CIS-TR-80-8

WHICH GRIDS ARE HAMILTONIAN?

by

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Abstract

A complete grid $G_{m,n}$ is a graph having m x n vertices which are connected to form a rectangular lattice in the plane, i.e. all edges of $G_{m,n}$ connect vertices along horizontal or vertical lines. A grid is a subgraph of a complete grid. We study the existence of Hamiltonian cycles in complete grids and complete grids with one or two vertices removed.

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\+ Research supported in part by the National Science Foundation under Grant MCS-79-03913

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1. Introduction

A complete grid $G_{m,n}$ is a graph having m x n vertices which are connected to form a rectangular lattice in the plane, i.e., all edges of $G_{m,n}$ connect vertices along horizontal or vertical lines. A grid is a subgraph of a complete grid.

Complete grids describe the basic pattern of streets in sections of virtually every city and town. Most street systems, however, correspond more closely to grids than complete grids since large hospital complexes or city parks block some of the streets, as if a vertex or two have been removed from a complete (sub-) grid. The French Quarter of New Orleans (Figure 1) provides a typical example of a grid.

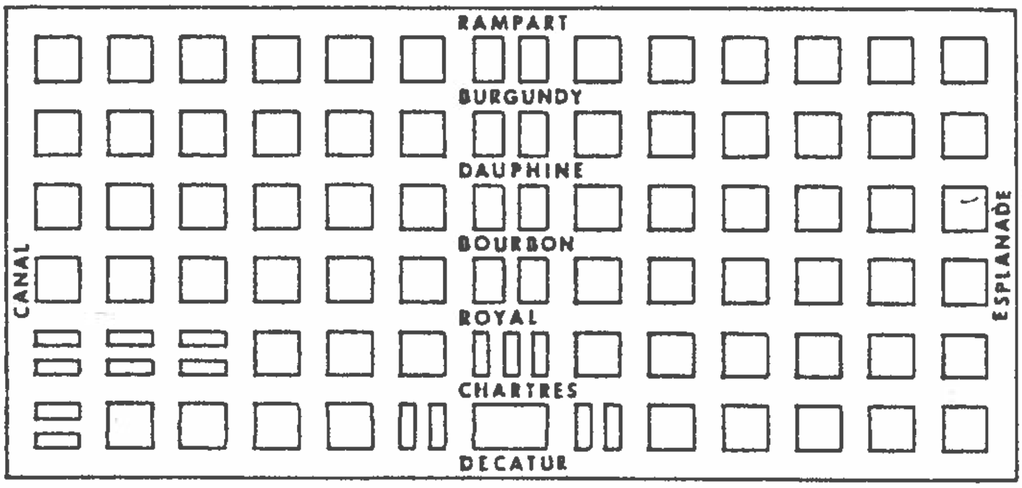

Figure 1. The French Quarter of New Orleans.

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‡ Research supported in part by the National Science Foundation under Grant MCS-79-09313.

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* This work was supported in part by the U. S. Department of Energy under Contract No. AT(29-1)-789.

In this paper we are interested in the existence of Hamiltonian cycles in complete grids and complete grids with one or two vertices removed. As such this work relates to that of Thompson [3], who determined which complete grids are Hamiltonian; Simmons [2], who studied which n-dimensional lattices are Hamilton-laceable; and Cull [1], who studied which complete grids (checkerboards) have knight's tours.

We determine for most values of $m,n \geq 1$, which grids $G_{m,n} - \{u\}$ and $G_{m,n} - \{u,v\}$, are Hamiltonian.

2. Preliminary results

Let $G_{m,n}$ be a complete grid where $V(G_{m,n}) = \{v_{i,j}: 1 \leq i \leq m, 1 \leq j \leq n\}$. Let $S \subseteq V(G)$, with $S = \emptyset$ a possibility, and consider when $G_{m,n} - S$ will be Hamiltonian.

Let $V_1 = \{v_{i,j}: i + j \text{ is even}\}$, and $V_2 = \{v_{i,j}: i + j \text{ is odd}\}$. If we associate the vertices of $G_{m,n}$ with the squares of an $m \times n$ checkerboard, then the set of vertices V_1 will correspond to the red squares and V_2 will correspond to the black squares of the checkerboard. The upper left red square is $v_{1,1}$ and there are m rows and n columns.

<u>Lemma 1.</u> If $G_{m,n} - S$ is Hamiltonian, then $|V_1 - S| = |V_2 - S|$.

<u>Proof.</u> $G_{m,n}$ is a bipartite graph, as is any subgraph $G_{m,n} - S$, where the vertices form a bipartition $V_1 - S$, $V_2 - S$. Since the vertices in any (Hamiltonian) cycle must alternate between $V_1 - S$ and $V_2 - S$, if $G_{m,n} - S$ is Hamiltonian, then $|V_1 - S| = |V_2 - S|$.

Stated in terms of checkerboards, Lemma 1 simply says that if $G_{m,n} - S$ is Hamiltonian, then the number of red squares must equal the number of black squares.

<u>Theorem 2</u> [Thompson]. For $m,n \geq 2$, $G_{m,n}$ is Hamiltonian if and only if m times n is even.

<u>Proof.</u> The necessity follows immediately from Lemma 1; the sufficiency follows by the simple construction of a Hamiltonian cycle as indicated in Figure 2 where it is assumed that n is even, $n = 2k$.

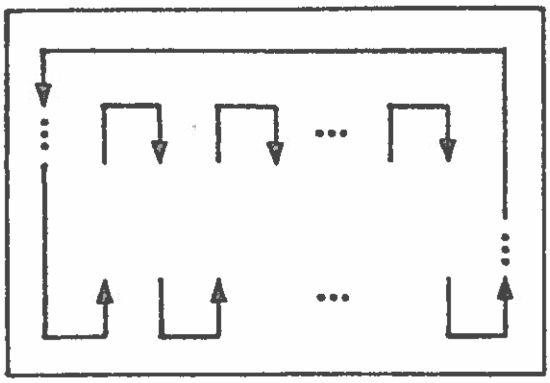


Figure 2. Hamiltonian cycles in complete grids $G_{m,2K}$.

3. Complete grids with one vertex (square) removed

We now consider which grids of the form $G_{m,n} - \{v\}$ are Hamiltonian. The following result is immediate from Lemma 1.

Proposition 3. If $G_{m,n} - \{v\}$ is Hamiltonian, then m times n is odd.

Theorem 4. For $m,n \geq 1$, $G_{2m+1,2n+1} - \{v\}$ is Hamiltonian, if and only if $v \in V_1$.

Proof. In $G_{2m+1,2n+1}$, the cardinality of V_1 is one greater than the cardinality of V_2. Therefore, by Lemma 1, v must be in V_1.

The sufficiency can be shown by the constructions of Hamiltonian cycles indicated in Figure 3. If $v \in V_1$, then there are two cases to consider: (a) v lies in an even numbered row and column (in which case there are an odd number of rows and columns in each direction from v (cf. Figure 3a)) or (b) v lies in an odd numbered row and column (cf. Figure 3b).

Consider for example Figure 3a. First, locate in $G_{m,n}$ the smallest 3×3 grid G containing v such that there are an odd number (≥ 1) of rows and columns above and below, to the left and right of the square for v. We first construct the simple Hamiltonian cycle for $G - v$. This cycle can then be augmented, as indicated in Figure 3a, for every additional two columns (3 rows high) on either side. It can then be augmented in the same way for every two additional rows above and below.

This manner of augmentation can also be applied in Figure 3b.

(a)

(b)

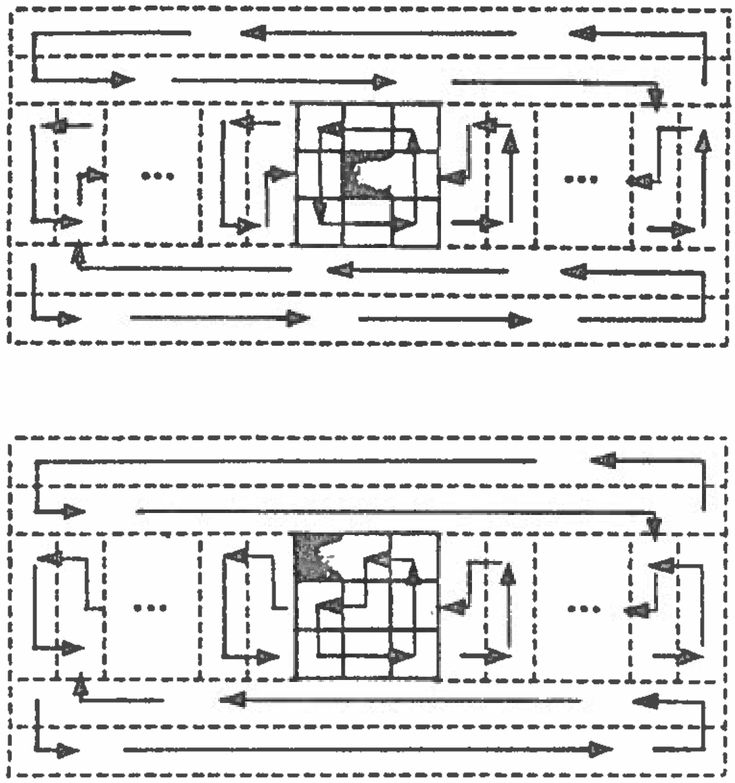


Figure 3. Complete grids with one square removed

4. Complete grids with two vertices (squares) removed

We next consider grids $H = G_{m,n} - \{u,v\}$ with 2 vertices removed. Clearly such a grid H is not Hamiltonian if H is not 2-connected. From Lemma 1 and the previous statement the following necessary conditions for H to be Hamiltonian are obvious.

<u>Lemma 5.</u> If $G_{m,n} - \{u,v\}$ is Hamiltonian, then

(i) m times n is even

(ii) $v \in V_1$ and $u \in V_2$, and

(iii) $G_{m,n} - \{u,v\}$ is 2-connected.

The next result is immediate by construction and Lemma 5.

<u>Theorem 6.</u> Let $S = \{u,v\}$. Then the grid $G_{2,n} - S$ is Hamiltonian if and only if $n \geq 3$ and $S = \{v_{1,n}, v_{2,n}\}$ or $S = \{v_{1,1}, v_{2,1}\}$.

We have two general cases to consider, either both m and n are even, or one of m or n is odd. We begin with the case where both m and n are even, and we first consider Hamiltonian paths in grids of the form $G_{2h,4}$. The following result can easily be shown.

Lemma 7. Let $G_{m,4}$ be an even by four grid. If $v \in V_1$, i.e., $v = v_{i,j}$ such that $i + j$ is even, then $G_{m,4} - \{v\}$ has a Hamiltonian path from $v_{m,1}$ to $v_{1,4}$ and a Hamiltonian path from $v_{m,1}$ to $v_{m,3}$.

We use this result to prove the following.

Theorem 8. Let $G_{m,n}$ be a grid such that $m,n \geq 4$ are both even. Consider $G_{m,n} - \{v\}$ where $v \in V_1$ and v is not $v_{1,n-1}$ or $v_{2,n}$. Then in $G_{m,n} - \{v\}$ there is a Hamiltonian path from $v_{m,1}$ to $v_{m,n-1}$.

Proof. Let v be located in column j where $1 \leq j < n-1$. Partition $G_{m,n}$ into three blocks of columns such that block 2 contains four columns, one of which is j. Furthermore, j is even numbered or odd numbered in the block according to whether j is even numbered or odd numbered in $G_{m,n}$. This implies that both blocks 1 and 3 have an even number of columns. Construct three paths in $G_{m,n} - \{v\}$: (a) from $v_{m,1}$ to $v_{m,k}$, where k is the leftmost column in block 2; (b) from $v_{m,k}$ to $v_{1,k+4}$; and (c) from $v_{1,k+4}$ to $v_{m,n-1}$. Path (b) can be constructed by Lemma 7. Paths (a) and (c) can be constructed as in Figure 4.

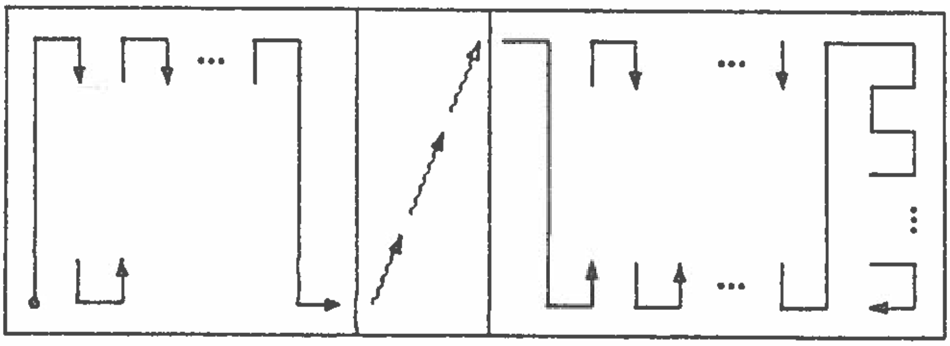


Figure 4. Paths for the solution of Theorem 8 with $j < n-1$.

Finally, let v be in column $n-1$ or in column n. Then $G_{m,n}$ can be partitioned into two blocks of columns. Block 1 will consist of the first $n-2$ columns and hence will have an even number of columns. Block 2 will be the last two columns. Create two paths - one from $v_{m,1}$ to $v_{2,n-1}$ and a path from v_{2n-1} to $v_{m,n-1}$. The constructions will be as in Figures 5a or 5b if v is in column $n-1$ or in column n, respectively (note the excepted vertices,

vertices $v_{1,n-1}$ and $v_{2,n}$).

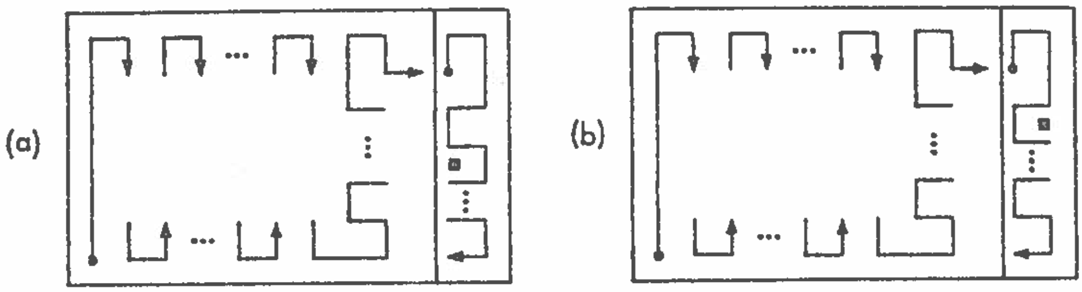


Figure 5. Paths for Theorem 8 with $j = n-1$ or n.

<u>Theorem 9.</u> Let $G_{m,n}$ be an even by even grid where $m,n \geq 4$, $S = \{v_1,v_2\}$, where $v_1 \in V_1$ and $v_2 \in V_2$. Then $G_{m,n} - S$ is Hamiltonian if and only if $G_{m,n} - S$ is 2-connected. That is, we exclude the case where a vertex is removed which is adjacent to a "corner" vertex without removing the "corner" vertex itself.

<u>Proof.</u> $G_{4,4} - S$, $G_{6,4} - S$ and $G_{6,6} - S$ can be shown to satisfy the theorem by exhaustively considering all possibilities for S. The cases for $m,n \geq 8$ will be handled by induction using Theorem 8.

Assume $m \geq 8$. If the top four rows or the bottom four rows do not contain an element of S, then we can assume that the bottom four rows do not. Easily handled special cases are $v_1 = v_{m-4,2}$ and/or $v_2 = v_{m-4,n-1}$. Otherwise, by induction, $G_{m-4,n} - S$ has a Hamiltonian cycle that must contain edge $(v_{m-4,1}, v_{m-4,2})$. Replace this edge in the cycle and the edge $(v_{m-3,1}, v_{m-3,2})$ in a Hamiltonian cycle in the last four rows obtained by using the pattern in Figure 2 with the edges $(v_{m-4,1}, v_{m-3,1})$ and $(v_{m-4,2}, v_{m-3,2})$. We then have a Hamiltonian cycle for $G_{m,n} - S$.

Assume therefore that both the top four rows and the bottom four rows contain a member of S. By symmetry, it can be assumed that the top four rows contain the element $v_1 \in V_1$. Apply Theorem 8 to $G_{m-4,n} - \{v_1\}$ to get a Hamiltonian path P_1 from $v_{4,1}$ to $v_{4,n-1}$. By symmetry, Theorem 8 can also be applied to the bottom four rows minus v_2 to get a Hamiltonian path P_2 from $v_{m-3,1}$ to $v_{m-3,n-1}$. For the even number of intermediate rows between the top and bottom four rows, paths from $v_{m-3,1}$ to $v_{4,1}$ and from $v_{m-3,n-1}$ to $v_{4,n-1}$ can easily be constructed, as indicated in Figure 6 to produce a Hamiltonian cycle in $G_{m,n} - S$.

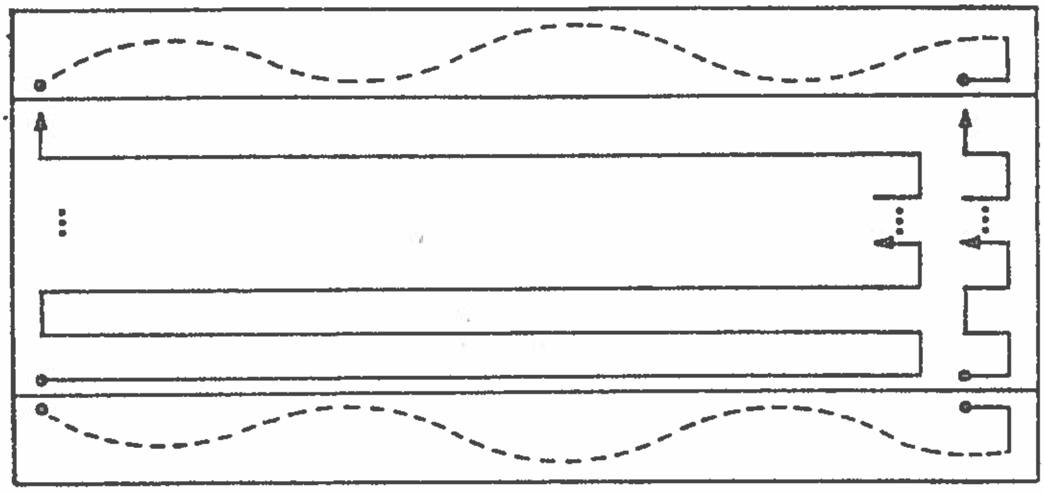


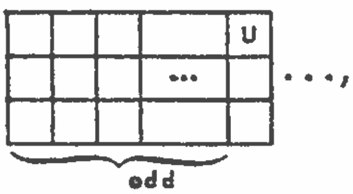
Figure 6. Completion of the Hamiltonian paths in Theorem 9.

The next generalization follows easily from the proof of Theorem 9; it can be proved by induction in much the same way having verified the result for $G_{4,4}$, $G_{4,5}$ and $G_{4,6}$ by exhaustion. In the interest of brevity, we will omit the details.

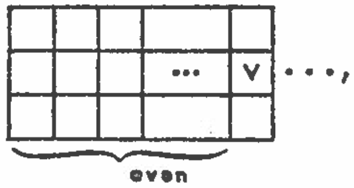
Theorem 10. Let $m \geq 4$ be any integer (including odds) and $S = \{v_1, v_2\}$ with $v_1 \in V_1$ and $v_2 \in V_2$. Then $G_{m,4} - S$ is Hamiltonian if and only if $G_{m,4} - S$ is 2-connected.

Our second general case is grids $G_{m,n} - S$ where m is odd and n is even; $S = \{v_1, v_2\}$, where $v_1 \in V_1$ and $v_2 \in V_2$, i.e., v_1 is a red square and v_2 is a black square on a checkerboard.

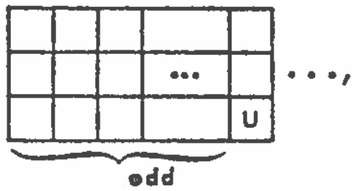
Theorem 11. Any 2-connected grid of the form $G_{3,2n} - \{u,v\}$, where u is black and v is red is Hamiltonian if and only if it does not begin (or end) with one of the following three forbidden patterns:

(i) 

$u = v_{1,2h}$
with v to the right of u,

(ii) 

$v = v_{2,2h+1}$
with u to the right of v, or

(iii) 

$u = v_{3,2h}$
with v to the right of u

Proof. Clearly none of the forbidden patterns could result in a Hamiltonian graph since the cycle would cover all of the squares in the pattern by entering and leaving through the two squares in the column containing the removed square. But this is impossible since, in cases (i) and (iii) there are an odd number of squares in the pattern, and the only way the path could cover all of them would be to enter and leave on the same parity. In case (ii), the opposite is true, there are an even number of squares in the forbidden pattern, but the two squares by which the Hamiltonian cycle must enter and leave have the same parity.

It only remains to show how to construct a Hamiltonian cycle if none of the forbidden patterns are present.

We will only present this construction in the case where u occurs in the first row and v occurs in some column to the right of u. Since none of the forbidden patterns occur we can assume that an even number of columns occur to the left of the column contining u.

If v occurs in row 1 or 3 (resp. 2), then by the same reasoning an even (resp. odd) number of columns must appear to the right of v. Figure 7 illustrates the construction of a Hamiltonian cycle in each of the three possible cases.

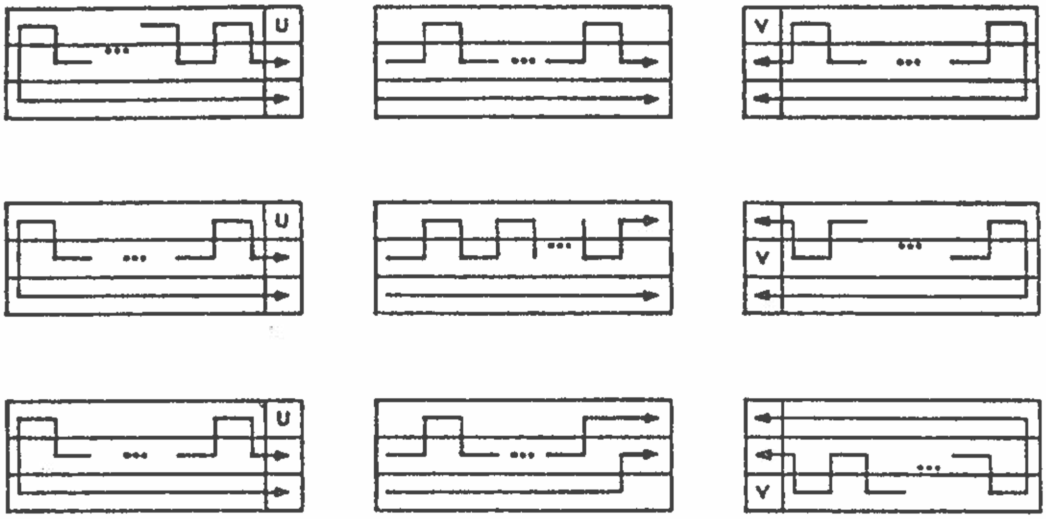
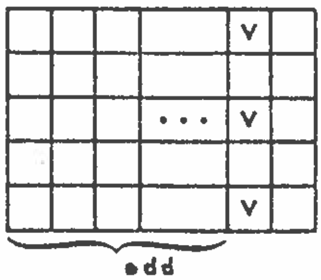


Figure 7. Patterns for cases in Theorem 11.

In the interest of brevity we will only sketch the proof of the next result.

Theorem 12. Any 2-connected grid of the form $G_5 = G_{5,2n} - \{u,v\}$, where u is red and v is black is Hamiltonian if and only if it does not begin (or end) with one of the following five forbidden patterns:

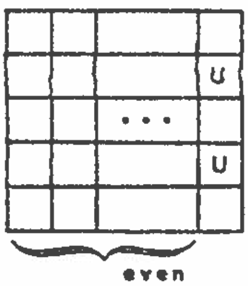
(a)



. . .

$v = v_{1,2h}$, $v_{3,2h}$ or $v_{5,2h}$, and
$u = v_{i,j}$ with $j \geq 2h+2$,

(b)



. . .

$u = v_{2,2h+1}$ or $v_{4,2h+1}$, and
with v to the right of u.

Proof. It is easy to show, using a simple parity argument, that any grid of the form G_5, which begins or ends with one of the five forbidden patterns, is not Hamiltonian. For example, in pattern (a) above, there are an odd number of squares to the left of the column containing v. Any Hamiltonian cycle would therefore have to enter and leave this pattern (through the column containing v) on rows 2 and 4. (See, for example, Figures 8f, h and j.) But then all of the squares in the column to the right of v cannot be covered. Similarly, in pattern (b), any Hamiltonian cycle would have to enter and leave the columns to the left of u in rows 1 and 4 (or rows 2 and 5), in which case all of the squares in the column containing u cannot be covered.

If none of the forbidden patterns exist, then it remains to show how to construct a Hamiltonian cycle. We do this by partitioning the even number of columns into blocks of 2. We then determine the leftmost (L) and rightmost (R) blocks containing a deleted square (possibly these are the same (L=R)). Using the diagrams in Figure 8, and observing the obvious symmetries between left and right, we can then construct a Hamiltonian cycle for the part of the grid between (and including) blocks L and R. It is then a simple matter to augment this cycle by adding on the remaining columns, two at a time, until a Hamiltonian cycle for the entire grid is constructed (cf. Figure 9).

In Figure 8 we indicate beneath each block that all the squares can be covered by entering and leaving various rows.

Furthermore, since we are assuming that G_5 is 2-connected, and possibly these are the 2 leftmost columns in G_5, in cases b, d, f and j in Figure 8 we can assume that there are two additional columns to the left. However, in cases b and d this produces a forbidden pattern.

We leave the remaining details to the reader, including the cases where L = R.

It is somewhat surprising that for 2-connected grids of the form $G_7 = G_{7,2n} - \{u,v\}$ there are no forbidden patterns.

Theorem 13. Every 2-connected grid of the form $G_7 = G_{7,2n} - \{u,v\}$ where u is red and v is black, is Hamiltonian.

Proof. Again we will only sketch the proof. As in the proof of Theorem 12, we partition the columns in blocks of two, and determine the leftmost

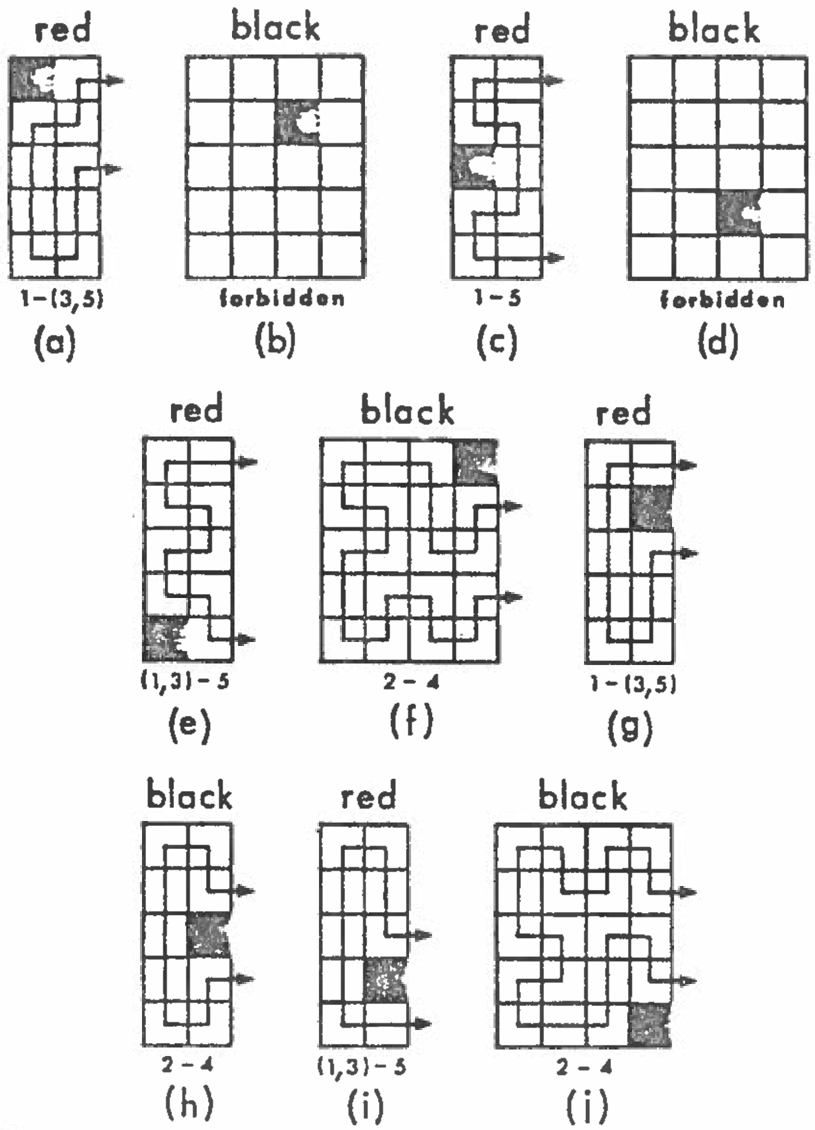


Figure 8. Leftmost blocks in Theorem 12.

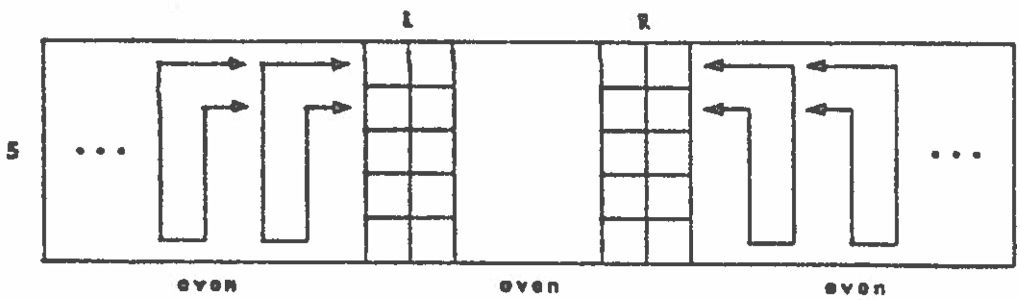


Figure 9. Construction of Hamiltonian cycle for Theorem 12.

(L) and rightmost (R) block containing a deleted vertex. Figure 10 illustrates this partitioning, where possibly any of the blocks labeled I, II, or III are empty.

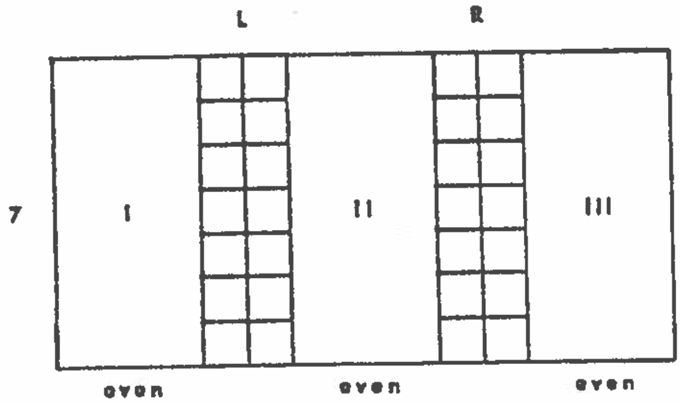


Figure 10. Partitions of G_7.

Figure 11 illustrates all possible cases for block L, along with paths which traverse every square in these blocks.

The corresponding rightmost blocks R can easily be constructed by symmetry. It is then an easy matter to construct a Hamiltonian cycle containing all the squares in blocks L, II and R. Finally, by augmenting this cycle two blocks at a time to include all the columns in blocks I and III, one can construct a Hamiltonian cycle for G_7. We again leave the details to the reader, including the cases where L = R, i.e., both u and v belong to the same block of two columns.

Although we have not to date been able to generalize the proof of Theorem 13 in any simple manner, we strongly conjecture that the obvious generalization is true.

Conjecture: Every 2-connected grid of the form $G_{2m+1,2n} - \{u,v\}$, where u is red and v is black, and $m \geq 3$, is Hamiltonian.

Finally, Figure 12 provides an example of a grid $H = G_{m,n} - S$ where $|S| = 8$,

(i) $|V(H)|$ is even,

(ii) $|V_1 - S| = |V_2 - S|$, and

(iii) H - S is 2-connected,

yet H is not Hamiltonian.

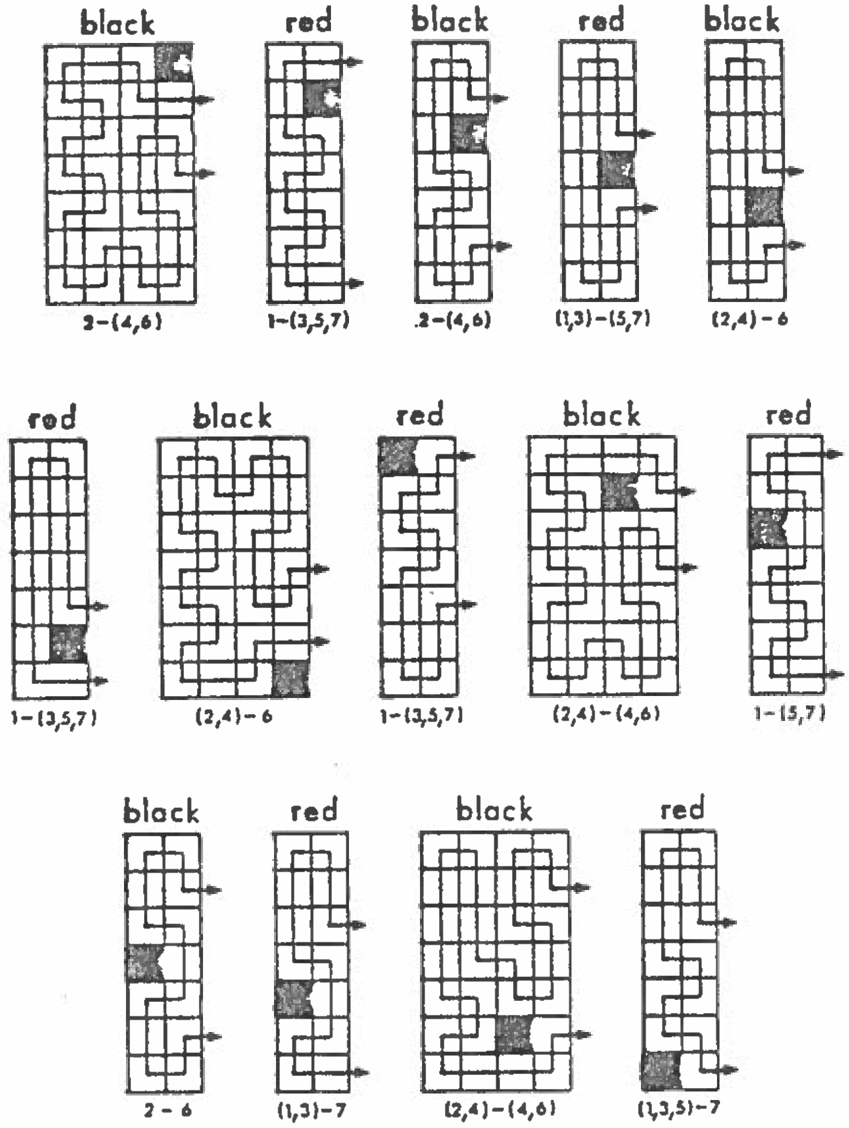


Figure 11. Leftmost blocks L.

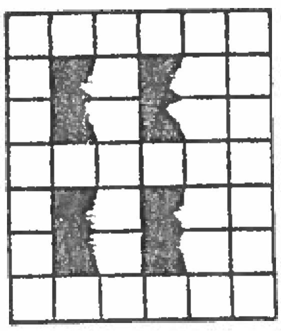


Figure 12. A non-Hamiltonian grid where $|S| = 8$.

It would be interesting to know if there are smaller examples, i.e., $|S| \leq 7$, where conditions (i), (ii) and (iii) do not imply that H is Hamiltonian.

5. Bibliography

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