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CENTERS AND MEDIANS OF $C_{(N)}$-TREES*

by

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ABSTRACT

Given a connected graph $G = (V,E)$, let $d(u,v)$ denote the distance, or length of a shortest path, between vertices u and v. The center of G consists of the set of vertices u for which the value $\max \{d(u,v): v \in V(G)\}$ is a minimum. The median of G consists of the set of vertices u for which the sum $\Sigma\, d(u,v)$, $v \in V(G)$, is a minimum; and the center and median subgraphs of G are the subgraphs induced by these two sets of vertices, respectively. Recently, the center subgraphs of a variety of different classes of graphs have been characterized; included among these are maximal outerplanar graphs, 2-trees, unicyclic graphs, cacti and $C_{(4)}$-trees. This paper generalizes the result for $C_{(4)}$-trees by completely characterizing the center subgraphs of all $C_{(n)}$-trees. Several results about the possible median subgraphs of $C_{(n)}$-trees are also presented.

A $C_{(n)}$-tree is a graph which can be constructed from a cycle of length n by a finite number of applications of the operation of adding another cycle of length n and identifying one of its edges with an edge already in the graph. These graphs are frequently used to describe classes of chemical compounds.

1. Introduction

In general, a "center" of a graph $G = (V,E)$ is a vertex or a set of vertices which minimizes some function involving the distance between an arbitrary vertex and a vertex in the center. For example, one may want to find the set of vertices $U = \{u_i\}$ each element of which minimizes the sum $D(u) = \Sigma\, d(u,v)$, $v \in G$, where $d(u,v)$ denotes the distance, or length of a shortest path, between vertices u and v. The set U is called the <u>median</u> of a graph G, and the subgraph which it induces is called the <u>median subgraph</u>. Alternately, one may want to find a minimax, e.g. the set of vertices $X = \{x_i\}$ each element of which minimizes

$$\max \{ d(v,x) : v \in G \}.$$

The set X is called the (Jordan) <u>center</u> of G and the subgraph which it induces is called the <u>center subgraph</u>.

For a representative sample of the notions of centrality see [1,3-7,12-16].

Until recently, the only known result which characterized the center subgraphs or median subgraphs of a given class of graphs was one obtained by Jordan [8] in 1869.

<u>Theorem 1</u>. (Jordan) If T is a tree, then the center subgraph (and the median subgraph) of T is either a single vertex or two adjacent vertices.

In 1979, Proskurowski characterized the center subgraphs of maximal outerplanar graphs [11] and of 2-trees (equivalently, $C_{(3)}$-trees) [10]. Subsequently, Mitchell and Hedetniemi [9] characterized the center subgraphs of unicyclic graphs, cacti and $C_{(4)}$-trees, and Buckley, Miller and Slater [2] examined the problem of embedding an arbitrary graph into a supergraph with required properties. Slater [17] then presented the result for medians corresponding to that of Proskurowski by characterizing the median subgraphs of 2-trees, and it was demonstrated that any graph G is the median subgraph of some supergraph.

In this paper the results in [9] which characterize the center subgraphs of $C_{(4)}$-trees are generalized to $C_{(n)}$-trees. Results are also provided about the median subgraphs of $C_{(n)}$-trees.

Informally, a $C_{(n)}$-tree is a tree of cycles, each having length n, where two cycles are either disjoint or have one edge in common. More formally, a graph G is a $C_{(n)}$-tree if and only if it can be constructed from a cycle of length n by a finite number of applications of the following operation: add a new cycle of length n and identify an edge of this cycle with an edge of the existing graph. Every cycle of length n in a $C_{(n)}$-tree is called an elementary cycle.

2. Center subgraphs of $C_{(n)}$-trees

In [9] the following results were presented concerning the center subgraphs of $C_{(4)}$-trees and $C_{(n)}$-trees.

Theorem 2. (Mitchell and Hedetniemi) The graphs in Figure 1 are the only center subgraphs of $C_{(4)}$-trees.

Figure 1

Theorem 3. (Mitchell and Hedetniemi) Let $\mathcal{S}_n$ be the set of graphs which are center subgraphs of $C_{(n)}$-trees, for any $n \geq 3$. Then $\mathcal{S}_n$ contains (a) K_1, (b) K_2, and (c) the graph containing four vertices w_1,u,v,w_2, where w_1,u and v are on one elementary cycle and u,v and w_2 are on another elementary cycle.

We next show that the subgraph in Theorem 3(c) is the only center subgraph of a $C_{(n)}$-tree which is not contained in one of the elementary cycles of G. In order to do this we will need the following definitions. A geodesic between two vertices u and v is any shortest path between them. The eccentricity of vertex u, denoted e(u), is the length of a longest geodesic between vertex u and another vertex. The eccentricity of a graph G, denoted e(G), equals the minimum eccentricity of any vertex in G.

Theorem 4. If G is a $C_{(n)}$-tree with $n \geq 4$, and the center of G is not contained in one of the elementary cycles of G, then the center consists of four vertices w_1, u, v, w_2, where vertices w_1, u, v are on one elementary cycle and vertices u, v, w_2 are on another elementary cycle.

Proof. Let $C_1, C_2, \ldots, C_k$ denote the elementary cycles containing a given edge (u,v). Let $\tilde{C}_i$ denote the component of G - {u,v} containing vertices of C_i. Assume that the center of G contains vertices $w_1 \in \tilde{C}_1$ and $w_2 \in \tilde{C}_2$, i.e. $e(w_1) = e(w_2) = e(G)$.

We will show that $d(u,w) \leq e(G)$ and $d(v,w) \leq e(G)$ for all vertices w in G. Let $w \in \tilde{C}_2 \cup \tilde{C}_3 \cup \ldots \cup \tilde{C}_k$, and let P_{w_1w} be a w_1 to w geodesic. Clearly, P_{w_1w} contains u and/or v. If, for example, it contains v but not u, then the path P_{uw} = u, v, ..., w (where the v to w section of P_{uw} is the same as in P_{w_1w}) shows that $d(u,w) \leq d(w_1,w)$. Clearly in all cases, $d(v,w) \leq d(w_1,w) = e(G)$ and $d(u,w) \leq d(w_1,w) = e(G)$. Similarly, if $w \in \tilde{C}_1$ then $d(v,w) \leq d(w_2,w) = e(G)$ and $d(u,w) \leq d(w_2,w) = e(G)$. Consequently, $e(u) = e(v) = e(G)$, and the center also contains u and v.

Select $v' \in V(G)$ such that $d(v,v') = e(v)$, and assume that v' is not in $\tilde{C}_2$. If $P_{w_2v'}$ is a geodesic, then $v \notin P_{w_2w}$ or else $d(w_2,v') > d(v,v') = e(v) = e(w_2)$. Thus $u \in P_{w_2v'}$. Furthermore, $d(u,v') = d(v,v')$ would also imply that $d(w_2,v') > e(w_2)$, and so $d(u,v') = d(v,v') - 1$. Now if $(w_2,u) \notin E(G)$, then $d(w_2,v') \geq d(u,v') + 2 = d(v,v') + 1 > e(v) = e(w_2)$.

Hence, $(w_2,u) \in E(G)$.

Let $u' \in V(G)$ be such that $d(u,u') = e(u)$. If u' is not in $\tilde{C}_2$, then a similar argument would also show that $(w_2,v) \in E(G)$. This is a contradiction since $n \geq 4$. Hence $u' \in \tilde{C}_2$. Note that if v' is not in $\tilde{C}_1$, then the argument above would imply that $(w_1,v) \in E(G)$ and $(w_1,u) \in E(G)$, again contradicting the assumption that $n \geq 4$.

In short, at this point one can conclude that the center of G consists of u, v, some vertices of $\tilde{C}_2$ adjacent to u, and some vertices of $\tilde{C}_1$ adjacent to v.

Let x_2 denote the vertex on C_2 adjacent to u. It will be shown that $w_2 = x_2$.

Since $d(v,u') = d(u,u') - 1$ by the argument above, it is easy to see that u' must be in the component of $G - \{u,x_2\}$ containing v. And clearly $d(x_2,u') \geq d(u,u') - 1$. Let y be any vertex of $\tilde{C}_2$ adjacent to u (and different from x_2). Since $n \geq 4$, one has $d(y,u') = \min\{1+d(u,u'), d(y,x_2) + d(x_2,u')\} \geq \min\{1 + d(u,u'), 2 + d(x_2,u')\} \geq 1 + d(u,u') > e(G)$. Hence, y is not in the center.

Thus, $w_2 = x_2$ and similarly w_1 must be on C_1, and the center is w_1, v, u, w_2, proving the theorem.

<u>Theorem 5</u>. Let $C = \{v_1, v_2, \ldots, v_{2n}\}$ be a cycle of even length. Let $S \subseteq C$ be an arbitrary subset of C. Then there exists a $C_{(2n)}$-tree G, $n \geq 2$, with cycle C and center S.

<u>Proof</u>. Let $S = \{w_1, \ldots, w_k\}$ be a subset of cycle $C = \{v_1, \ldots, v_{2n}\}$ of even length. For each vertex $w \in S$, go clockwise around C to vertices w' and w" which are distances n-1 and n, respectively, from w. Add cycle C_1 of length 2n so as to share edge (w',w") with C. Select the edge in C_1,

but not C, which is incident with w"; add cycle C_2 of length 2n to this edge (cf. Figure 2a). Note that max $\{d(w,y): y \in C_1 \cup C_2\} = 2n$, and if $u \in C$ with $u \neq w$, then max $\{d(u,y): y \in C_1 \cup C_2\} \leq 2n-1$.

For each vertex $w \in C$, $w \notin S$, add cycle C_1 of length 2n, so as to share edge (w',w") in C, where w' and w" are as previously defined. Let x = (w",y) denote the edge incident with w" that is in C_1, but not in C. Add cycle C_2, of length 2n, sharing edge (y,y') where y' is not w" (cf. Figure 2b). Note that max $\{d(w,z): z \in C_1 \cup C_2\} = 2n+1$, and if $v \in C$, $v \neq w$ ($v \in S$ or $v \notin S$ is possible when $v \neq w$), then max $\{d(v,z): z \in C_1 \cup C_2\} \leq 2n$.

The graph G obtained by the above construction has eccentricity e(G) = 2n and the center of G is S, since any vertex not in S has eccentricity at least 2n+1.

Figure 2

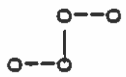
<u>Theorem 6</u>. Let $C = \{v_1, v_2, \ldots, v_{2n+1}\}$ be a cycle of odd length. Let $S \subseteq C$ be an arbitrary subset of C. Then there exists a $C_{(2n+1)}$-tree G, $n \geq 2$, with cycle C and center S.

<u>Proof</u>. Let $S = \{w_1, \ldots, w_k\}$ be a subset of cycle $C = \{v_1, \ldots, v_{2n+1}\}$ of odd length. For each vertex $w \in S$, add the configuration of cycles C_1, C_2 and C_3, of length 2n+1, as shown in Figure 3a, to the edge in C opposite w. Note that max $\{d(w,y): y \in C_1 \cup C_2 \cup C_3\} = 3n$ and if $v \neq w$ with $v \in C$, then max $\{d(v,y): y \in C_1 \cup C_2 \cup C_3\} \leq 3n-1$.

Let $u \in C$, $u \notin S$. Add the configuration of cycles C_1, C_2, C_3 and C_4, of length 2n+1, as shown in Figure 3b, to the edge in C opposite u. Note that max $\{d(u,y): y \in C_1 \cup C_2 \cup C_3 \cup C_4\} = 3n+1$ and if $v \in C$ with $v \neq u$, then max $\{d(v,y): y \in C_1 \cup C_2 \cup C_3 \cup C_4\} \leq 3n$.

Any vertex not on cycle C clearly has eccentricity at least 3n+1. Thus the graph G obtained by the above construction has eccentricity e(G) = 3n, and the center of G consists precisely of S.

Figure 3

In summary, for $n \geq 4$, S_n consists of the collection of <u>all</u> subgraphs of C_n and the subgraph  as described in Theorem 4.

3. Median subgraphs of $C_{(n)}$-trees

We next present results about the median subgraphs of $C_{(n)}$-trees. Let u be a vertex in a graph G. Denote by D(u) the sum of d(u,v) for all other vertices v in G. The median of G consists of the set of vertices for which the sum D(u) is minimum. We will first show that for even values of n, the median is contained in an elementary cycle.

<u>Theorem 7</u>. Let $(u,v) \in E(G)$ for a graph G where $\{u,v\}$ is a cutset. Let $R_1, R_2, \ldots, R_n$ be the components of $G - \{u,v\}$ with vertex sets $S_1, S_2, \ldots, S_n$, respectively, and assume $|S_1| \leq \sum_{i=2}^{n} |S_i|$. If $w \in S_1$ with $d(w,u) < d(w,v)$, then $D(u) < D(w)$.

<u>Proof</u>. Let $d(w,u) = k \geq 1$. Then $d(w,v) = k+1$. If $x \in S_i$ with $i \geq 2$, then clearly $d(w,x) = d(u,x) + k$; if $x \in S_1$, then $d(u,x) \leq d(w,x) + k$. Hence we have the following:

$$D(w) = d(w,v) + d(w,u) + \sum_{x \in S_1} d(w,x) + \sum_{\substack{x \in S_i \\ i \geq 2}} d(w,x)$$

$$\geq D(u) + 2\cdot k - k \cdot |S_1| + k \cdot \left|\bigcup_{i=2}^{n} S_i\right|$$

$$> D(u).$$

Corollary 8. The median of a $C_{(2k)}$-tree is contained in an elementary cycle of G.

Proof. $C_{(2k)}$- trees are bipartite. But $(u,v) \in E(G)$ with $d(w,u) = d(w,v)$ implies an odd cycle.

It is shown in [17] that the median of a $C_{(3)}$-tree is contained in an elementary cycle, i.e. a triangle. The $C_{(5)}$-tree in Figure 4, however, has two median vertices in different cycles (as identified by squares). Such an occurrence can be found in any $C_{(2k+1)}$-tree for any $k \geq 2$. Some relatively complicated applications of Theorem 7 can be used to prove the following result.

Figure 4

Theorem 9. The median M of a $C_{(2k+1)}$-tree is contained in either one elementary cycle or two adjacent elementary cycles. If M has vertices from $C_1 - \{u,v\}$ and $C_2 - \{u,v\}$ (as in Figure 5), then exactly one of u_k and v_k is in M, and if $u_k \in M$ then $M \cap \{u_1, u_2, \ldots, u_{2k-1}\} = \{u_k\}$.

Figure 5

To date we have not completed a proof characterizing the structure of $M \cap \{v_1, v_2, \ldots, v_{2k-1}\}$.

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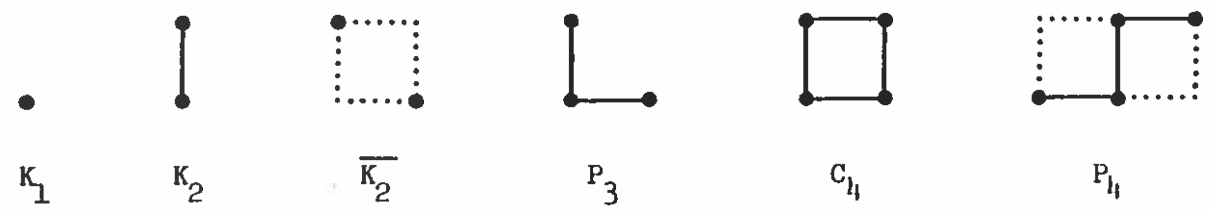


Figure 1.

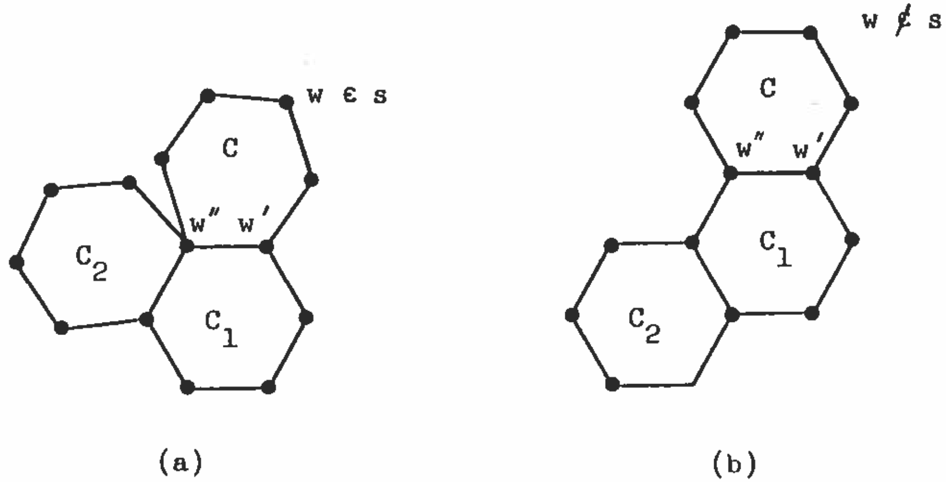
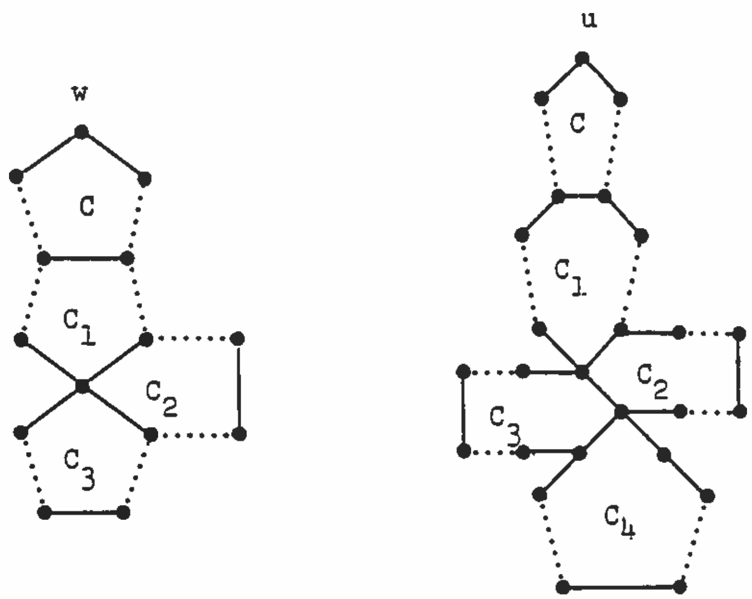


Figure 2.



(a) (b)

Figure 3.

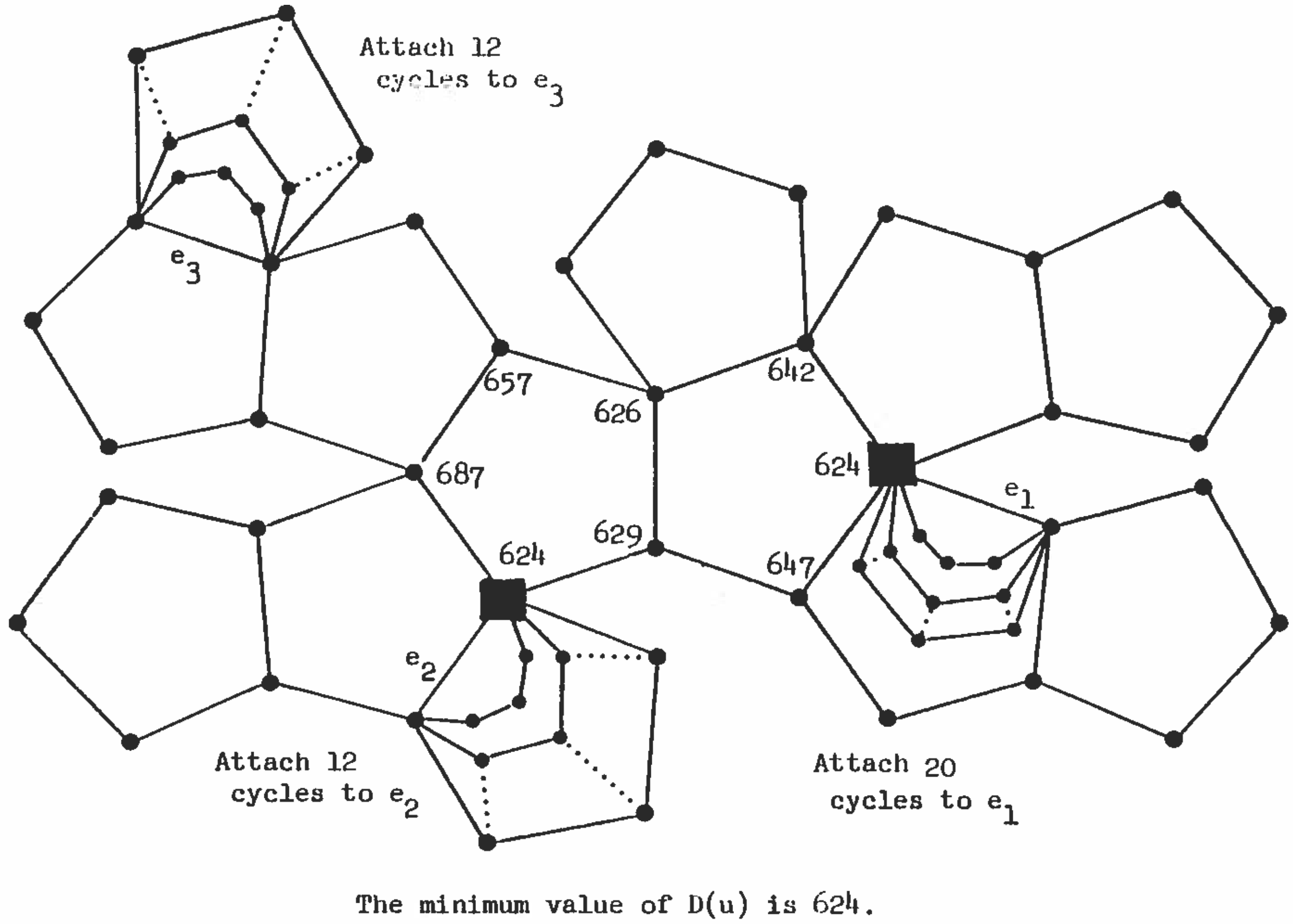


Figure 4. A $C_{(5)}$-tree whose median is not contained in an elementary cycle.

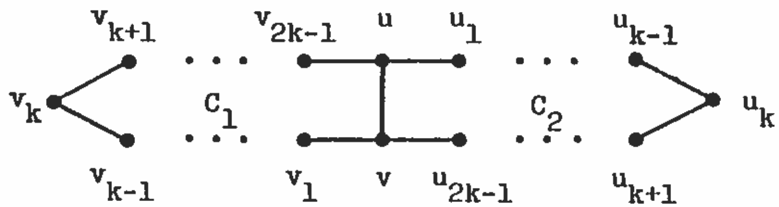


Figure 5. Adjacent cycles in a $C_{(2k+1)}$-tree.