ON HALIN GRAPHS*

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ABSTRACT

The paper deals with a subfamily of those planar graphs which have outerplanar intersection of their MacLane cycle basis. These graphs have been known as Halin graphs. Their connectivity properties, structure of cycles, and feasible embeddings in the plane are discussed here. This paper also presents some initial investigations of NP-complete problems restricted to the family of Halin graphs.

1. INTRODUCTION

In view of the apparent intractability of the NP-complete problems [3], a popular approach to such problems is to restrict their domain. Several algorithmic problems on graphs remain NP-complete even when restricted to planar graphs. Our research has been motivated in part by the existence of such problems.

Almost all problems which are hard even for planar graphs become easy when their domain is further restricted to acyclic graphs: trees and forests. This is in part due to the fact that many hard problems for arbitrary graphs deal with cycles. Also, several parameters associated with graphs can be easily calculated in the absence of cycles. Notable exceptions from this rule are, NP-complete even for trees, the subgraph isomorphism problem and the bandwith problem.

*This paper presents results of research done when the first author was visiting Department of Computer and Information Science at the University of Oregon in June 1980.

Restricting the domain of NP-complete problems on planar graphs to outerplanar graphs often yields algorithmically tractable problems. It has been shown that all standard questions such as characterization, coding, isomorphism testing, and counting have simple answers for outerplanar graphs. Also the two following hard problems have straightforward solutions in that case. (i) The Hamiltonian cycle: outerplanar graphs are at most 2-connected, and every 2-connected outerplanar graph has exactly one Hamiltonian cycle. (ii) The chromatic number: if an outerplanar graph has at least one edge and is not bipartite, then its chromatic number is 3. Both problems have linear time solution algorithms.

Recently, we were able to construct an efficient algorithm solving the minimum dominating cycle problem for outerplanar graphs [7]. We also classified the subgraph isomorphism problem for these graphs [10]. The (induced) subgraph isomorphism problem is to find whether a graph G is an (induced) subgraph of another graph H. When G is a forest and H is a tree, this problem is NP-complete [3]. We have proved that it remains NP-complete, even with the strongest connectivity constraints imposed on both graphs, when they are outerplanar. The same result holds for the induced subgraph isomorphism problem for outerplanar graphs, except when both graphs are 2-connected. In that case, there exists a polynomial time algorithm which verifies the isomorphism condition [10].

The decreased difficulty of general problems when restricted to outerplanar graphs seems to be brought about by the close relation of such graphs to acyclic graphs. We define the following two ways of assigning a tree to an outerplanar graph. Let G be a 2-connected outerplanar graph embedded in the plane, with all vertices belonging to the exterior face. We call such an embedding of G an *outerplane graph*. If G^* is the geometric dual of G, and v denotes the vertex of G corresponding to the exterior face of G, then $G^w = G^* \setminus v$ has no cycles. Graph G^w is called a *weak dual* of G [2]. Independently, graph G^w has been introduced in [9] as the intersection graph of the interior faces of G over the set of its edges. If G is a 2-connected plane graph, then the set C_p of the interior faces of G forms a cycle basis of G. It has been called the *plane cycle basis*, or *MacLane basis* of G. G^w is called in [9] the *cycle graph of G with respect to* C_p.

There exists a 1-1 correspondence between the interior faces of G and the vertices of G^w. Also interior edges of G are mapped uniquely onto the edges of G^w. However, the exterior face and exterior edges of G are not represented in G^w. They appear in another tree associated with an outerplane graph G. Instead of removing v from G^*, let us split this vertex into a number of copies, one for each edge incident with v in G^*. We denote the resulting plane tree by G^t and call it the *associated tree of G* [7]. The plane tree associated with an outerplanar graph determines the graph up to the isomorphism.

We refer the reader to the literature (for instance, [7], [8], [9]) for detailed description of relations between outerplanar graphs and trees. The main goal of this paper is to explore a class of graphs which are related to outerplanar graphs as outerplanar graphs are related to trees. In the next section we shall consider this class of planar graphs with which we will uniquely associate outerplanar graphs. We hope that this relationship between outerplanar graphs and the new class of graphs will aid us in studying problems algorithmically hard for general planar graphs.

2. HALIN GRAPHS

Let G be a 2-connected outerplane graph and let G^w and G^t denote its weak dual and the associated tree, respectively. Since G is a simple graph, neither of the two derived graphs contains vertices of degree 2. Let G^s denote the plane graph obtained from G^t by adding edges between each pair of end-vertices (leaves) of G^t such that their pendant edges represent adjacent edges of the exterior face of G. This corresponds to constructing a cycle through all the leaves of G^t. Graph G^s has been called a *skirted tree* by Malkevitch [6] and a *Halin graph* by Bondy [1]. (Halin [4] considered these graphs as minimally 3-connected.)

Every plane tree without vertices of degree 2 is the associated tree of some 2-connected outerplane graph. Therefore, Halin graphs may be defined alternatively as graphs which can be obtained from plane trees by the skirting construction described above. We will consider Halin graphs as plane, i.e., embedded in the plane. The Halin graph built on a tree T is sometime denoted by $H(T)$, and T is called its *interior tree.*

Unlike the outerplanar graphs which can be 2-connected, strictly 1-connected, or disconnected, all Halin graphs are strictly 3-connected.

PROPOSITION 2.1. *Every Halin graph is strictly 3-connected.*

Proof. Let H be a Halin graph and T be its interior tree. H can be at most 3-connected since it contains vertices of degree 3. We have to show that if u and v are two vertices of H, then there exist in H three vertex-dosjoint paths connecting u and v.

One of these paths is the unique path p connecting u and v in T. If both u and v belong to the exterior cycle of H (i.e., are leaves of T) then they partition the exterior cycle into two paths, both disjoint from p. Otherwise, let us assume, that u is an interior vertex of T. It has degree at least 3 and is the only common element of otherwise disjoint three or more subtrees of T. Thus, there exist two disjoint paths connecting u with two leaves of T, and also disjoint from p. Let us call

these leaves u' and u''. Similarly, if v is an interior vertex of T, there exist paths (v,v') and (v,v'') connecting it with two leaves v' and v''. Because of the separation property, v' and v'' lie outside a path (u',u'') along the exterior cycle; let us assume that paths (u',v') and (u'',v'') along the exterior cycle of H are disjoint. Then the paths: p, (u,u',v',v), and (u,u'',v'',v) are mutually disjoint.□

If G is a 2-connected outerplane graph and G^s denotes the Halin graph constructed from G, then G is the cycle graph of G^s with respect to its MacLane cycle basis.

PROPOSITION 2.2. *A planar 3-connected graph is a Halin graph if and only if it has a MacLane cycle basis* C_p *in which every cycle has an exterior edge.*

Proof. The necessity of this condition follows immediately from the definition. To prove its sufficiency, let us assume that H is a 3-connected plane graph with a MacLane cycle basis C_p in which every cycle has an exterior edge. Since H is 3-connected, every cycle in C_p has at most one exterior edge. It also follows from the connectivity condition that the removal of all exterior edges (i.e., edges of the entire exterior face) does not disconnected H, but breaks all its cycles. Therefore, H without the exterior edges is a tree. By 3-connectivity of H, every non-leaf vertex of this tree has degree greater than 2.□

COROLLARY 2.3. *The intersection graph of the MacLane cycle basis of a Halin graph over its set of edges is a 2-connected outerplanar graph.*

A counterexample to the statement converse to that of Corollary 2.3 is given by the graph in Figure 1. It is a plane, 3-connected graph, not a Halin graph, and the intersection of its plane cycle basis is outerplanar. Therefore, the class of Halin graphs is only a subfamily of the family of all 3-connected plane graphs whose cycle graphs are outerplanar.

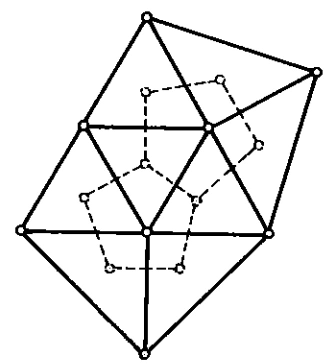

Fig. 1. A graph and its outerplanar weak dual.

Halin graphs were introduced as an example of a family of minimally 3-connected graphs.

PROPOSITION 2.4. *The removal of any edge or vertex of a Halin graph results in a graph which is at most 2-connected.*

Proof. Let H be a Halin graph. Each face of H shares exactly one edge with the exterior face of H. The removal of an interior edge or vertex merges two or more interior faces into one that has at least two edges in common with the exterior one. Next, the removal of an exterior edge causes some interior faces to share another edge with the expanded exterior face. If we remove an exterior vertex then two faces are merged with the exterior one and vertices of degree two appear. In all the cases, the resulting graph has two edges which disconnect it, therefore it is at most 2-connected.□

One can easily modify the above proof to show the following corollary.

COROLLARY 2.5. *Every proper subgraph of a Halin graph is at most 2-connected.*

3. CYCLES IN HALIN GRAPHS

The exterior face of a Halin graph H has size $\kappa(H) = m-n+1$, where m and n are the number of edges and vertices of H, respectively. In a 3-connected planar graph, removal of any face results in a connected graph. Also, in such graphs the faces of their plane embeddings are unique. Hence the following characterization of Halin graphs.

PROPOSITION 3.1. *A planar 3-connected graph H is a Halin graph if and only if one of its faces is of size $\kappa(H)$.*

COROLLARY 3.2. *If H is a Halin graph, then $\kappa(H)$ is the size of its largest face.*

Proof. If H has a face of size $k > \kappa(H)$, then removal of its edges would leave less than n-1 edges, thusly disconnecting H, a contradiction.□

Bondy proved (unpublished, see [5]) that all Halin graphs are 1-*Hamiltonian*, i.e., both H and $H\backslash v$ have Hamiltonian cycles, for any vertex v of H. Skowrońska (private communication) also proved hamiltonicity of $H\backslash v$. Her proof exploits the following properties of Halin graphs.

PROPOSITION 3.3. *Every edge of a Halin graph H belongs to a Hamiltonian cycle of H.*

PROPOSITION 3.4. *In a Halin graph H, every two adjacent edges, of which at least one is an exterior edge, belong to a Hamiltonian cycle of H.*

In the above Proposition 3.4, one of the edges must be exterior. For instance, two spokes of a wheel (which is a Halin graph) may not belong to its Hamiltonian cycle.

It follows from the existence of Hamiltonian cycles in Halin graph that every such graph has a dominating cycle. The algorithmic problem of finding a minimum-length such cycle is still open. The Hamiltonian cycle problem remains NP-complete even when restricted to 3-regular, 3-connected planar graphs [3]. However, when restricted to 3-connected Halin graphs, this problem is trivial.

Bondy and Lovász proved (unpublished, see [1] and [5]) that every Halin graph with n vertices is *almost pancyclic*, i.e., it contains cycles of all sizes between 3 and n except, possibly, one even size. This result is in a sense best possible, as showed by the following statement due to Malkevitch [6].

PROPOSITION 3.5 ([6]). *If m is any even integer ($m \geq 4$), then there exists a 3-regular Halin graph which is almost pancyclic of order m (i.e., has no cycle of size m).*

Malkevitch conjectured that this singular cycle size causing incomplete pancyclicity disappears for higher degrees of the interior tree of a Halin graph.

CONJECTURE 3.6. *Every k-regular Halin graph ($k \geq 4$), i.e., a Halin graph with all interior vertices of degree k, is pancyclic.*

The bound on k is necessary, as Figure 2 shows it.

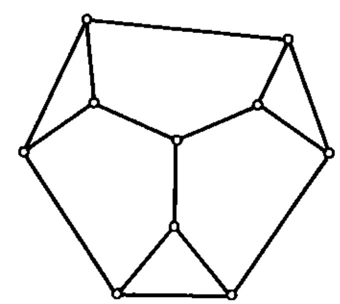

Fig. 2. A Halin graph with no cycle of length 4.

4. FEASIBLE EMBEDDINGS OF HALIN GRAPHS

An embedding of a planar graph is called *H-feasible* if the resulting plane graph is a Halin graph. We show that only few graphs have more than two H-feasible embeddings, and characterize the class of graphs with exactly two such embeddings.

PROPOSITION 4.1. *A planar graph can have at most 4 different H-feasible embeddings.*

P r o o f. Let H be a planar graph which has an H-feasible embedding. Since H is 3-connected, the set of faces of H is unique. If H has an H-feasible embedding, it has a face of size $\kappa(H)$. This is the number of interior faces of H, which equals n^*-1, where n^* is the number of vertices of H^*, the geometric dual of H. Thus, if H has k H-feasible embeddings, then H^* has k vertices of degree n^*-1, and it therefore contains K_k as a subgraph. Since H^* is planar, we have $k \leq 4$.□

COROLLARY 4.2. *The only graph which is a Halin graph in four of its embeddings is* K_4.

P r o o f. If G is a Halin graph with 4 H-feasible embeddings, then G^* contains K_4 as an induced subgraph. Since G^* is planar, there is no other vertex adjacent to all vertices of this K_4, and thus $n^* = 4$. Hence, G^* is isomorphic to K_4, and so is G.□

PROPOSITION 4.3. *The only Halin graph with exactly three H-feasible embeddings is the geometric dual of* $K_5 \backslash e$ *(where e is an edge of* K_5*)*.□

P r o o f. The dual graph G^* of a graph G with 3 H-feasible embeddings has three vertices of degree n^*-1. Any such graph with $n^* \geq 6$ would contain $K_{3,3}$ as a subgraph, therefore $n^* \leq 5$. For $n^* = 4$ we have $G = K_4$, and for $n^* = 5$ $G^* = K_5 \backslash e$. Graph G for this case is shown in Figure 3.□

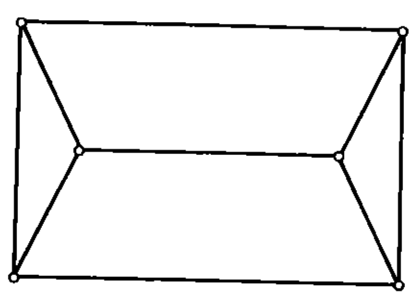

Fig. 3. The graph with three H-feasible embeddings.

To identify graphs with exactly two H-feasible embeddings we notice that the geometric dual of such a graph has two vertices adjacent to all other vertices which must form a path. Let us call a graph H with such a dual a *necklace*. Figure 4 shows such a graph and its dual.

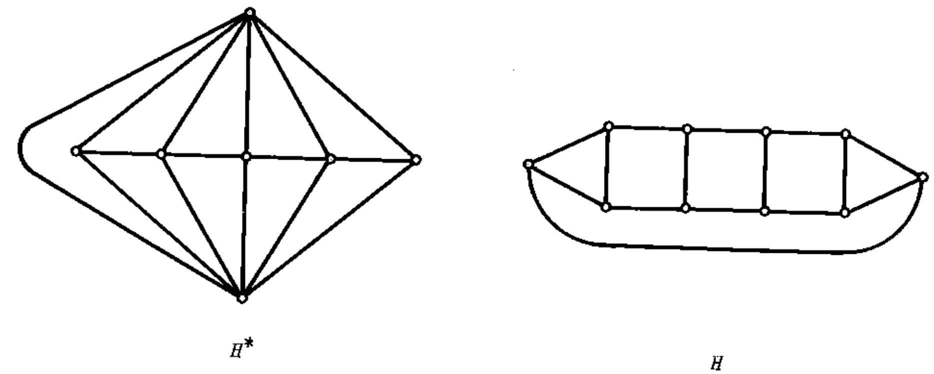

Fig. 4. A graph with two H-feasible embeddings and its dual.

PROPOSITION 4.4. *The only graphs with exactly two H-feasible embeddings are necklaces.*

All other Halin graphs have exactly one H-feasible embedding. The connectivity and the face structure of Halin graphs makes it possible to test efficiently whether a planar graph is a Halin graph, and also to test isomorphism of Halin graphs.

5. CONCLUSIONS

In this paper we have presented some properties of Halin graph. These graphs form a subfamily of graphs with outerplane intersection of MacLane cycle bases. We anticipate that further research will address the following issues: (i) asymptotic behavior of algorithmically difficult problems when they are restricted to Halin graphs, and (ii) generalization of these results to all graphs with outerplanar intersection graphs of MacLane cycle bases.

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