CS-TR-79-9

Centers of Two-Trees

by

Andrzej Proskurowski
Department of Computer Science
University of Oregon, Eugene, OR

1. Introduction

In a graph G, the distance d(u,v) between vertices u and v of G is defined as the length of a shortest path connecting u with v (i.e., the number of edges in the path). Eccentricity e(v) of a vertex v in graph G is the largest distance from v to any vertex of G ($e(v) = \max d(u,v) | u$ in G). The minimum eccentricity of vertices in G is called the radius r(G). Center C(G) of a graph G is the subgraph of G induced by the set of vertices with the smallest eccentricities:

$C(G)= v$ in $G | e(v)=r(G) \leq e(u)$ for all u in G .

In [2] we investigated possible shapes of centers for a class of graphs called maximal outerplanar graphs (mops).

Def.1.1 A graph is a mop iff it is isomorphic to a triangularization of a polygon.

All mops can be constructed according to the following recursive rule (see, for instance, [1]).

Fact 1.2 The "triangle" is the only mop with three vertices. To construct a mop with n vertices, given a mop M with n-1 vertices (n>3), add a vertex by making it adjacent to any two vertices of M adjacent along the Hamiltonian cycle of M.

The finitness of the set of all centers of mops follows from two facts: (i) that all centers of mops are non-separable, and (ii) that there are some forbidden subgraphs for centers of mops.

Prop.1.3[2] The center of a mop is non-separable.

Lemma 1.4[2] The graphs in Figure 1 cannot be subgraphs of a center of any mop.

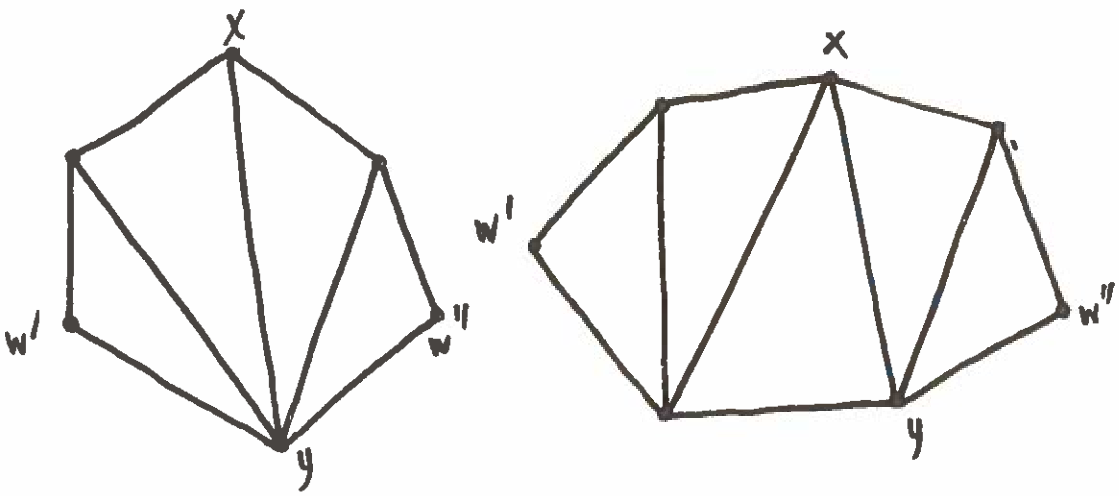


Figure 1 Forbidden subgraphs of centers of mops.

From these facts follows the main result of [2].

Theorem 1.5[2] Every mop G has a center isomorphic to one of the seven graphs in Figure 2.

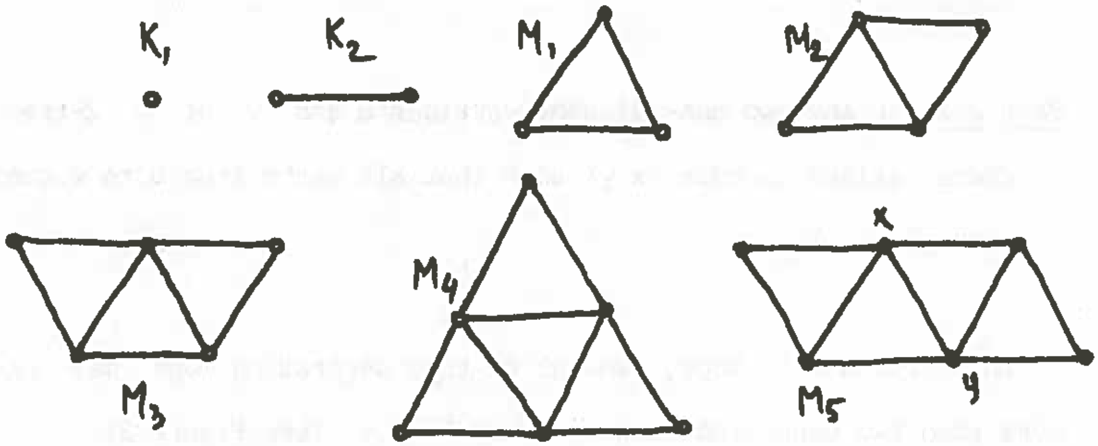


Figure 2 All centers of mops.

2. Centers of 2-trees

We will generalize the above result for so called 2-trees. We define this class of graphs by giving a recursive construction process anologous to that of Fact 1.2 for mops.

Def.2.1 A 2-tree is a graph that can be obtained in the following recursively defined construction process:

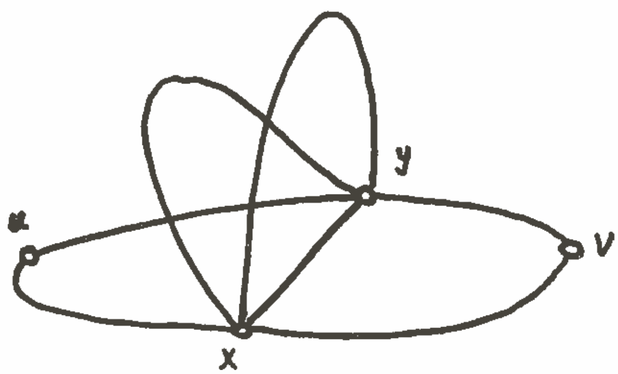
(i) The triangle is the only 2-tree with 3 vertices.

(ii) Given a 2-tree T with n vertices ($n \geq 3$), add a vertex adjacent to any two adjacent vertices of T.

The main difference between mops and general 2-trees is that two adjacent vertices of a 2-tree may have more than two common neighbors. 2-trees preserve the property of mops that for any two non-adjacent vertices u and v there exist two adjacent vertices whose removal separates u and v.

<u>Fact</u> <u>2.2</u> For any two non-adjacent vertices u and v of a 2-tree T, there exists an edge (x,y) such that all paths from u to v contain x or y.

In difference to mops, removal of this separating edge may create more than two connected components of T- x,y (see Figure 3).



<u>Figure</u> <u>3</u> Schematic representation of a 2-tree.

This fact allows us to state properties of centers of 2-trees analogous to those of centers of mops.

<u>Lemma</u> <u>2.3</u> The center of a 2-tree is not separable.

<u>Proof</u> <u>2.3</u> Let us assume that u and v belong to the center of a 2-tree T while a vertex x of a separating them edge (x,y) does not. This implies that there is a vertex z such that d(x,z)>r(G). If after removal of (x,y) z is not in the same connected component as v, then $e(v)\geq d(z,v)>\min(d(y,v),d(x,v))$ and therefore $e(v)\geq d(x,z)$.

This contradicts our assumption. []

Lemma 2.4 Every 2-tree has a center consisting of K1, K2, or a 2-tree.

Proof 2.4 Any non-separable induced subgraph of a 2-tree T with more than two vertices is a 2-tree. Hence the Lemma. []

The separation property expressed by Fact 2.2 allowes a generalization of Lemma 1.4.

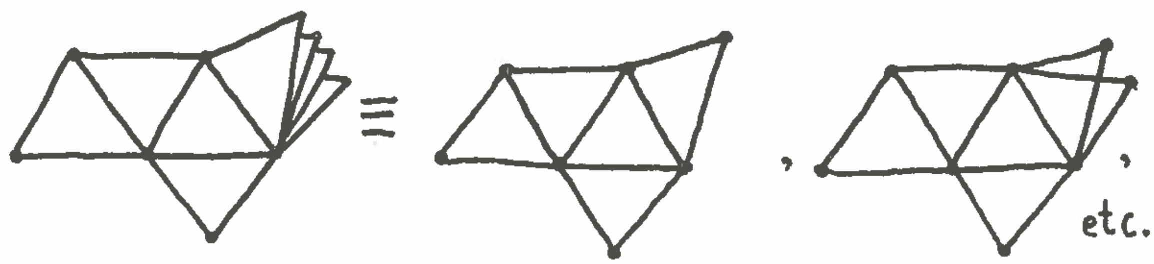
Lemma 2.5 The graphs in Figure 1 cannot be subgraphs of a center of any 2-tree.

Proof 2.5 Let us assume that a graph G (Ga or Gb from Figure 1) is the center of a 2-tree T. Let us assume that a vertex z such that $d(y,z)=r(T)$ is not in the same connected component of T- x,y as w, w w',w" . Then $e(w)\geq d(w,z)>d(y,z)=r(T)$ contradicting the assumption that w is in the center of T. []

We will now describe all 2-trees which do not have graphs of Lemma 2.5 as subgraphs. To facilitate the description we will use the following notion of a set of vertices commonly adjacent to end-vertices of an edge.

<u>Def.2.6</u> For a given edge (x,y) of a 2-tree T, we define <u>deck(x,y)</u> to be the set of all vertices of degree 2 of T adjacent to both x and y.

A schematic representation of a family of 2-trees parametrized by the size of a deck is given in Figure 4.



<u>Figure 4</u> A family of 2-trees.

Let us consider outerplanar subgraphs of such family. We readily observe that all 2-trees in the family have the same set of mop subgraphs, as decks of increased sizes do not introduce any new outerplanar subgraphs (the only exception are the 2-trees with 3 and 4 vertices which belong to the same family). Therefore, all graphs which can be centers of mops may give origin to families of 2-trees which avoid the forbidden subgraphs of Lemma 2.5 as well. These families are obtained by substituting decks for all vertices of degree 2 in the mops. Only one of the mops in Figure 2 has an internal edge (not on the Hamiltonian cycle) with end-vertices not adjacent to a vertex of degree 2 (viz. (x,y) of M5). Appending a deck to this edge does not introduce any of the forbidden subgraphs. By inspection we find that the addition of vertices adjacent to the introduced decks either results in a 2-tree of another family of this group, or introduces a forbidden subgraph of Lemma 2.5. This proves our main theorem.

Theorem 2.7 Every 2-tree has a center isomorphic to a member of the families of 2-trees in Figure 5.

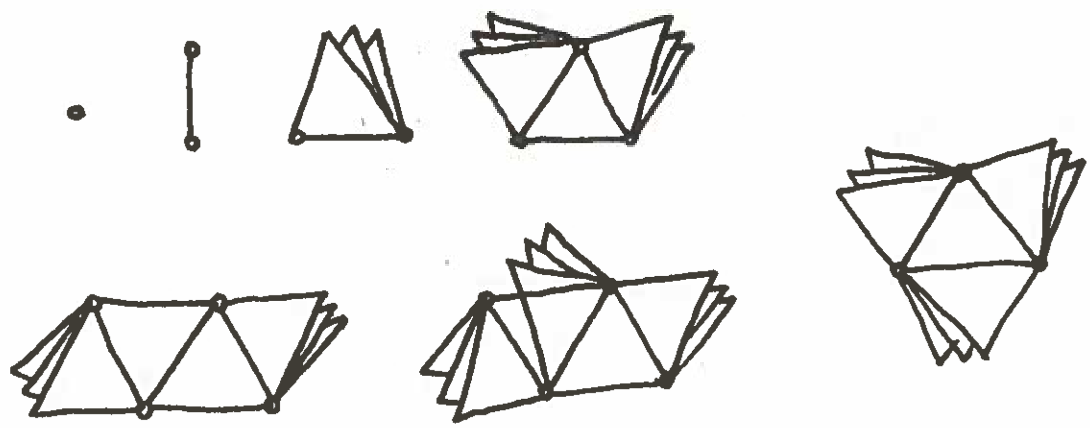


Figure 5 Families of centers of 2-trees.

6. References

[1] A.Proskurowski: Minimum dominating cycles in maximal outerplanar graphs, CS-TR-77-4, University of Oregon, to appear in Int'l J. of Comp. Info. Sci. 8, 4(1979).

[2] A.Proskurowski: Centers of maximal outerplanar graphs, CS-TR-79-4, University of Oregon.