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A NOTE ON THE COMPLEXITY OF VERTEX AND EDGE PARTITION PROBLEMS FOR GRAPHS*

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Abstract

In this brief note we make a simple observation which relates the maximum matching problem to the family of vertex partition problems, many members of which are known to be NP-complete. Based on this observation we suggest a large number of unsolved problems, algorithmic solutions to any one of which would generalize standard matching algorithms, yet could still run in polynomial time.

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1. Introduction

The general vertex partition problem for graphs can be stated as follows: given a graph $G = (V,E)$, partition the vertices $V(G)$ into a minimum number of subsets $V_1, V_2, \ldots, V_k$ so that the subgraph $\langle V_i \rangle$ induced by each subset V_i has property P.

A closely related problem is the vertex maximization problem: given a graph $G = (V,E)$, determine the largest order of a set of vertices $W \subseteq V$ such that the subgraph $\langle W \rangle$ induced by W has property P. Table 1 provides a list of typical properties P that one might want to consider.

Unfortunately, for most of these properties P, the corresponding vertex partition and vertex maximization problems are NP-complete; as indicated in Table 1, where the citations refer to NP-complete problems mentioned in the book by Garey and Johnson [2].

Similar edge partition and edge maximization problems can also be defined for these properties P. Table 2 summarizes the situation for these problems. For definitions of terms indicated in braces, cf. Harary [3].

2. Restricted vertex problems

Although the majority of these vertex partition problems are NP-complete, the situation becomes somewhat different when we place the added requirement that each block V_i of the partition be restricted in size, i.e. the restricted vertex partition problem is to partition the vertices $V(G)$ into a minimum number of subsets $V_1, V_2, \ldots, V_k$ of size R or less, such that the subgraph $\langle V_i \rangle$ induced by each subset V_i has property P.

It is interesting to note that for the value $R = 2$, all restricted vertex partition problems are either trivial or equivalent to the well-known maximum matching problem.

The maximum matching problem is usually stated as follows: given a graph $G = (V,E)$, find a maximum set of edges M, no two of which have a vertex in common. However, it can also be stated equivalently as:

partition the vertices V(G) into a minimum number of subsets $V_1, V_2, \ldots, V_k$ where the subgraph $\langle V_i \rangle$ induced by each subset V_i is a complete graph with $R \leq 2$ vertices.

Table 3 summarizes this situation for the previously mentioned vertex partition problems. Thus, it can be seen that, in this sense, matching lies at the root of a large number of NP-complete combinatorial problems.

In this light it is interesting to focus attention on the problems that result when $R = |V_i| \leq 3$. In this category there appear to be four distinct types of problems, as indicated in Table 3, depending on the particular subset of subgraphs with ≤ 3 vertices which are allowed in a block of a partition. Given that the general matching problem (for $R \leq 2$) can be solved in $O(n^{5/2})$ time (cf. [1]), it is interesting to consider what the time complexity of any of these ($R \leq 3$) problems must be. We suspect, however, that even for $R \leq 3$ some of these problems might be NP-complete.

3. Restricted edge problems

The situation for restricted edge partition problems is equally interesting, i.e. partition the edges E(G) into a minimum number of subsets $E_1, E_2, \ldots, E_k$ of size R or less, such that the subgraph $\langle E_i \rangle$ induced by each subset E_i has property P. Certainly for $R \leq 1$ all of these problems are trivial, but for $R \leq 2$ or $R \leq 3$ most of these problems become challenging.

To the best of our knowledge no one has designed an algorithm for partitioning the edges of an arbitrary graph into a minimum number of subsets, each of which has property P <u>and</u> size $R \leq 3$, for any of the properties mentioned in Table 4.

Of particular interest would be an efficient "edge matching" algorithm for partitioning the edges of a graph G into a minimum number of adjacent pairs of edges or singleton edges. One approach to solving this problem is to apply a 'standard' matching algorithm to the edge-graph of G (or line graph L(G) in Harary's terminology [3]). However, there may very well be a simpler, more direct approach.

Similarly, an algorithm for partitioning the edges of G into a minimum number of pairs of non-adjacent edges or singleton edges (cf. P_{11}, $R \lesssim 2$, Table 4) can be obtained by applying a 'standard' matching algorithm to the complement of the edge-graph of G. Again, it might be possible to construct a simpler algorithm.

4. Summary

The preceding observations raise a large number of questions and challenges concerning the design and complexity of restricted or unrestricted vertex or edge partition algorithms for a variety of properties P. It would be interesting to see efficient algorithms for solving any of these problems, and it would be interesting to see which of these problems are NP-complete.

5. Bibliography

[1] S. Even and O. Kariv, An $O(n^{2.5})$ algorithm for maximum matching in general graphs, 16th Annual Symp. on Foundations of Computer Science, IEEE, 1975, 100-112.

[2] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman and Co., San Francisco, 1979.

[3] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

PROPERTY	VERTEX-PARTITION PROBLEM	VERTEX-MAXIMIZATION PROBLEM
P_1: is a complete graph	[GT15] < clique-partition >	[GT19]. < largest clique >
P_2: is a totally disconnected graph	[GT4] < chromatic no. >	[GT20] < vertex-independence no. >
P_3: is a tree		
P_4: is a connected graph	trivial	trivial
P_5: is a star	< star partition no. >	[GT20]
P_6: contains a spanning star	[GT2] domination no.	trivial
P_7: is a path		[GT23]
P_8: contains a spanning (or Hamiltonian) path		[GT34]
P_9: is an acyclic graph	[GT14] < vertex arboricity >	[GT21]
P_{10}: is a planar graph	< vertex thickness >	[GT21]
P_{11}: is a disconnected graph		< vertex connectivity >
P_{12}: has diameter ≤ 2		[GT21]
P_{13}: is a (connected) bipartite graph	< biparticity >	[GT22]
P_{14}: has maximum degree $\leq k$	[GT16]	[GT26]

Table 1. Vertex Problems

PROPERTY	EDGE-PARTITION PROBLEM	EDGE-MAXIMIZATION PROBLEM
P_1: is a complete graph		[GT19]
P_2: is a totally disconnected graph		trivial
P_3: is a tree		trivial
P_4: is a connected graph	trivial	trivial
P_5: is a star	[GT1] vertex covering no.	trivial
P_6: contains a spanning star		trivial
P_7: is a path		[GT39]
P_8: contains a spanning (or Hamiltonian) path		[GT39]
P_9: is an acyclic graph	arboricity	trivial
P_{10}: is a planar graph	thickness	[GT27]
P_{11}: is a disconnected graph		edge-connectivity
P_{12}: has diameter $\leq k$		
P_{13}: is a (connected) bipartite graph		[GT25]
P_{14}: has maximum degree $\leq k$		[GT26]

Table 2. Edge Problems

PROPERTY	POSSIBLE SUBGRAPHS FOR $R = \lvert V_i \rvert \leq 2$	$R = \lvert V_i \rvert \leq 3$	PROBLEM TYPE $\lvert V_i \rvert \leq 3$
P_1: is a complete graph	 matching		I
P_2: is a totally disconnected graph	 matching		I*
P_3: is a tree	 matching		II
P_4: is a connected graph	 matching		III
P_5: is a star	 matching		II
P_6: contains a spanning star	 matching		III
P_7: is a path	 matching		II
P_8: contains a Hamiltonian path	 matching		III
P_9: is an acyclic graph	 trivial		IV
P_{10}: is a planar graph	 trivial		trivial
P_{11}: is a disconnected graph	 matching*		III*
P_{12}: has diameter ≤ 2	 matching		III
P_{13}: is a (connected) bipartite graph	 matching		II
P_{14}: has maximum degree $\leq k$			

Table 3. Restricted vertex problems

PROPERTY	POSSIBLE SUBGRAPHS FOR $R = \lvert E_i \rvert \leq 2$	$R = \lvert E_i \rvert \leq 3$
P_1: is a complete graph	trivial	I Δ
P_2: is a totally disconnected graph	not applicable	I II III
P_3: is a tree	I L	I L Y ⊔
P_4: is a connected graph	I L	I L Y ⊔ Δ
P_5: is a star	I L	I L Y
P_6: contains a spanning star	I L	I L Y ⊔ Δ
P_7: is a path	I L	I L ⊔
P_8: contains a Hamiltonian path	I L	I L ⊔ Δ
P_9: is an acyclic graph	trivial	I L I⊔ ⊔ Y LI
P_{10}: is a planar graph	trivial	trivial
P_{11}: is a disconnected graph (* or a K_2)	I II	I II III LI
P_{12} : has diameter ≤ 3	I L	I L Y ⊔ Δ
P_{13}: is a (connected) bipartite graph	I L	I L Y ⊔
P_{14}: has maximum degree $\leq k$		

Table 4. Restricted edge problems