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EXTREMAL GRAPHS WITH NO DISCONNECTING INDEPENDENT VERTEX SET OR MATCHING*

by

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Abstract

We describe classes of graphs with the minimum number of edges for a given number of vertices which have no disconnecting, independent set of vertices or edges. An independent vertex set has no adjacent pair of vertices. An independent edge set (i.e., a matching) has no pair of edges incident to the same vertex. A disconnecting set is one which, by its removal, transforms the given, connected graph into one having at least two connected components. We note a relationship to previously determined extremal graphs having certain connectivity and forbidden subgraph properties.

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A k-tree with k vertices is a graph whose vertex set forms a k-clique (i.e., a complete graph with k vertices); given any k-tree T with n vertices, $n \geq k$, a k-tree with n+1 vertices is obtained when the (n+1)st vertex is made adjacent to each vertex of a k-clique in T. The result of Rose [3] restricted to 2-trees states that a graph is a 2-tree iff every minimal separator is an edge. As no set of independent vertices induces an edge, a 2-tree cannot be disconnected by an independent vertex set. 2-trees are also minimal such graphs, i.e., one cannot remove an edge of a 2-tree and preserve the property. The following theorem follows from [3, Proposition 3.3].

Theorem 2.1 Any graph of order n and size less than 2n-3 can be disconnected by some independent vertex set.

Actually, a graph with any minimal separator containing an edge has the property of remaining connected after removal of any set of independent vertices and edges. Let us call this property R. Any 2-tree has property R. In fact, the class of 2-trees is exactly the class of minimum size graphs with this property.

We first state a technical lemma.

Lemma 2.2 Let G_1 and G_2 be two 2-trees and (x,y_1) and (z,y_2) be edges, respectively in G_1 and G_2. The graph obtained by identifying y_1 and y_2 as a vertex y and adding a vertex v adjacent to x, y, and z is a 2-tree.

Proof There is a perfect elimination order [3] reducing G_1 to (x,y_1) and G_2 to (z,y_2). By definition, the subgraph induced by vertices x, y, z, v is a 2-tree. Thus, a construction of G starting with this subgraph and using the reversed elimination order of adding vertices defines a 2-tree. []

Theorem 2.3 A graph G of order n > 2 is a minimum size graph with no disconnecting independent set of vertices and edges iff it is a 2-tree.

Let us define the following reduction rules. We denote them according to the subgraphs they eliminate, see Figure 1.

(C2) Contract any two parallel edges (Figure 1a). Any two vertices u and v adjacent through more than one edge are collapsed to result in a single vertex w preserving all external adjacencies.

(C3) Contract any triangle, C_3. (Figure 1b). The three vertices of such a triangle collapse into one preserving all external adjacencies.

(C4) Contract two independent edges of any C_4. (Figure 1c). Let u_1, u_2, u_3, and u_4 be vertices of a C_4 with sets of external adjacent vertices N_1, N_2, N_3, and N_4, respectively. The vertices u_1 and u_2 collapse into w_1 whose neighborhood is the union of N_1 and N_2. The vertex w_2 results from collapsing u_3 and u_4 and also inherits their neighborhoods, $N_3 \cup N_4$. Vertices w_1 and w_2 are connected by a new edge.

(P2) Eliminate a vertex u of degree 2 and its neighbor v of degree 3 (Figure 1d). Connect the other two neighbors of v by an edge and remove u, v and all edges incident to them.

By a direct count of removed and introduced vertices and edges, we see that the application of any of the four above rules reduces the size and the order of a given graph by three edges and two vertices (C3, C4, and P2), or, even at a greater ratio, by at least two edges and a vertex (C2). We have to prove that these reductions preserve property P.

Lemma 3.1 Given a graph G, let G' be a graph resulting from application of any of the reduction rules C2, C3, C4, or P2 to G. If G has property P, then G' also has this property.

[Figure 1]

Lemma 3.2 In a graph G with property P and no C_3 or C_4, any vertex of degree d can be adjacent through simple edges to at most d-2 vertices of degree 2.

Proof A matching disconnecting a vertex v of degree d and adjacent to it d-1 vertices of degree 2 from the rest of G is shown in Figure 2. Thus G cannot have property P.

[Figure 2]

This observation allows us to establish a lower bound on the size-to-order ratio for non-trivial non-reducible graphs with property P.

Lemma 3.4 A graph G with property P which does not admit application of any of the reduction rules (C2)-(P2) either has only one vertex or has the ratio of its size to its order at least 3/2.

Proof If G has more than one vertex, then all of its vertices of degree 2 (if there are any) are adjacent to vertices of degree at least 4. Let S be the set of vertices of degree 2 and F the set of adjacent to them vertices and let us denote their cardinalities by s and f, respectively. Then $\sum_{v \in F} \deg(v) \geq 4f$ and, by Lemma 3.2, $\sum_{v \in F} (\deg(v)-2) \geq 2s$. These inequalities give us $2s+\sum_{v \in F}\deg(v) \geq 3(s+f)$, and thus

$$\sum_{v \in V}\deg(v)=\sum_{S}\deg(v)+\sum_{F}\deg(v)+\sum_{V-(F \cup S)}\deg(v) \geq 3(s+f)+3(n-(s+f))=3n$$

The size of G is thus at least 3n/2, as postulated. []

Considering that all our reductions decrement the size and the order of a graph with a constant ratio of 3/2, we have finally obtained a lower bound on the minimum size of graphs with property P.

Theorem 3.5 A graph of order n with no disconnecting matching must have at least $\lceil 3(n-1)/2 \rceil$ edges.

Proof Let G be a graph with property P with n vertices and m edges, and let successive

(AUG1) Augment a vertex v by a pair of mutually adjacent vertices adjacent only to v.

(AUG2) Replace an edge (w_1,w_2) by a pair of non-adjacent vertices, each adjacent only to both w_1 and w_2.

Lemma 4.1 A graph G' obtained from a graph G by application of rules AUG1 and AUG2 has property P iff G has property P.

Proof In a graph G' obtained by application of AUG1 to a graph G, two edges (not in the added triangle) are independent iff they are independent in G. Also, the vertices of this triangle are non-separable by any independent set of edges in G'. Hence, the preservation of property P by AUG1. In a graph G' obtained from a graph G by application of the rule AUG2 to an edge (w_1,w_2), two edges (not adjacent to the new vertices) are independent iff they are independent in G. Any matching M' in G' disconnects w_1 and w_2 iff $M'=\{(u_1,u_2),(u_3,u_4)\}\cup M$ (or, equivalently, (u_2,u_3) and (u_1,u_4)) such that $M\cup\{(w_1,w_2)\}$ is a matching in G disconnecting w_1 and w_2. []

In the case of reduction by rule C2 when the degree of u (or v) is 2, the ratio of discarded edges to dropped vertices is 2. Therefore, an inverse augmentation operation may preserve the minimum size of a graph with property P only if applied to a graph with odd order. We note that to preserve property P, the added vertex of degree 2 can be made adjacent to any two vertices of an original graph with this property. Hence the following rule.

(AUG3) Add a vertex of degree 2 adjacent to any one or two vertices of the original graph.

Lemma 4.2 A graph G' obtained from a graph G by application of rule AUG2 has property P if G has property P.

We have thus shown that the lower bound on the size of graphs with property P which have a given order is attained by an infinite class of graphs. Our augmentation rules appear to be necessary to construct a graph guaranteeing existence of a reduction giving equi-independence between the original and the reduced graphs. This constitutes a strong evidence to indicate that the above class is indeed exactly the class of extremal graphs with property P. We were however unable to prove the following statement.

Conjecture 4.4 A graph G or order $n \geq 3$ and size $m = \lceil 3(n-1)/2 \rceil$ has property P iff it can be obtained from a single vertex by a finite combination of applications of rules AUG1 and AUG2, and exactly one application of rule AUG3.

5. Conclusions

We have establisned a minimum size of graphs of a given order with no disconnecting independent set of vertices, edges, or both. The notion of a disconnecting matching and the associated numerical results seem to be intriguingly related to the results for graphs having bounded local connectivity, see Bollobás [1, Sec. I 5]. Local connectivity is defined for a graph G as the greatest minimum number of vertices (edges) that have to be removed to disconnect a given pair of vertices of G. The maximum size of a graph with local vertex connectivity at most 2 is $\lfloor 3(n-1)/2 \rfloor$. Such an extremal graph is connected, with blocks which are triangles with the exception of at most one block which is an edge or a C_4. A similar result involves extremal graphs not having a subgraph isomorphic to a cycle C, a vertex x not on C and two edges joining x to C. This is a particular case of a semi-topological subgraph; for definition and discussion see [1, Section VII 3]. Exploring these relationships may help to prove Conjecture 4.4.

Acknowledgment

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References

[1] B. Bollobás: Extremal Graph Theory, Academic Press, 1978.

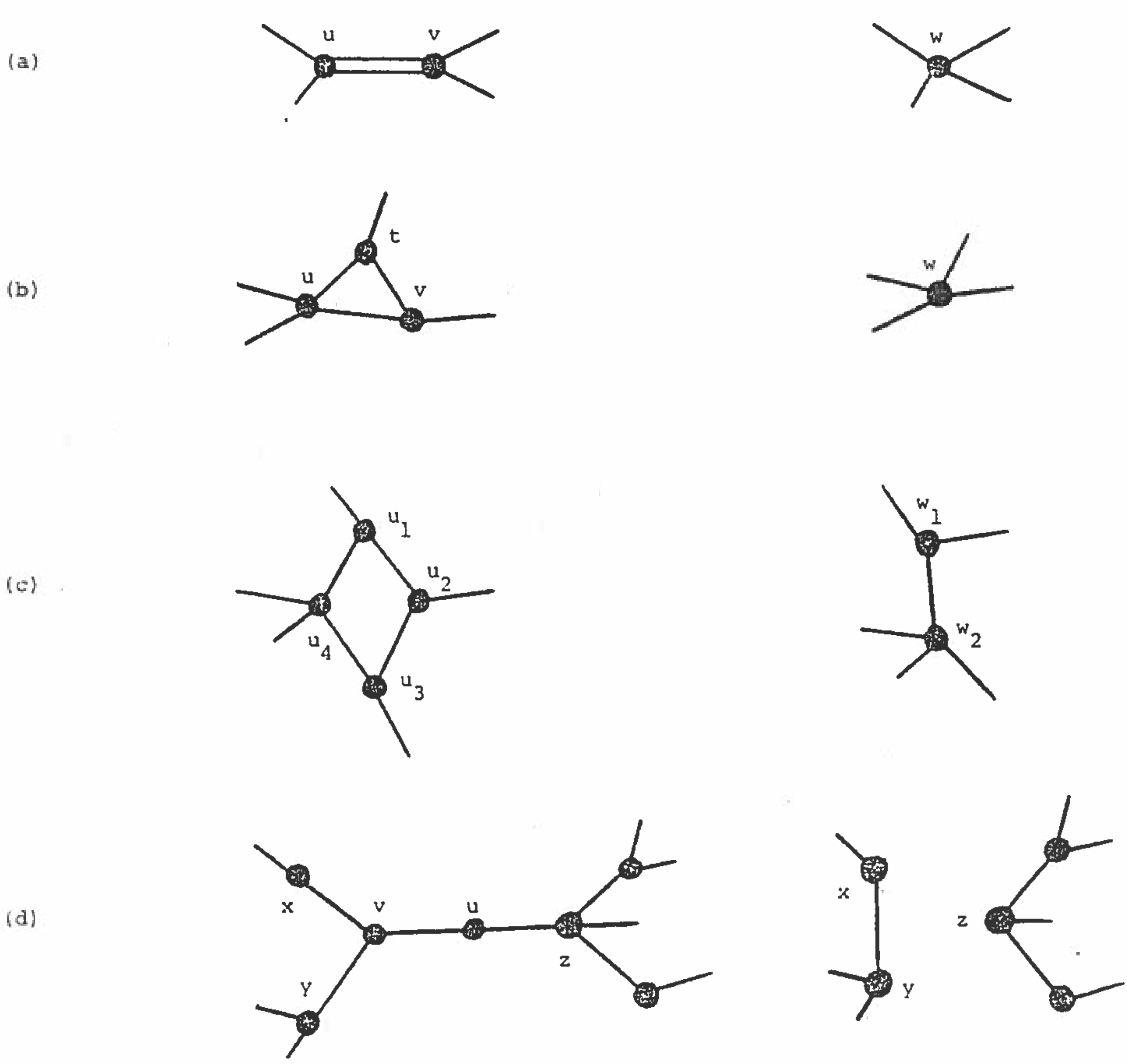

Figure 1 Four reductions preserving property P.

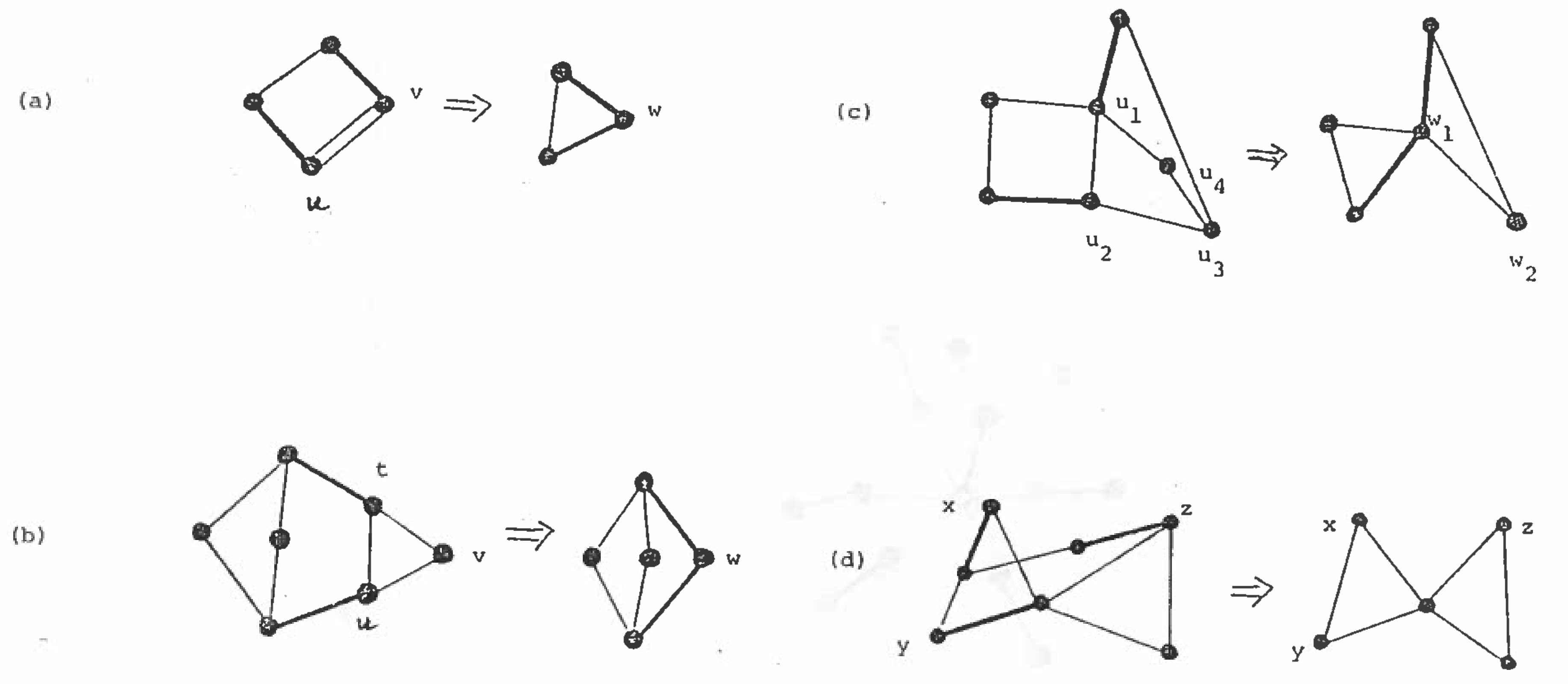

Figure 3 Rules C2-P2 transform graphs without property P into graphs with this property: (a)-(d), respectively.

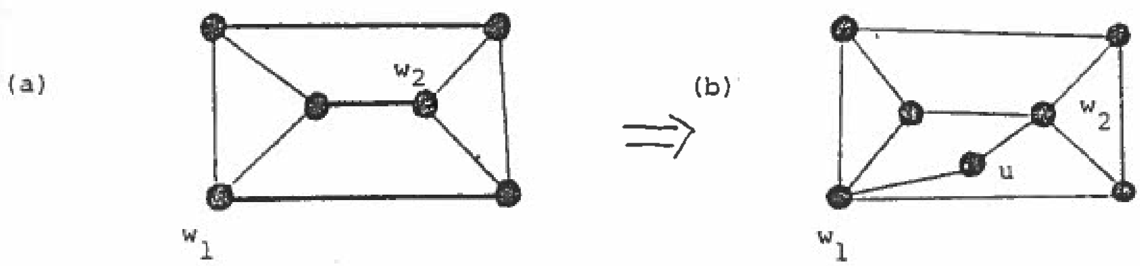

Figure 4 (a) A graph G, and (b) the augmented graph G'.