bijective relation between the set of free trees with m internal nodes and k leaves and the set of twin graphs with k degree 2 vertices and m pairs of twins of higher degree (*cf.* Figure 2). The existence of a tree describing a given twin graph indicates the possibility of an efficient algorithmic treatment of these graphs. We exploit here the concept of the *treewidth* of a graph (*cf.* [7, 1]) which is defined by its *tree-decomposition*: a tree with vertex subsets as nodes such that every vertex is in some nodes, for every edge there is a bag containing its end vertices, and for every vertex, the set of nodes containing it induces a (connected) subtree of the tree-decomposition. The treewidth is one less than the maximum size of a node in a tree-decomposition minimizing this size.

Given a twin graph, the above mentioned bijection determines a tree-decomposition with at most 4 vertices per node: each degree 2 vertex is in a node together with its twin vertices, and each pair of twin vertices of higher degree is in a node with twins to which they were connected in the construction process. Thus, twin graphs have treewidth at most 3.

Actually, only twin graphs $K_{2,i}$, $i > 2$, (represented by trees of star shape) have treewidth 2. (This follows from the property of such graphs that excludes subgraphs homeomorphic to K_4.) We notice that, though most twin graphs have treewidth 3, their minimal separators are of size 2. Thus, many optimization algorithms for twin graphs, guided by the tree-decomposition of the input graph, should be more efficient than the corresponding algorithm for generic treewidth 3 graphs (see, for instance [1]).

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