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MINIMUM BROADCAST GRAPHS

By

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1. Introduction

Let a graph $G=(V,E)$ represent a computer network. The vertices V represent computer centers or sites and the edges E represent communication links or leased lines between various sites. Condsider the following problem: the requirements for using the computing facilities at a given vertex u change and it is necessary to inform every vertex in the graph of this change. This is to be accomplished as quickly as possible by a series of phone calls over the leased lines in the network. We adopt the constraints that (i) each call requires one unit of time, (ii) a vertex can participate in only one call per unit of time, and (iii) a vertex can only call an adjacent vertex. How much time is required to inform every vertex in the graph? We refer to this process as broadcasting from vertex u to all other vertices, and we define the broadcast time for u, $b(u)$, to equal the minimum time required to broadcast from vertex u.

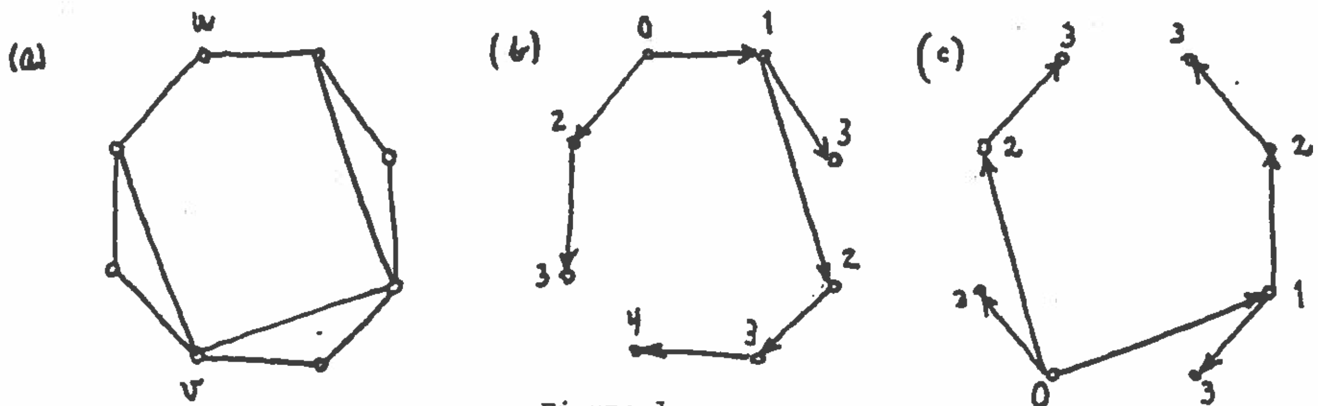


Figure 1

For example, for the graph in Figure 1(a), $b(u)=4$, but $b(v)=3$. The rooted trees in Figures 1(b) and 1(c) indicate how the broadcasts from u and v can take place in times 4 and 3; we refer to these rooted trees as (minimum) broadcast trees for u and v, respectively.

We define the broadcast time of a graph G, $b(G)$, to equal the maximum broadcast time of any vertex in G, i.e. $b(G)=\max\{b(u) \mid u \in V(G)\}$.

For the tree T in Figure 2, $b(u_1)=b(u_2)=b(u_3)=b(u_4)=3$, while $b(v)=4$. Hence $b(T)=4$.

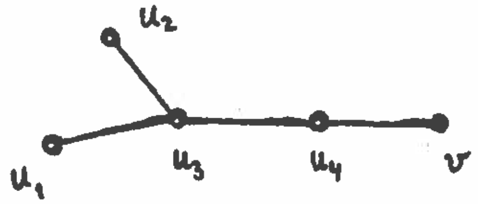


Figure 2

It is easy to see that for any vertex v in a connected graph G with n vertices, $b(v)\geq \lceil \log n \rceil$. At time 1 vertex v can call one other vertex; at time 2 these two vertices can call two others; at time 3 these four vertices can call 4 others; etc.

It is also easy to see that for the complete graph K_n with n vertices, $b(K_n)=\lceil \log n \rceil$, yet K_n may not be minimal with respect to this property. That is, we may be able to remove some edges from K_n and still have a graph whose broadcast time is $\lceil \log n \rceil$. A graph G with n vertices is a <u>minimal broadcast graph</u> if $b(G)=\lceil \log n \rceil$, but for every proper, spanning subgraph $G'\subset G$, $b(G') > \lceil \log n \rceil$. For example it can be seen that the graph C_4 consisting of a cycle with 4 vertices satisfies $b(C_4)=2$, yet $b(G')=3$ for every proper spanning subgraph G' of C_4.

Define the <u>broadcast function</u> B(n) to equal the minimum number of edges in any minimal broadcast graph on n vertices.

A <u>minimum broadcast graph</u> is a minimal broadcast graph on n vertices having B(n) edges. From the point of view of applications, minimum broadcast graphs represent the cheapest possible communication networks (in terms of number of leased lines) in which broadcasting can be accomplished from any vertex as fast as theoretically possible.

In this paper we initiate a study of minimum broadcast graphs by determining the value of B(n) for $n\leq 16$ and $n=2^k$, and constructing an example of a minimum broadcast graph for each value of $n\leq 16$. Several papers have recently been written on broadcasting. In [8], the broadcast center (the set of vertices having minimum broadcast times) of a tree is determined. In [3], broadcasting in complete networks and techniques for constructing <u>minimal</u> broadcast graphs are discussed. And in [6], minimum broadcast trees are studied. A subsequent paper will determine the number of distinct minimum broadcast graphs on n vertices, for all values of $n\leq 12$.

We should also point out that a number of other papers have appeared which study the process of <u>gossiping</u>. (e.g. [1], [2], [4], [5] and [7]).

In gossiping every vertex has a unique piece of information and it is necessary to inform every vertex of every piece of information. Broadcasting is therefore a one-to-all process, while gossiping is an all-to-all process.

2. The values of $B(n)$ for $n \leq 6$

In the next three sections we will determine the values of $B(n)$ for $n \leq 16$ and $n=2^k$. In this section we consider the first six values of $B(n)$, as indicated in Figure 3.

Vertices n	Edges $B(n)$	Broadcast time $\lceil \log n \rceil$	Minimum broadcast graph with n vertices
1	0	0	K_1
2	1	1	$K_{1,1}$
3	2	2	$K_{1,2}$
4	4	2	C_4
5	5	3	C_5
6	6	3	C_6

Figure 3

$B(1)=0$

Certainly it requires no time and no edges for a vertex to call itself.

$B(2)=1$

It is obvious that a minimum broadcast graph must be a connected graph. The only connected graph with two vertices has one edge.

$B(3)=2$

There are only two connected graphs with three vertices: the triangle K_3 and $K_{1,2}$. Both of these graphs are minimal broadcast graphs, but only $K_{1,2}$ with 2 edges is a minimum broadcast graph.

$B(4)=4$

Since the four-cycle C_4 is a minimal broadcast graph, we know that $B(4) \leq 4$. There are only two connected graphs with 4 vertices and 3 edges. Since both of these graphs have a broadcast time of $3 > \lceil \log 4 \rceil$, neither of them in a minimal broadcast graph. Hence, $B(4)=4$.

$B(5)=5$

Since the five-cycle C_5 is a minimal broadcast graph, we know that $B(5)\leq 5$. There are only three connected graphs with 5 vertices and 4 edges, each of which has a broadcast time of $4 > \lceil \log 5 \rceil = 3$. Hence there are no minimal broadcast graphs with 5 vertices and 4 edges, and $B(5)=5$.

$B(6)=6$

Since the six-cycle C_6 is a minimal broadcast graph, we know that $B(6)\leq 6$. Of the six connected graphs (trees) with 6 vertices and 5 edges, four have broadcast times of 5 and two have broadcast times of $4 > \lceil \log 6 \rceil = 3$. Hence there are no minimal broadcast graphs with 6 vertices and 5 edges, and $B(6)=6$.

3. The values of $B(n)$ for $7\leq n\leq 12$

As one might expect, the next six values of $B(n)$ are more difficult to establish. Since a simple, exhaustive argument will no longer be effective, we must use a different method of proof. This method will consider what a broadcast tree for an arbitrary vertex might look like.

Each proof will proceed by first presenting a minimal broadcast graph with n vertices and l edges. This establishes that $B(n)\leq l$. Then it is shown that no minimal broadcast graph with n vertices exists with $l-1$ edges. Therefore, $B(n)\geq l$, completing the proof that $B(n)=l$.

$B(7)=8$

The graph in Figure 4a is a minimal broadcast graph with 7 vertices and 8 edges; hence, $B(7)\leq 8$.

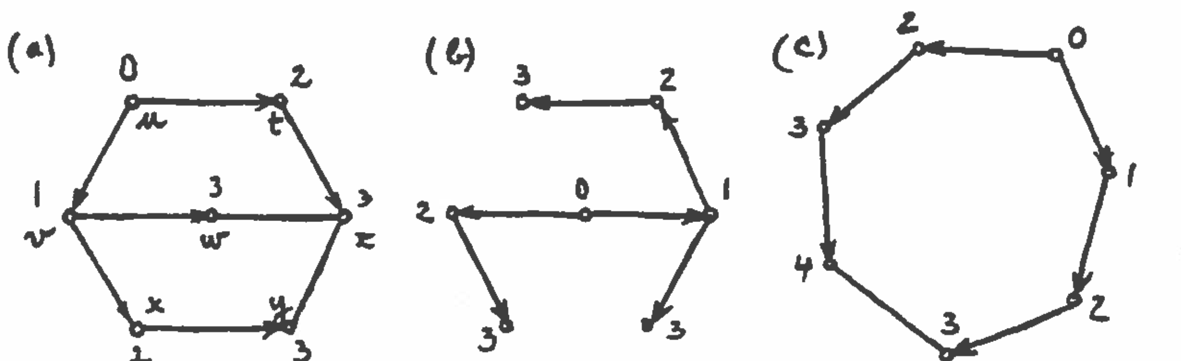


Figure 4

It is instructive to show that the graph G in Figure 4a has a broadcast time of $3=\lceil \log 7 \rceil$. In Figure 4a we indicate with directed edges a minimum broadcast tree for vertex u. Notice that if we relabel u and v as 1 and 0, respectively, we will obtain a minimum broadcast tree for vertex v, i.e. if the broadcast time for a vertex u is $\lceil \log n \rceil$, then the broadcast time for the vertex labeled 1 in a minimum broadcast tree for u is also $\lceil \log n \rceil$.

Notice also that vertices u, t, x and y are all similar to each other, and that vertices v and z are similar. Therefore, having demonstrated that b(u)=3, we have simultaneously demonstrated that b(u)=b(t)=b(x)=b(y)=3 and that b(u)=b(v)=b(z)=3. It only remains to show that b(w)=3; this is done in Figure 4b.

We will now show that there are no minimal broadcast graphs with 7 vertices and 7 edges, and hence that B(7)=8.

Let G be a graph with 7 vertices, and let u be a vertex of degree 1 in G. Then in any minimum broadcast tree for u only 4 other vertices can be called by time $\lceil \log 7 \rceil$=3 (cf. Figure 5). Thus G cannot be a minimal broadcast graph.

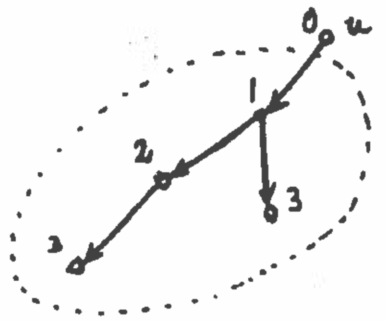


Figure 5

Consequently, in a minimal broadcast graph G with seven vertices, the minimum degree of any vertex is $\geq$ 2. Since the sum of the degrees of all vertices in G must equal twice the number of edges, we can conclude that G must have at least 7 edges, i.e. B(7)$\geq$7. The only connected graph with 7 vertices, 7 edges and no vertex of degree 1 is the seven-cycle. Since the broadcast time for the seven-cycle is 4 > $\lceil \log 7 \rceil$= 3, it is not a minimal broadcast graph (cf. Figure 4c). Therefore, there are no minimal broadcast graphs with 7 vertices and 7 edges.

<u>B(8)=12</u>

In a minimal broadcast graph with 8 vertices, every vertex must have degree $\geq$ 3. Thus, G must have $\geq$ 3x8/2=12 edges, i.e. B(8)$\geq$12. Since the 3-cube G in Figure 6 is a minimal broadcast graph with 8 vertices and 12 edges, B(8)$\leq$12.

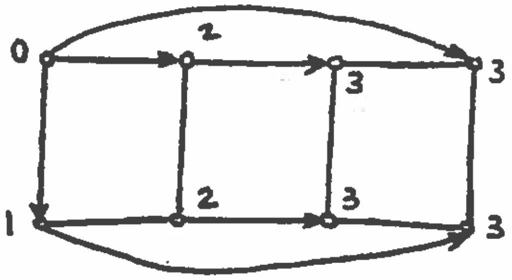


Figure 6

B(9)=10

This result may come as something of a surprise since B(9)<B(8)! However it is less surprising when one considers that there is an additional time unit available during which calls can be made. ($\lceil \log 9 \rceil = 4$, while $\lceil \log 8 \rceil = 3$).

The graph in Figure 7 is a minimal broadcast graph with 9 vertices and 10 edges. Figures 7a, 7b and 7c demonstrate the (directed) broadcast trees from three of the vertices.

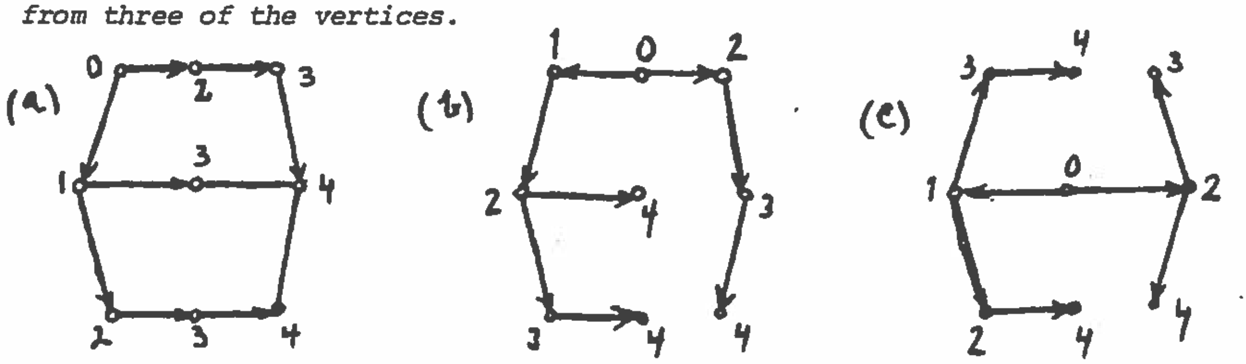


Figure 7

We will now show that there are no minimal broadcast graphs with 9 vertices and 9 edges. Such a graph G must be connected and have exactly one cycle. The only such graph with minimum degree=2 is the nine-cycle, which is not a minimal broadcast graph. Thus G must have a vertex u of degree 1, and the broadcast tree for u must be isomorphic to the tree T_u in Figure 8. Consequently u must be adjacent to a vertex v of degree $\geq$ 4. Therefore every vertex of degree 1 must be adjacent to a vertex of degree $\geq$ 4. Furthermore G must be obtained by adding one edge to the tree T_u in Figure 8. Consider therefore the vertices w, x, y in Figure 8. No matter how one edge is added to T_u, in the resulting graph either w or x or y will be an endvertex which is not adjacent to a vertex of degree $\geq$ 4. Thus, G cannot be a minimal broadcast graph.

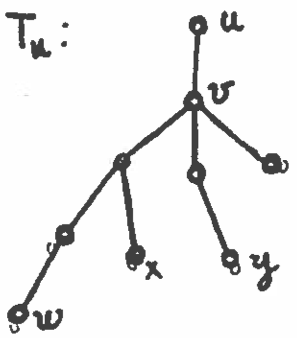


Figure 8

<u>B(10)=12</u>

The graph in Figure 9 in a minimal broadcast graph having 10 vertices and 12 edges; it consists of a 10-cycle with 2 symmetrically placed chords. Hence, $B(10)\le 12$.

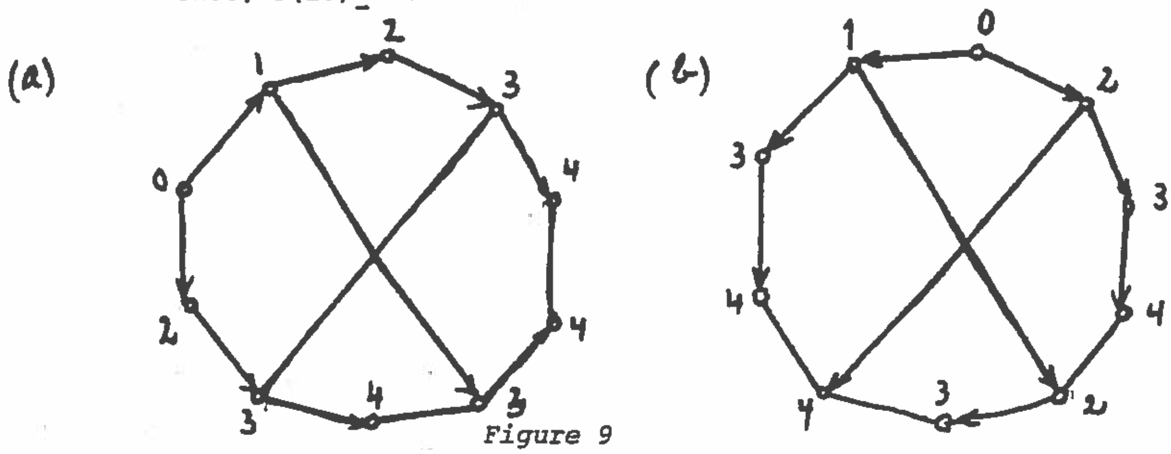


Figure 9

We will now show that there are no minimal broadcast graphs with 10 vertices and 11 edges. Suppose that G were such a graph. The broadcast tree in T_u in Figure 8 shows that no vertex in G can have degree 1, since then only 9 of the 10 vertices could be called by time $\lceil \log 10 \rceil = 4$. Thus the minimum degree of any vertex in G is ≥ 2, and the only possible degree sequences for the vertices in G are:

(a) 2 2 2 2 2 2 2 2 2 4

(b) 2 2 2 2 2 2 2 2 3 3

The only connected graphs having degree sequence (a) look like those in Figure 10. They consist of two cycles I and II, of lengths $m+1$ and $n+1$, respectively.

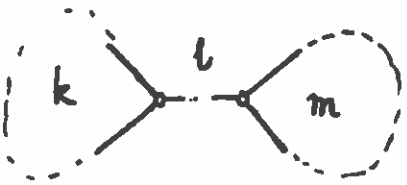
Figure 10

where $m+n=9$, $m,n\geq 2$, and the two cycles are joined at a cutvertex u of degree 4. It can easily be shown that none of the possible values for (m,n) (i.e. (7,2), (6,3), (5,4)) produces a minimal broadcast graph (cf. (5,4) in Figure 10b).

Therefore, let G have degree sequence

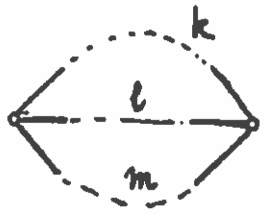
$$2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3,$$

and let u and v be the two vertices of degree three. There are two kinds of connected graphs having this degree sequence, as indicated in Figure 11, where $k+l+m=8$. The possible values of k, l and m for each kind of graph indicated in Figure 11 reveal that we must consider eighteen possible graphs. A simple examination of these graphs shows that none of them are minimal broadcast graphs. Thus there are no minimal broadcast graphs with 10 vertices and 11 edges, i.e. $B(10) \geq 12$.



k	l	m
2	0	6
3	0	5
4	0	4
2	1	5
3	1	4
2	2	4
3	2	3
2	3	3
2	4	2

(a)



k	l	m
0	1	7
0	2	6
0	3	5
0	4	4
1	1	6
1	2	5
I	3	4
2	2	4
2	3	3

(b)

Figure 11

B(11)=13

In Figure 12 we present a minimal broadcast graph with 11 vertices and 13 edges. The interested

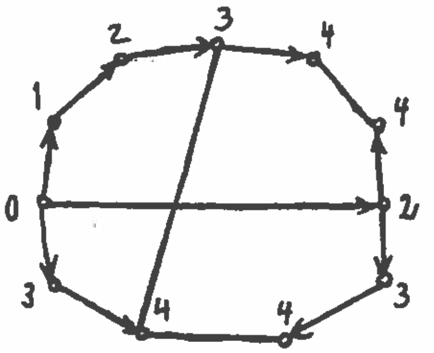


Figure 12

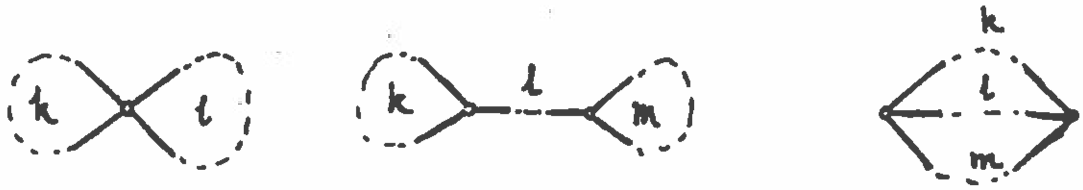
reader can verify that the broadcast time for each vertex is $\lceil \log 11 \rceil = 4$. Hence, $B(11) \leq 13$.

We now show that there are no minimal broadcast graphs with 11 vertices and 12 edges. Using the same argument as in the case for 10 vertices, above, one can show that no minimal broadcast graph on 11 vertices can have a vertex of degree 1. Thus the only possible degree sequences, as before, are

(a) 2 2 2 2 2 2 2 2 2 2 4

(b) 2 2 2 2 2 2 2 2 2 3 3

Again, as in the case for 10 vertices, the only types of graphs with these degree sequences are those in Figure 13.



k	l
2	8
3	7
4	6
5	5

k	l	m	k	l	m
2	0	7	2	2	5
3	0	6	3	2	4
4	0	5	2	3	4
2	1	6	3	3	3
3	1	5	2	4	3
4	1	4	2	5	2

k	l	m	k	l	m
0	1	8	1	3	5
0	2	7	1	4	4
0	3	6	2	2	5
0	4	5	2	3	4
1	1	7	3	3	3
1	2	6			

Figure 13

A simple check of each of these 27 graphs shows that none of them are minimal broadcast graphs. Thus there are no minimal broadcast graphs with 11 vertices and 12 edges, and $B(11)\geq 13$.

<u>$B(12)=15$</u>

The minimal broadcast graph in Figure 14 has 12 vertices and 13 edges. Hence, $B(12)\leq 15$. We will show that $B(12)\geq 15$ by showing that there are no minimal broadcast graphs with 12 vertices and 14 edges.

From previous arguments we know that any minimal broadcast graph with 12 vertices must be connected and have no vertex of degree 1. Assume that G has a vertex u of degree 2 and consider the possible broadcast trees for u. They are the six trees obtained by deleting one endvertex from the tree T in Figure 15.

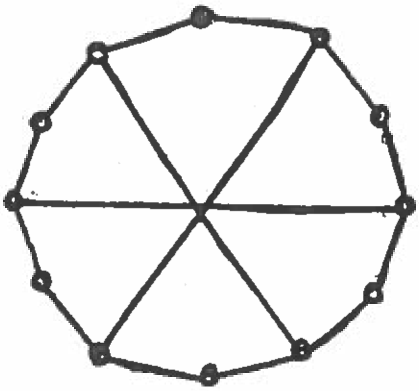


Figure 14

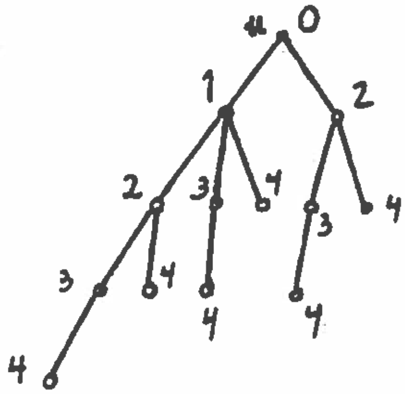


Figure 15

Assume further that G has two adjacent vertices u, w of degree 2. Then the broadcast tree for u must be isomorphic to the tree in Figure 16, and the graph G can be obtained by adding several edges to T_u.

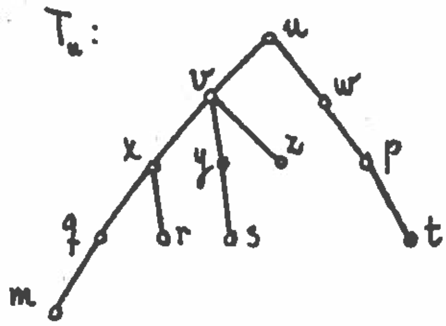


Figure 16

Since vertex w has degree 2, its broadcast tree must also be isomorphic to T_u. Consequently, vertex p must have degree ≥ 4 (like vertex v). Furthermore, since no vertex in G can have degree 1, we must add at least one edge to m, r, s, z and t. The two additional edges needed for p can be joined to, say, m and r. A third edge can be added between s and z, leaving us in need of a fourth edge to be added to t. Thus if G has two adjacent vertices of degree 2, then it must have $\geq 11+4=15$ edges.

Consider therefore the six possible broadcast trees for a vertex u of degree 2. We will show in each case that if G does not have 2 adjacent vertices of degree 2, then it must have ≥ 15 edges.

Case 1.

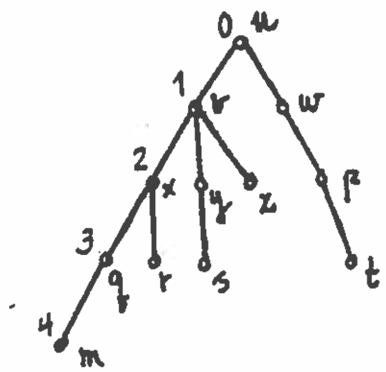


Figure 17

We must add at least one edge to w (else it has degree 2), and to endvertices m, r, s,z and t. Furthermore, we must add at least one edge to either p or t (else we have two adjacent vertices of degree 2). Hence, we must add at least 4 edges to T_u, i.e. G has $\geq$ 11+4 edges.

Case 2.

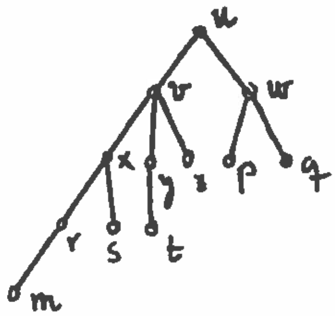


Figure 18

We must add at least one edge to each of the six endvertices m, s, t, z, p and q, and at least one additional edge to either y or t (else we have two adjacent vertices of degree 2). Thus, we must add at least 4 edges to T_u and G has $\geq$ 11+4 edges.

Cases 3, 4, 5 and 6 are all similar to cases 1 and 2 are are omitted in the interests of brevity.

Thus we have shown that if G is a minimal broadcast graph with 12 vertices, then it has either

a) two adjacent vertices of degree 2, or

b) at least one vertex of degree 2, but no two of them are adjacent, or

c) no vertices of degree 2, i.e. the minimum degree of any vertex is $\geq$ 3.

In either case above G must have $\geq$ 15 edges. Thus, B(12)=15.

4. The values of $B(n)$ for $13 \le n \le 16$.

The effort required to establish each of the next five values of $B(n)$ is very similar to that required for the previous four values. Therefore we will only sketch the corresponding proofs.

$B(13)=18$

The graph G in Figure 19 is a minimal broadcast graph with 13 vertices and 18 edges. Hence, $B(13) \le 18$. We next outline the proof that $B(13) \ge 18$.

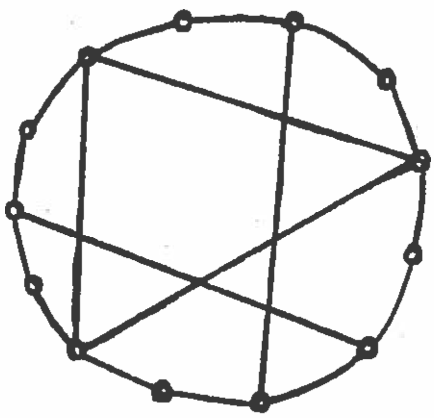


Figure 19

Using previous arguments one can show that no minimal broadcast graph G on 13 vertices can have a vertex of degree 1. Hence every vertex in G has degree ≥ 2. If G has a vertex u of degree 2, then its broadcast tree must be isomorphic to the broadcast tree T_u in Figure 20.

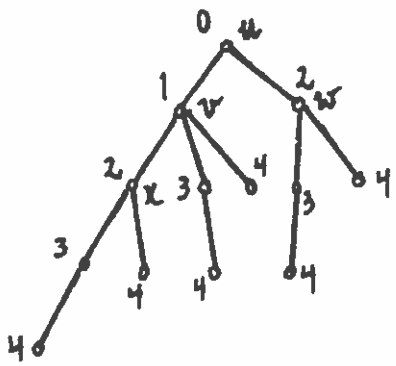


Figure 20

From the tree T_u it follows that if vertex u has degree 2, then it must be adjacent to a vertex v of degree ≥ 4, and a vertex $w \neq v$ of degree ≥ 3. Furthermore, vertex v must be adjacent to a vertex x, $x \neq w$, of degree ≥ 3.

It only remains to consider how many vertices of degree 2 are in G. The conditions above indicate that for every 3 vertices of degree 2 there must be one vertex of degree 4 and two vertices of degree ≥ 3. (cf. Figure 21). It follows

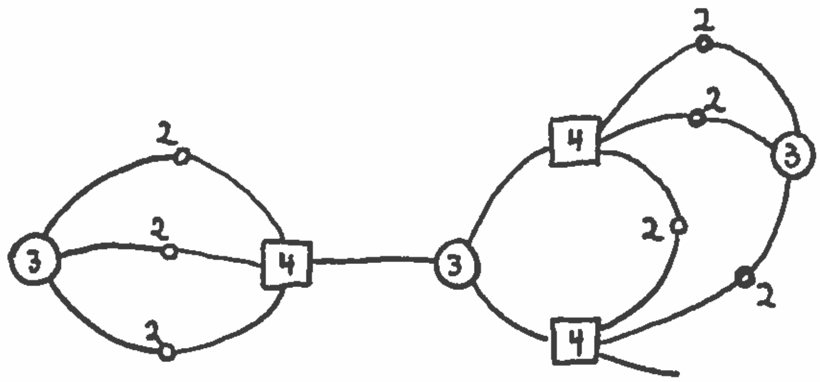


Figure 21

therefore that G can have at most 7 vertices of degree 2, and furthermore, if G has 7 vertices of degree 2 then G must have ≥ 18 edges.

Using similar arguments one can show that if G has k vertices of degree 2, for $1 \leq k \leq 6$ then G must have at least 18 edges. Thus, $B(13) \geq 18$.

<u>B(14)=21</u>

The graph G in Figure 22 is a minimal broadcast graph with 14 vertices and 21 edges. Hence, $B(14) \leq 21$.

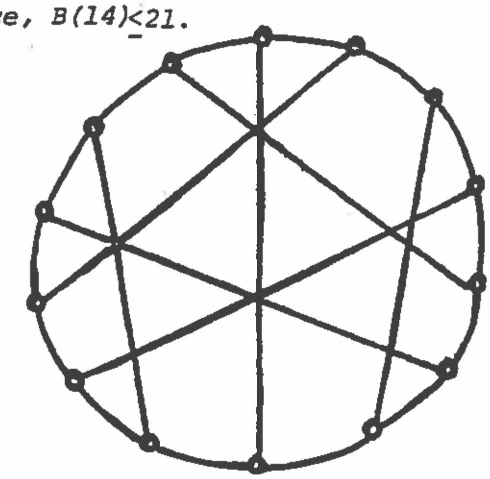


Figure 22

In order to show that $B(14) \geq 21$ one only needs to observe that in a minimal broadcast graph G with 14 vertices, no vertex can have degree 2 (cf.Figure 20). Thus if the minimum degree of any vertex in G is ≥ 3 then G must have $\geq 14 \times 3/2 = 21$ edges.

<u>B(15)=24</u>

The minimal broadcast graph with 15 vertices and 24 edges in Figure 23 shows that $B(15) \leq 24$.

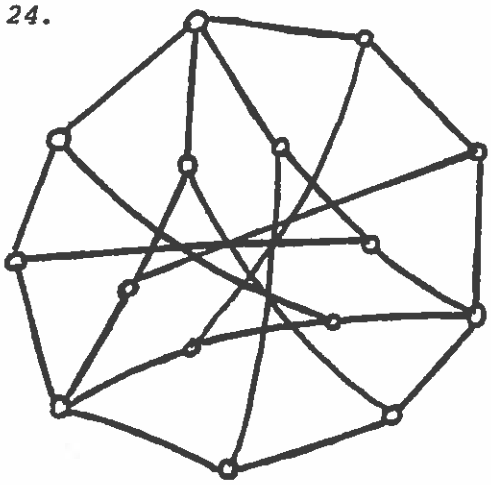


Figure 23

In order to show that $B(15) \geq 24$, consider a broadcast tree for any vertex in a minimal broadcast graph G with 15 vertices. Previous arguments suffice to show that G cannot have a vertex of degree ≤ 2. Consider therefore a broadcast tree for a vertex u of degree 3 in G; it must be isomorphic to the broadcast tree in Figure 24. It follows therefore that vertex u must be adjacent to a vertex v of degree ≥ 4. Thus every vertex

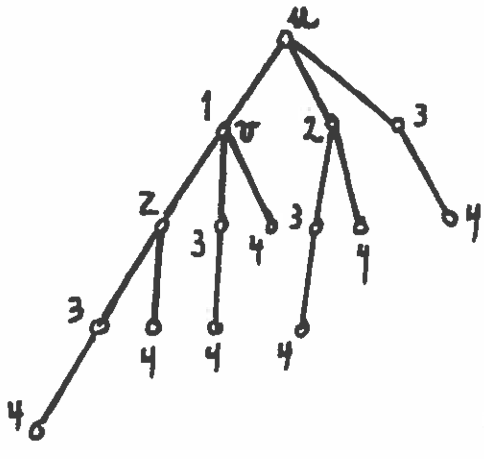


Figure 24

in G of degree 3 must be adjacent to a vertex of degree $\geq$ 4. Stated in other words, for every 4 vertices of degree 3 in G there must be at least one vertex of degree $\geq$ 4. Therefore, G can have no more than 12 vertices of degree 3 and must have at least 3 vertices of degree $\geq$ 4. It follows then that G must have $\geq (12x3+3x4)/2=24$ edges.

$B(16)=32$ (and $B(2^k)= k.2^{k-1}$)

In a minimal broadcast graph G with 2^k vertices, every vertex must have degree at least k in order to call all 2^k vertices in time k. Thus, G must have $k.2^k \,/\, 2=k.2^{k-1}$ edges and $B(2^k) \geq k.2^{k-1}$

In order to show that $B(2^k) \leq k.2^{k-1}$ it will suffice to construct a minimal broadcast graph with 2^k vertices and $k.2^{k-1}$ edges for every $k \geq 3$. But this is easily done by taking any two minimum broadcast graphs G,H with 2^{k-1} vertices and adding 2^{k-1} additional edges in a 1-1 fashion between the vertices in G and the vertices in H. Figure 25 illustrates this construction with a minimum broadcast graph with 16 vertices.

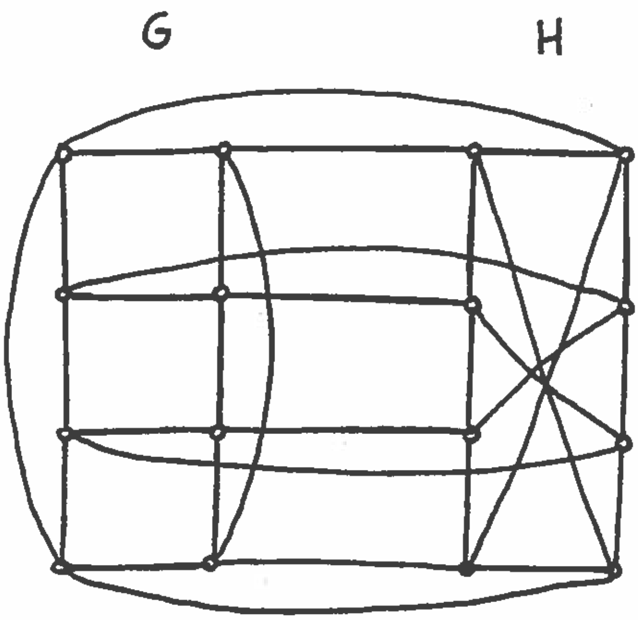


Figure 25

5. Summary

In this paper we have initiated a study of minimum broadcast graphs by determining the number of edges required in such graphs for all values of $n \leq 16$ and $n=2^k$, and for each of these values we have constructed an example of a minimum broadcast graph with n vertices. We have not yet determined the value of B(n) for any other value of n. In a later paper, however, it will be shown (among other things) that the minimum broadcast graphs shown earlier with 7, 11 and 12 vertices, are unique, i.e. there are no other minimum broadcast graphs with 7, 11 or 12 edges.

B I B L I O G R A P H Y

[1] B. Baker and R. Shostak

Gossips and telephones,
Discrete Math. 2 (1972), 191-193.

[2] G. Berman

The gossip problem
Discrete Math. 4 (1973), 91

[3] A. Farley

Minimal broadcast graphs, submitted.

[4] F. Harary and A. J. Schwank

The communication problem on graphs and digraphs.
J. Franklin Inst. 297 (1974), 491-495.

[5] W. Knodel

New gossips and telephones
Discrete Math. 13 (1975), 95.

[6] A. Proskurowski

Minimum broadcast trees, in preparation.

[7] P. Schmitt

Spreading information by conferences
Discrete Math. 15 (1976), 305-306.

[8] P. J. Slater, E. J. Cockayne and S. T. Hedetniemi

The gossip center of a tree, submitted.