82-05

EFFICIENT VERTEX- AND EDGE-COLORING OF OUTERPLANAR GRAPHS

by

Andrzej Proskurowski+
and
Maciej M. Syslo*

Abstract

The problems of finding values of the chromatic number and the chromatic index of a graph are NP-hard even when restricted to planar graphs. Every outerplanar graph has an associated tree structure which facilitates algorithmic treattment. Using that structure, we give an efficient algorithm to color vertices of an outerplanar graph with the minimum number of colors. We also establish algorithmically the value of the chromatic index of outerplanar graphs. The algorithms is based on an edge-coloring procedure preserving a property of a partial coloring by a systematic edge-coloring of adjacent faces.

+ Department of Computer and Information Sciences, University of Oregon, Eugene, Oregon.
* Department of Computer Science, Washington State University, Pullman, Washington (on leave from Institute of Computer Science, University of Wroclaw, Wroclaw, Poland).

UO-CIS-TR - 82-5

EFFICIENT VERTEX- AND EDGE-COLORING OF OUTERPLANAR GRAPHS

by

Andrzej Proskurowski+ and Maciej M. Syslo*

Department of Computer and Information Sciences, niversity of Oregon, Eugene, Oregon.
Department of Computer Science, Washington State niversity, Pullman, Washington (on leave from Institute of omputer Science, University of Wroclaw, Wroclaw, Poland).

1. Introduction

For a graph G, its chromatic number, X(G), is the minimum number of colors needed to color vertices of G in such a way that no two adjacent vertices are assigned the same color. Related to coloring of edges of G is its chromatic index, X'(G). It is defined as the minimum number of colors needed to color edges of G so that no two adjacent edges are assigned the same color. An assignment of at most k colors to the vertices (edges) of a graph G is called a k-vertex- (k-edge-) coloring of G. The problems of finding the value of the chromatic number (index) of a graph G is NP-complete even when G is planar (see [Garey and Johnson], [Holyer]). As for some positive results, [Gabow and Kariv] give an efficient edge-coloring algorithm for bipartite graphs. [Mitchell and Hedetniemi] edge-color trees and unicyclic graphs. Recently, [Widgerson] presented an efficient approximation algorithm for vertex-coloring of general graphs. Applying a method that follows the recursive construction of series-parallel graphs, we can easily color vertices of such graphs using the minimum number of colors. Here, we present efficient (linear in the size of the graph) algorithms for both problems when restricted to a subclass of series-parallel graphs, outerplanar graphs. A planar graph G is outerplanar if and only if there exists a plane embedding of G in which all vertices lie on the exterior (unbounded) face. Such an embedding is referred to as an outerplane graph. Without loss of generality, we may restrict our discussion of coloring vertices and edges to 2-connected outerplane graphs.

For every outerplane graph G there is unique associated tree T(G). This tree has internal nodes corresponding to the interior (bounded) faces of G, and external nodes (leaves) all corresponding to the exterior face, one leaf for each edge of G on the exterior face. To avoid confusion, we refer to nodes of T(G) and vertices of G. The edges of T(G) correspond uniquely to edges of G in such a way that there is an edge between nodes of T(G) if and only if the two corresponding faces of G share an edge. We consider T(G) to be a plane tree, in which the neighborhood of each node is ordered (see [Proskurowski and Syslo]). A choice of a node of T(G) as its root induces a natural father-son relation between adjacent nodes, and also a left-to-right ordering of brother nodes (sons of a common father). See Figure 1 for an example of an outerplane graph, its associated tree and a rooting.

The four color theorem [Appel and Haken] ensures that the chromatic number of outerplanar graphs (as planar graphs) is at most 4. We will constructively prove that it is at most 3.

The chromatic index of a graph G is bounded by the maximum degree, $\Delta(G)$, of a vertex of G. The Vizing's theorem (see for instance [Fiorini and Wilson]) states that

$\Delta(G) \leq X'(G) \leq 1+\Delta(G)$. [Firorini] proves that for an outerplanar graph G, $X'(G)=\Delta(G)$ unless G is an odd cycle. His proof is an existential one and does not provide a method for finding an optimal edge-coloring. Another proof of the above equality given in [Fiorini and Wilson] contains a flaw. We present an algorithm optimally edge-coloring an outerplanar graph, which may be considered as yet another proof of the above equality.

In the remainder of our paper we follow the standard texts [Fiorini and Wilson], [Garey and Johnson] and [Harary] as references for, respectively, edge coloring, complexity analysis, and general graph theory.

2. Vertex-coloring

The fact that the chromatic number of an outerplanar graph is at most 3 is implied by the following observation. Every outerplanar graph has a vertex of degree 2. Every subgraph of an outerplanar graph is outerplanar. Hence, applying the Szekeres-Wilf's bound on the chromatic number $X(G) \leq 1+\max \delta(G')$, where maximum is over all subgraphs G' of G, and $\delta(G')$ is the minimum vertex degree of G', we have the following result.

<u>Theorem 1</u> The chromatic number of an outerplanar graph is at most 3. []

We now give a procedure for producing an optimal vertex-coloring of a 2-connected outerplane graph G. Our method makes use of a traversal of the associated tree T(G) rooted at an arbitrary node. We assume that the traversal is monotonic, that is, no node other than the root is visited before its father.

Visiting a node C of T(G) we color the vertices of the corresponding face of G with two or three colors, depending on its length. If C is not the root, two of its adjacent vertices are already colored. These colors are subsequently used to color the cycle C. It is clear that an outerplanar graph containing an odd-length face is not bipartite. Our algorithm will produce a 3-coloring of such a graph. If all faces of G have an even length then a 2-coloring is produced.

The algorithm coloring vertices of a graph G takes time linear in the total size of all faces of G, and therefore proportional to the number of vertices in G.

3. Breadth-first edge-coloring algorithms

The arbitrary monotonic traversal of the arbitrarily rooted associated tree T(G), used in the optimal vertex-coloring of an outerplanar graph G fails in an attempt to edge-color G. The free choice of coloring edges along a cycle, when restricted by an algorithmic method may lead to an eventual coloring conflict. See Figure 1, where a monotonic traversal was used. Edges of the triangular face corresponding to node 8 cannot be colored without use of an additional color.

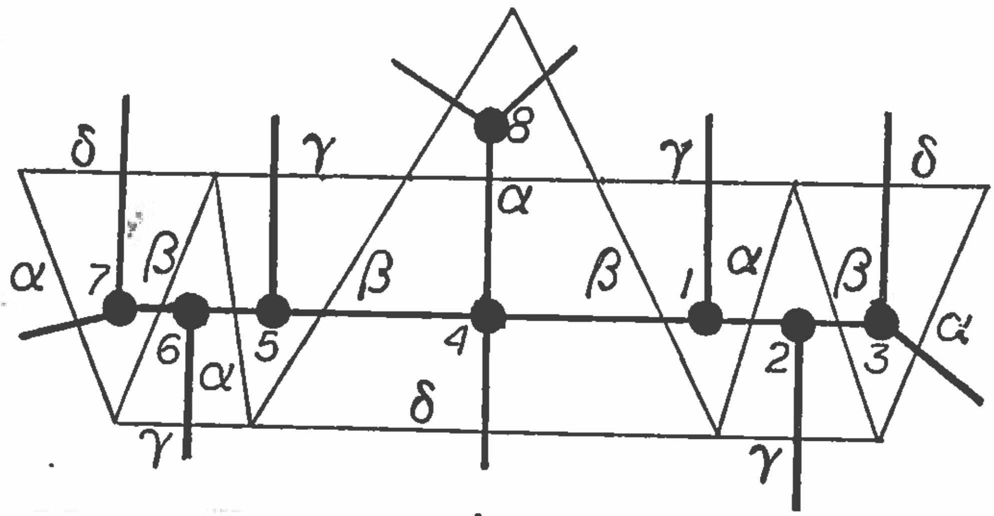

Figure 1 An outerplane graph, its associated tree with the depth-first traversal order, and a partial edge-coloring following that order.

We will give a traversal method of a carefully rooted associated tree T(G), and a judicial coloring of the corresponding cycles of G that lead to an optimal edge-coloring. We define the breadth-first traversal of internal nodes of a plane (extended) tree rooted at an external node as visiting the internal nodes in the left-to-right order in levels defined by the distance from the root. Figure 2 indicates order of node visits in the breadth-first traversal of a rooted plane tree. Visiting a node E during the traversal of T(G) brings a (completion of) edge-coloring of the corresponding face E of G. Although at most one edge e of E has a color already assigned (during a visit of the node's father, C), the color assignment to the two edges of E adjacent to e is restricted by other colored edges adjacent to e. This restriction may impair the optimal coloring if, in the case of a triangular face E with end-vertices u and v of the base e, u and v are incident each with $\Delta(G)-1$ edges already assigned colors, the same for both u and v. Fortunately, this cannot happen in the breadth-first traversal of T(G) for G with the maximum vertex degree $\Delta(G)\geq 5$.

<u>Lemma 1</u> Let a 2-connected, outerplane graph G with the maximum vertex degree (G)$\geq$5 be partially Δ-edge-colored by a breadth-first coloring algorithm. The Δ-coloring can be extended to a face E of G corresponding to the next-to-be visited node of T(G).

<u>Proof</u> By mathematical induction on the number of visited nodes of T(G). If E is the first face to be colored, then at most $3<\Delta(G)$ colors are needed. Therefore, let us assume that C is the father node of E and the corresponding faces of G share an edge e with end vertices u and v. At most one of these two vertices may be incident with $\Delta(G)-1$ previously colored edges: if v is in the face corresponding to some ancestor of the node C, then u may be in at most one colored face other than C, namely, that corresponding to the left brother of node E. Thus, the number of previously colored edges incident with v is at

most $3\langle\Delta(G)-1$, and edges of E can be colored using only (G) colors. []

The above Lemma does not translate directly for the case of $\Delta(G)=4$, because of the distinct possibility that the base edge of a not yet completely colored triangle face is adjacent to four colored edges forcing the same colors on both of the triangle's sides. The edge-coloring during the visit of the corresponding node's father must prevent an occurrence of this situation. The coloring process will have to preserve the following property.

<u>Property P4</u> A partial 4-edge-coloring of a 2-connected outerplanar graph G with $\Delta(G)=4$ has property P4 if and only if G does not have a colored edge (u,v) shared by a not yet colored face E such that u and v are incident with edges colored by three colors.

In the breadth-first edge-coloring algorithm, property P4 can be endangered only in two situations when coloring edges of the face corresponding to the father C of the node E. The first one, in which C's right brother would also be his leftmost brother, is eliminated by rooting the associated tree in a leaf node. The second situation can be reached only through the sequence of face coloring (tree traversal) illustrated in Figure 2. The node visiting order is A...BC...DE.

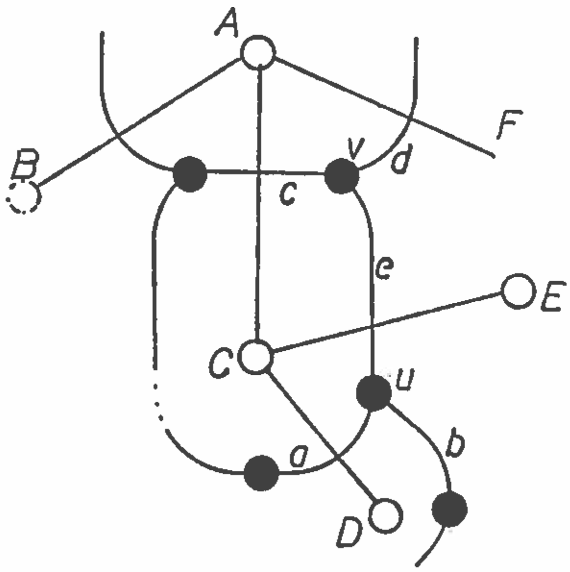

Figure 2. A paradigm of edge-coloring

Coloring edges of C we have to consider two cases of C's left brother B, which can be either external or internal (See Figure 2).

In the former case, there is a choice of two colors for the first (leftmost) edge of C. This guarantees that the next to the last (rightmost) edge a of C can be colored so as to preserve property P4, namely by assigning it a color different from those assigned to edges c and d of face A. In the latter case, when the color of the first edge of C is forced by the formerly colored edges of A and B, we have additionally to consider the length of C. If C is a triangle, then the property P4 might not be preserved. However this property is not necessary for maintaining the property P4 for coloring of E since its leftmost son node cannot be interior. If C has length greater than 3, then there is enough freedom in coloring its edges to preserve property P4. Thus, we have the following Lemma.

Lemma 2 Property P4 can be preserved in a partial coloring of an outerplane graph G with k=Δ(G)=4 colors following the breadth-first order coloring algorithm.[]

An immediate corollary gives the desired statement about edge-coloring of such graphs.

Corollary 3 Let a 2-connected outerplanar graph G with the maximum vertex degree Δ(G)=4 be partially Δ-edge-colored by a breadth-first coloring algorithm. The Δ-edge- coloring can be extended to a face C of G corresponding to the next-to- be-visited node of T(G). []

When (G)=3, the property of a partial 3-edge coloring required to avoid forced situations can be obtained directly from P4.

Property P3 A partial 3-edge-coloring of a 2-connected, outerplane graph G with Δ(G)=3 has property P3 if and only if G does not have a colored edge e shared by a not yet colored face such that the two colored edges adjacent to e have the same color.

Lemma 4 Let G' be a partially 3-edge-colored subgraph of a 2-connected, outerplane graph G with (G)=3 obtained through the breadth-first coloring algorithm which has property P3. Let C be the next-to-be-visited node of T(G). The 3-edge-coloring of G can be extended to C preserving property P3.

Proof First, let us assume that C is the first face of G to be colored and that it has n edges. Let e be an edge shared by C and another face, C'. Such an edge always exists since $\Delta(G)=3>2$. We assign color 2 to e and color other edges of C depending on the value of n. If $n \equiv 0 \mod 3$, then coloring edges by 1-2-3 (with e appropriately included in the sequence) ensures property P3. If $n \equiv 1 \mod 3$, then we color edges of C by 1-2-3 starting with an edge adjacent to the initially colored e, but excluding e. The only two edges that could violate property P are adjacent to e and thus belong only to C, since $\Delta(G)<4$. If $n \equiv 2 \mod 3$, we again color edges along C by 1-2-3 starting with an edge adjacent to e and excluding e. This time, however, the three last edges of C (i.e., f, g, h in Figure 3) are colored differently, depending on adjacencies of f. If f is shared with another face, then it is colored 3, with colors 2 and 3 assigned to the remaining edges. Otherwise, the three edges are colored 2-1-3, respectively. Beside the edges adjacent to e, the only possibly offensive edges in the former case are adjacent to f and thus in no other face than C. In the latter case, only f has adjacent edges assigned the same color. Again, f is in no other face and thus the property P3 holds.

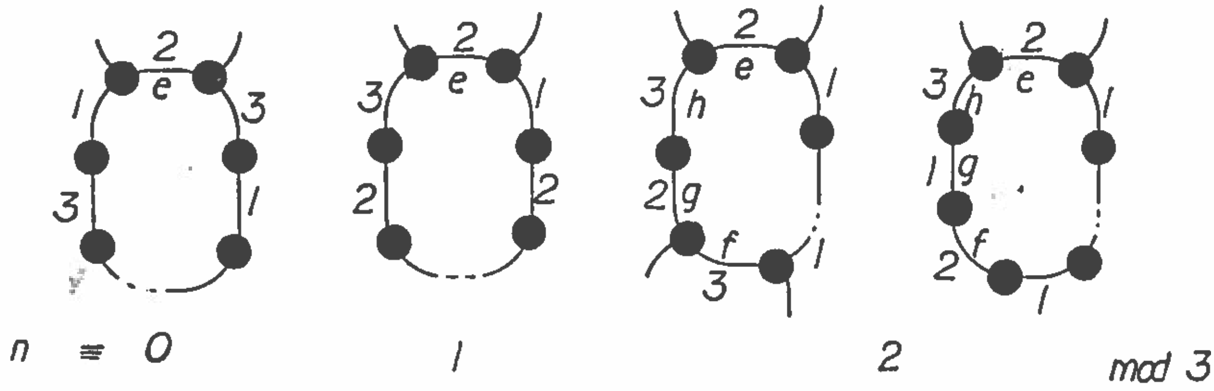

Figure 3 Coloring the first face of G.

Next, consider a G with property P3 and a face C corresponding to the next-to-be-visited node of T(G). Our inductive assumption yields that the three colored edges of C are assigned colors 3-2-1. The same case analysis as the one above proves that C can be colored to ensure property P3. Thus G can be colored with three colors. []

The amount of work to color edges of each face is proportional to the length of that face. Therefore, the total time spent on edge coloring of an outerplanar graph is bounded by a linear function of the graph's size. Preprocessing a given outerplanar graph to obtain its rooted associated tree can also be performed in linear time (see [Proskurowski and Syslo]). Collecting the results of this section, we finally have the following theorem.

Theorem 2 The breadth-first coloring algorithm produces an optimal edge-coloring of an outerplanar graph in time proportional to the size of the graph.

4. References

K.Appel and W.Haken, Every planar map is 4-colorable, Ill. J. Math. 21, 429-567;

S.Fiorini, On the chromatic index of outerplanar graphs, J. Combinatorial Theory (B) 18 (1975), 35-38;

S.Fiorini and R.J.Wilson, Edge-coloring of Graphs, Research Notes in Mathematics 16, Pitman, London 1977;

H.N.Gabow and O.Kariv, Algorithms for edge coloring bipartite graphs and multigraphs, SIAM J. of Computing 11, 1(1982), 117-129;

M.R.Garey and D.S.Johnson, Computers and Intractability, Freeman, San Francisco 1979;

F.Harary, Graph Theory, Addison-Wesley, Reading, Mass. 1969;

I.Holyer, The NP-completeness of edge-coloring, SIAM J. Computing 10 4(Nov. 1981), 718-720;

S.L.Mitchell and S.T.Hedetniemi, Linear algorithms for edge-coloring trees and outerplanar graphs, Information Processing Letters 9, 3(1979), 110-112;

A.Proskurowski and M.M.Syslo, Minimum dominating cycles in outerplanar graphs, Int'l J. Computer Information Sci. 10, 2(1981), 127-139.

A.Widgerson, A new approximate graph coloring algorithm, Proc. 14th Ann. ACM Symp. Th. Computing 1982, 325-329;