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Minimal Broadcast Networks

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Broadcast refers to the process of message dissemination in a communicatio[n] network whereby a message, originated by one member, is transmitted to all members of the network. A minimal broadcast network is a communication network in which a message can be broadcast in minimal time regardless of originator. This paper describes several classes of minimal broadcast networks. An algorithm is presented which constructs minimal broadcast networks which have approximately the minimum number of lines possible.

Introduction

Broadcast refers to the process of message dissemination in a communication network whereby a message, originated by one member, is transmitted to all members of the network. We model a communication network with a graph $G = (V,E)$ consisting of a set V of vertices, or members, and a set E of edges, or communication lines. We assume that all lines have equal cost (length). Such an assumption is reasonable for a model of a local, microcomputer network or a computer with parallel processors. A member can transmit a message to any adjacent vertex by making a call. We assume that each call requires one unit of time, and that a member can make only one call during any time unit.

Given a set of n members, three numbers are of interest with respect to the process of broadcast and the construction of optimal communication networks for broadcast. These are:

1) *the minimum number of calls required to complete the broadcast process,*
2) *the minimum number of time units required to complete the broadcast process,*
3) *the minimum number of communication lines such that the minumum time is required to complete the broadcast process, regardless of which member is message originator.*

The first two numbers are easily determined. The minimum number of calls required to broadcast a message from the originator to $n-1$ other members is clearly $n-1$. All broadcast networks and calling schemes to be discussed here require only the minimum number of calls. It can be shown by a straightforward inductive and (mo + 1) mod n calls (mo+1+2) mod n. Generally, during time unit t, each informed member i, calls uninformed member $(i + 2^{t-1}) \bmod n$. This process continues for $\lceil \log_2 n \rceil$ time units. For $n \neq 2^k$ (k a positive integer), only $n - 2^{\lfloor \log_2 n \rfloor}$ of the $2^{\lfloor \log_2 n \rfloor}$ members which are informed prior to the last time unit need place calls during the last time unit. Let these be the message originator and its $n-2^{\lfloor \log_2 n \rfloor}-1$ nearest clockwise neighbors. Figure 2 shows the calls which would be made under this calling scheme during a message broadcast originated by member 5 in a network with 12 members. Each directed arc represents a call made by the tail member to the head member. Each arc is labelled by an integer indicating the time unit during which the call would be made.

The above calling scheme can be used to construct a class of minimal broadcast networks. In a minimal broadcast network, each member must be able to serve equally well as message originator. In other words, each member must be able to place the necessary call during each potential time unit. Therefore, each member must have communication lines to $\lceil \log_2 n \rceil$ other members in order that the above calling scheme always realize the minimum time. A member, i, must have communication lines defined by the following set of edges

$$\left\{ (i,j) \mid j = i + 2^k \bmod n,\ 0 \le k \le \lceil \log_2 n \rceil - 1 \right\}.$$

These minimal broadcast networks are a subclass of the class of graphs known as star polygons. This is not the first time that a subclass of star polygons has satisfied a goal of communication network design. Boesch and Felzer [1] have reported a subclass of star polygons which satisfy a definition of invulnerability, an important concept in network reliability.

Figure 3 illustrates the first 12 of these star polygon minimal broadcast networks. Each network is labelled by the number of members and, in parentheses, original eight can complete broadcast in three time units. A fourth time unit is required for the new member to be called. Regardless of message originator, broadcast can be completed in the minimal four time units. Therefore, the network shown in Figure 4 is a minimal broadcast network.

This example demonstrates that the previous results as to the number of lines required for a minimal broadcast network of n members is not the best or minimum result. A better result can be obtained by a recursive construction algorithm which generalizes upon the method that has been employed above to produce the improved network of nine members. The algorithm constructs a minimal broadcast network of n members by combining minimal broadcast networks of fewer members. The process of combining the networks is subject to further specification, as follows:

CONSTPUCT_MINIMAL_BROADCAST_NETWORK(n)

CASE 0: $(n \le 2)$ If $n=1$, the network is a single node; If $n=2$, the network is 2 vertices connected by a single communication line.

CASE 1: $(2^{\lceil \log_2 n \rceil - 1} < n \le 3 * 2^{\lceil \log_2 n \rceil - 2})$

Step 1.1) Select integers i, j, and k such that $i, j, k \le 2^{\lceil \log_2 n \rceil - 2}$, $i + j + k = n$, $i \ge j \ge k$, and $i - k \le 1$.

Step 1.2) CONSTRUCT_MINIMAL_BROADCAST_NETWORK(i)
CONSTRUCT_MINIMAL_BROADCAST_NETWORK(j)
CONSTRUCT_MINIMAL_BROADCAST_NETWORK(k)

Step 1.3) If n is even then connect each member of the three components to a different member of a different component. (This requires $n/2$ lines). If n is odd then do as above for $n-1$ of the members. Then connect the remaining member to a member of a different component to which no member of its component is already connected. (This requires $\lceil n/2 \rceil$ lines).

Proof: The networks produced for one or two members are minimal broadcast networks by inspection (and by the prior, star polygon scheme). It must be shown that those produced by combining components (Case 1 and 2 networks) are also minimal broadcast networks. The proof is shown by defining calling schemes in such networks which realize minimal broadcast time regardless of message originator. The proof is inductive. Given n, we shall assume that the algorithm does construct minimal broadcast networks for any i, j, or k less than n.

Case 1 networks - The network of n members is realized by interconnecting three component networks. Broadcast proceeds in three phases:

Phase 1 - The message originator calls a member external to its component to which it is connected. This requires one unit of time. There is always at least one such call possible at this point.

Phase 2 - The two components with an informed member complete broadcast independently and in parallel. This requires at most $\lceil \log_2 n \rceil - 2$ time units, by the inductive assertion.

Phase 3 - Each member of the third component is called by an informed member of another component. This requires one time unit.

Case 2 networks - The network of n members is realized by interconnecting two component networks of i and j members respectively, $i \leq j$. There are two subcases to consider here. In each subcase broadcast proceeds in two phases.

Subcase a) - The originator is in the j member component.

Phase 1 - Broadcast is completed in the component of the originator. This requires at most $\lceil \log_2 n \rceil - 1$ time units, by inductive assertion.

Proof The proof is by induction. Inspection of the networks shown in Figure 5 indicate that the assertion is true for $n \le 12$. Suppose it is true for all networks with less than n members.

Case 1 networks:

The n member network is constructed from three components (with i, j, and k members, respectively). The number of lines required is less than or equal to

$$i/2\ (\lceil \log_2 n \rceil - 2) + j/2\ (\lceil \log_2 n \rceil - 2) + k/2\ (\lceil \log_2 n \rceil - 2) + \lceil n/2 \rceil$$

$$= \frac{(i + j + k)}{2} (\lceil \log_2 n \rceil - 2) + \lceil n/2 \rceil$$

$$= n/2\ (\lceil \log_2 n \rceil - 2) + \lceil n/2 \rceil = n/2 \lceil \log_2 n \rceil - \lfloor n/2 \rfloor .$$

Case 2 networks: The n member network is constructed from two components (with i and j members, respectively). The number of lines required is less than or equal to

$$i/2\ (\lceil \log_2 n \rceil - 1) + j/2\ (\lceil \log_2 n \rceil - 1) + n/2 = \frac{i+j}{2} (\lceil \log_2 n \rceil - 1) + \lfloor n/2 \rfloor$$

$$= n/2 \lceil \log_2 n \rceil - n/2 + \lfloor n/2 \rfloor$$

$$\le n/2 \lceil \log_2 n \rceil .$$ ||

Corollary In a Case 1 network of n members produced by the recursive construction algorithm, the number of lines is less than or equal to $n/2 \lceil \log_2 n \rceil - \lfloor n/2 \rfloor$.

Additional Considerations

It is interesting to observe that the number of lines in the constructed network does not monotonically increase with increasing n. (For example, see the networks with 8 and 9 members shown in Figure 5.) For an n just above a power of 2, the increase in minimal broadcast time introduces freedom or slack into the calling scheme. An informed member may not make a call during a given time unit and yet minimum time can be realized. This freedom in turn allows a decrease in the number the minimum broadcast time of four time units regardless of originator. If a uniform broadcast time of five time units is acceptable, a network with only 19 lines can be found, as shown in Figure 7.

There is a lower bound on the number of lines which are required for successful broadcast in a network of n members. The network must be connected. Therefore, at least n-1 lines are required. Such networks are trees. What trees produce the best results for broadcast time? What is broadcast time in such trees? In a tree, broadcast time may differ depending upon which member is message originator. Therefore, broadcast time for a tree is best characterized by three values: the minimum, the average, and the maximum number of time units required to complete the process.

A tree which has the minimum possible value for the minimum number of time units belongs to the class of minimum broadcast trees. A minimum broadcast tree is defined to be a tree with n vertices which contains a member from which broadcast can be completed in $\lceil \log_2 n \rceil$ time units. That such trees exist has been an implicit result of the earlier discussion of minimal broadcast networks. The graph induced by a completed broadcast in a minimal broadcast network is a minimum broadcast tree. A linear algorithm has been found for determining whether an arbitrary tree is a minimum broadcast tree [4]. A subclass of such trees can be constructed by a method based upon the calling scheme which led to the star polygon class of minimal broadcast networks. Let the members be numbered from 1 to n. Let member 1 be in the tree. Then, for each member i ($2 \le i \le n$), connect i to member $i-2^{\lceil \log_2 i \rceil - 1}$. This constructs the tree in layers, each successive layer being twice the size of the previous layer. During broadcast, an informed member calls the uninformed members to which it is linked in the order of increasing member number. Figure 8 presents a sample of such trees for $n \le 12$. Message broadcast originated by member 1 or member 2 in such trees requires the minimum, $\lceil \log_2 n \rceil$ time units. The maximum number of

Conclusion

Several problems relating to broadcast have yet to be solved. The problem of introducing reliability into broadcast networks is one. For $n \geq 4$, the recursively constructed networks are all minimally two-connected, which is at least a start. Questions concerning broadcast within an arbitrary network have not been answered. For example, an algorithm which determines a best calling scheme for any member of an arbitrary network has yet to be found. The potential importance of message broadcast in communication and computer networks and in parallel processors has motivated this research. This paper has reported some initial results and has suggested further questions for study.

TABLE 1 Number of Lines in Minimal Broadcast Network

n	complete graphs	minimal broadcast star polygons	recursively-constructed networks
4	6	6	4
6	15	12	6
8	28	20	12
12	66	36	18
16	120	56	32
24	276	96	48
32	496	144	90
48	1128	240	120
64	2016	352	192

TABLE 2 Number of Time Units to Complete Broadcast

n	recursively constructed networks		minimum broadcast trees	
	lines	time	lines	time min, avg, max
4	4	2	3	2 , 2.5 , 3
6	6	3	5	3 , 3.67 , 4
8	12	3	7	3 , 4 , 5
12	18	4	11	4 , 5.17 , 6
16	32	4	15	4 , 5.5 , 7
24	48	5	23	5 , 6.67 , 8
32	90	5	31	5 , 7 , 9
48	120	6	47	6 , 8.17 , 10
64	192	6	63	6 , 8.5 , 11

FIGURE 1

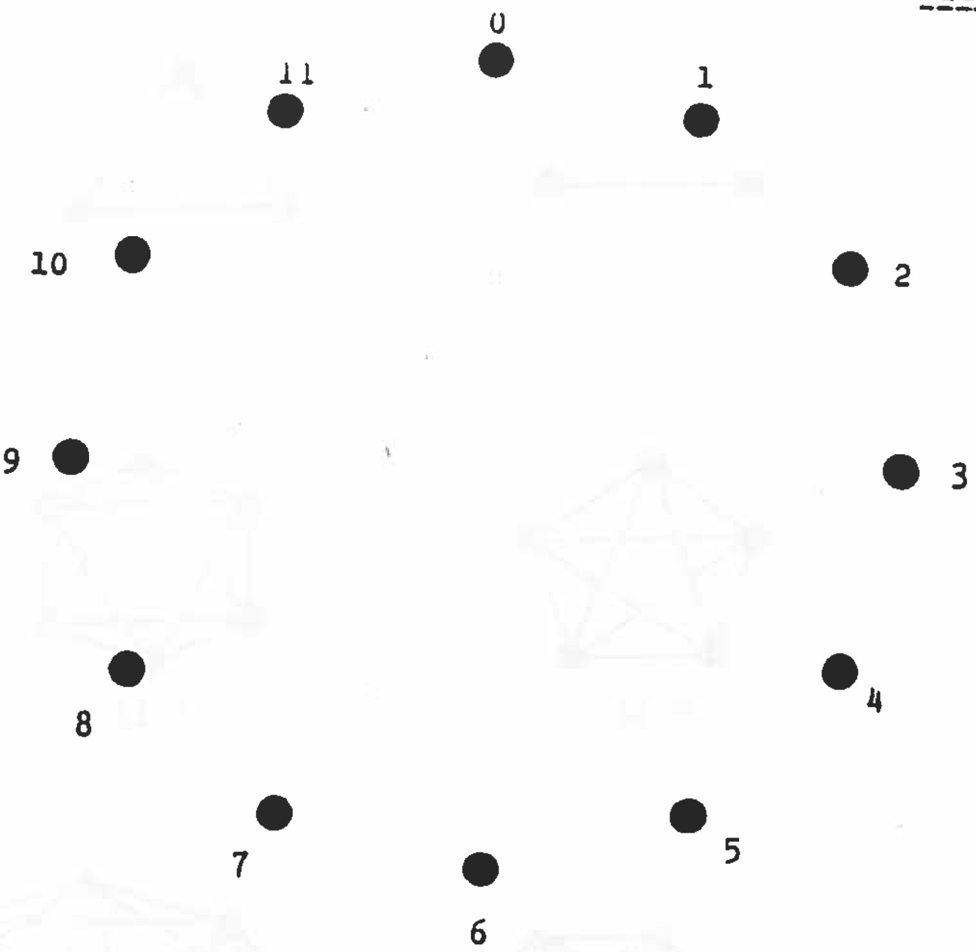


FIGURE 2

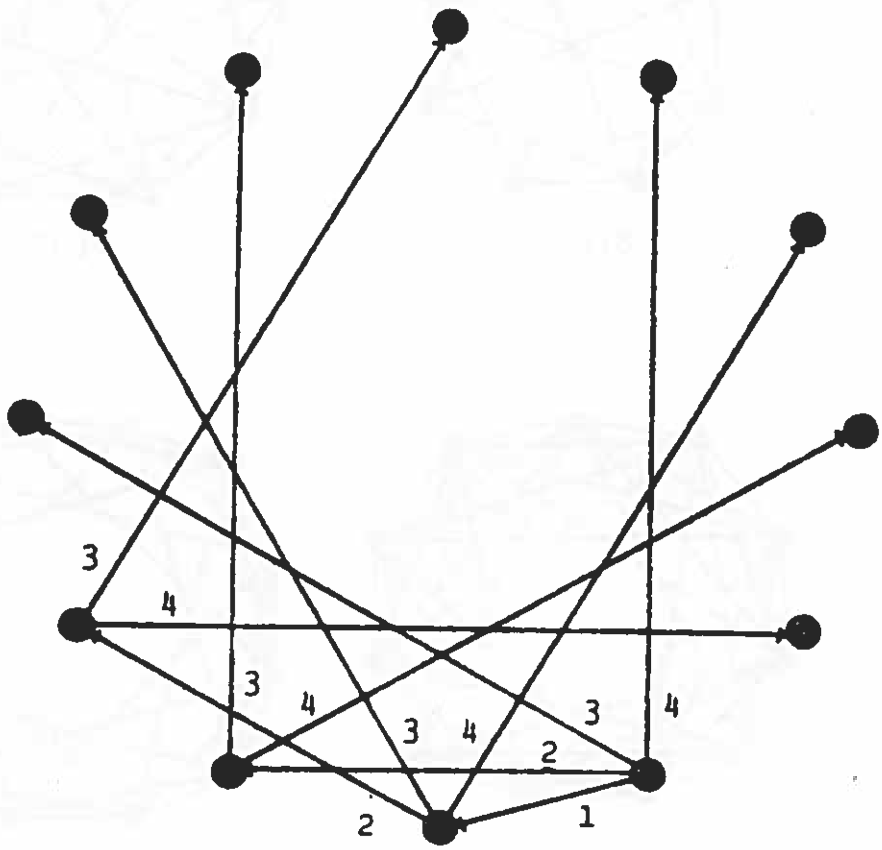
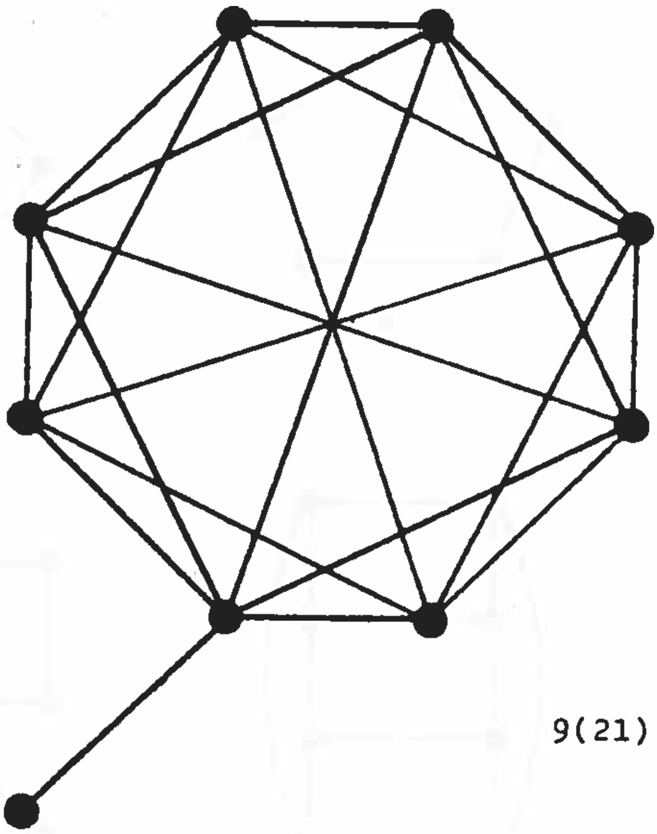
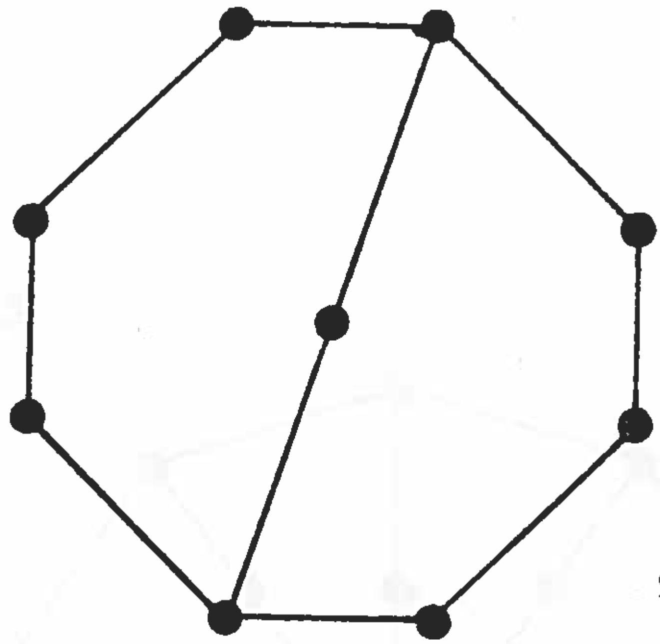


FIGURE 4

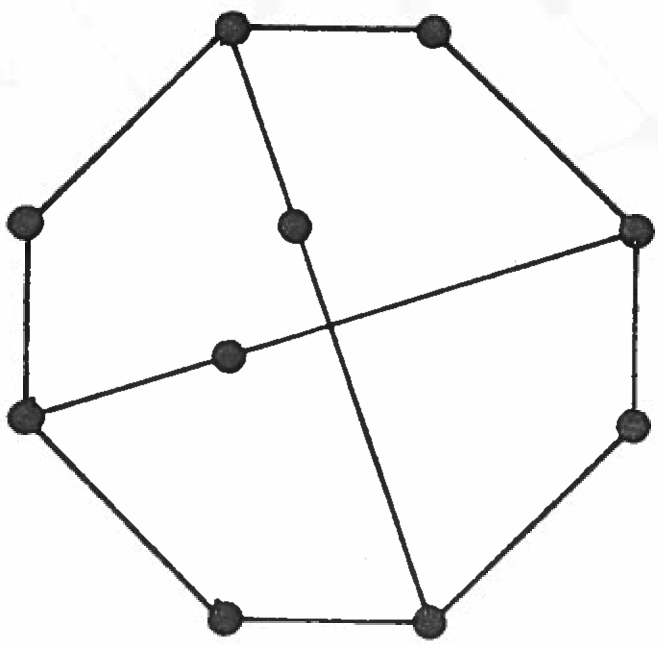


9(21)

FIGURE 6



9(10)



10(12)

FIGURE 8