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INTERIOR GRAPHS OF MAXIMAL OUTERPLANE GRAPHS

by

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Abstract

A maximal outerplane graph (mop) is a plane embedding of a graph in which all vertices lie on the exterior face, and the addition of an edge between any two vertices would destroy this outerplanarity property. Removing the edges of the exterior face of a mop G results in the interior graph of G. We give a necssary and sufficient condition for a graph to be the interior graph of some mop.

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1. Definitions and basic characterizations

With a plane embedding G of a planar graph there is associated its *geometric dual* graph G*, in which the vertex set corresponds to faces of G and vertex adjacency is equivalent to adjacency of the corresponding faces. Removing from G* the vertex v corresponding to the exterior (unbounded) face of G results in the *weak dual* graph Gw. Splitting v into the number of copies equal to the size of the exterior face of G so that each copy is adjacent to exactly one edge corresponding to an edge of the exterior face results in the *semidual graph* Gs. Figure 1 gives an example of a graph G and its dual graphs. To avoid confusion, we will refer to members of the vertex set of (geometric, weak, semi-) dual graphs as *nodes*.

A planar graph G is *outerplanar* if and only if there is an embedding of G in the plane in which every vertex of G lies on the exterior face. This embedding is called an *outerplane* graph. A *maximal outerplane graph* (hereafter called *mop*) is an outerplane graph with the maximum number of edges, i.e., such that addition of an edge between any pair of vertices destroys outerplanarity. Removing the edges in the exterior face of a mop G results in a number of isolated vertices and the connected *interior graph* of G, Gi.

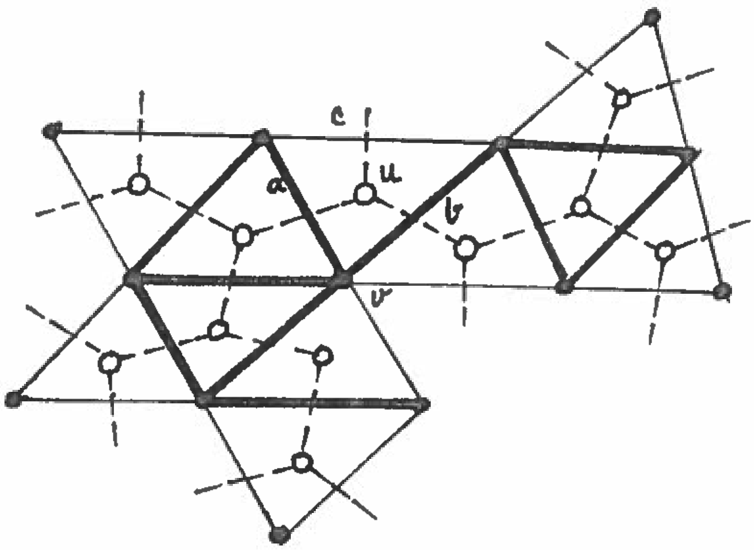

Figure 1 A mop G, its dual graphs, and its interior graph.

The weak dual graph of an outerplane graph (see Fleischner et al., [2]) is a tree called the associated tree of the graph (Proskurowski and Syslo [6]). A 3-regular tree has vertices of degree 3 and 1 only. For mops we have the following lemma.

Lemma 1 A graph G is a mop iff the semidual graph Gs is a 3-regular tree.

Proof (->) It is easy to see that a 3-regular tree has an even number of nodes. Therefore, assume that for all m, $2 \leq m < k$, every 3-regular tree with 2m nodes is the associated tree of a mop. (It is true for m=2, where G is a K3 and Gs is K13.) Let T be a 3-regular tree with 2k vertices. It has a node v of degree 3 adjacent to two leaves (nodes of degree 2) v1 and v2. Removal of these two nodes results in a 3-regular tree T′ which, by the inductive hypothesis, is a semidual of some mop H′. Let us define a mop H by adding to H′ a vertex w and two edges incident to it which form a triangular face corresponding to the node v. Edges (v,v1) and (v,v2) in T correspond to edges incident with w. Thus, T is the semidual graph of the mop H.

(<-) Every interior face of a mop is a triangle and thus the internal nodes of the weak dual graph have all degree 3. []

Removing the leaf nodes from the semidual graph of a mop G results in a cycleless connected graph which we will call the <u>associated tree</u> Ti of the interior graph Gi of G; Ti is isomorphic with the weak dual graph Gw.

<u>Lemma 2</u> A tree T is the associated tree of the interior graph of some mop if and only if nodes of T have degree at most 3.

<u>Proof</u> Sufficiency is obvious. To prove the necessity of the condition we extend the given tree T with node degree at most 3 to a 3-regular tree T′ by adding leaf nodes adjacent to all vertices of degree less than 3. By Lemma 1, T′ is a semidual of a mop G. It follows from the definition that the associated tree of the interior graph Gi is isomorphic to T. []

We observe that the the above extension procedure may yield trees non-isomorphic as plane trees and therefore associated with different mops as their semidual graphs.

2. A further characterization of interior graphs of mops

One subclass of interior graphs of mops is a special subclass of trees called caterpillars. A tree is a caterpillar if and only if removal of its end-nodes (leaves) results in a path. From our discussion of trees associated with interior graphs of mops we have the following property of caterpillars.

Lemma 3 Every caterpillar is the interior graph of some mop.

Proof With a caterpillar C we can associate a path P with the same number of edges such that each edge of P crosses exactly one edge of C (see Figure 2). P is a tree with nodes of degree less than 3 and thus, by Lemma 2, it is the associated tree of the interior graph of some mop G. []

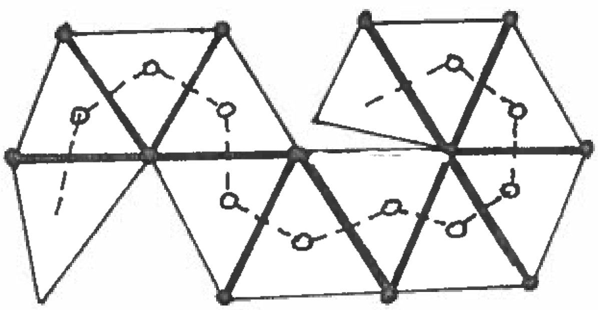

Figure 2 A caterpillar C, the path P, and the mop G.

Figure 2 illustrates the construction of the path P and the corresponding mop G. We observe that many non-isomorphic mops may have the same path P associated with their interior graphs. Only a caterpillar can be the interior graph of a mop if this interior graph is a tree.

Lemma 4 If the interior graph of a mop is a tree, then it is a caterpillar.

Proof Let us assume to the contrary, that the interior graph of a mop G is a tree and not a caterpillar. By a characterization of caterpillars of Harary and Schwank [3], T must contain the subdivision graph S(K13) (see Figure 3a) as an induced subgraph. Without loss of generality, we can represent the vertices of S(K13) on the Hamiltonian cycle of G as in Figure 3b. Since every interior face of a mop is a triangle, there must exist a vertex x on the arc of the Hamiltonian cycle of G between v1 and w2 not containing u which is adjacent to both u and v2. But the triangle {u, x, v2} consists of interior edges of G which contradicts the assumption that the interior graph of G is a tree. []

We have thus obtained an additional characterization of caterpillars.

Theorem 1. A tree is a caterpillar if and only if it is the interior graph of some maximal outerplanar graph.

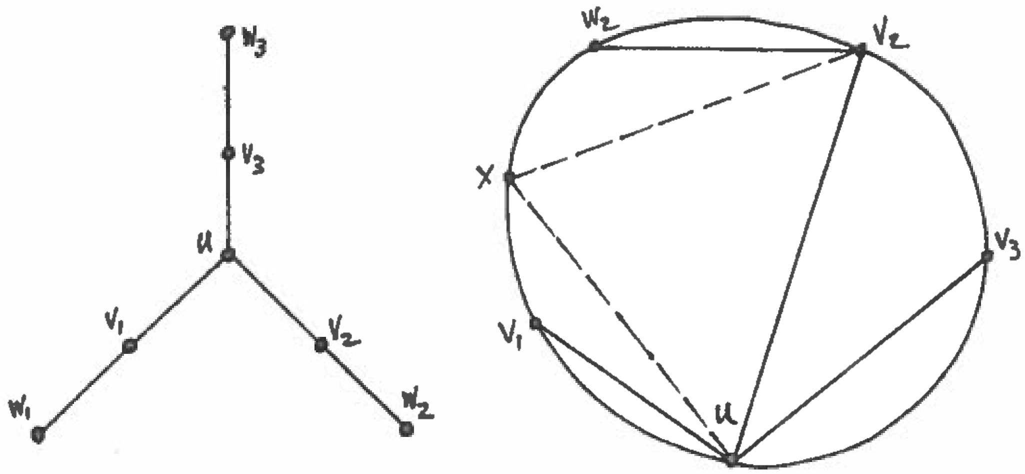

Figure 3 (a) The subdivision graph S(K13) and (b) the interior graph of a mop G.

Another subclass of interior graphs of mops are the mops themselves.

Lemma 5 Every mop is the interior graph of some mop.

Proof By Lemma 2, the semidual graph T of a given mop G (T is a tree by Lemma 1) can be extended to a 3-regular tree by adding two pendant edges to each leaf node of T. This new tree determines (uniquely, if considered a plane tree) a mop H, for which G is the interior graph. []

In fact, mops and caterpillars are the building blocks of any graph which is the interior graph a mop. We can see it directly from the associated tree of the interior graph of a mop, which can be partitioned into 3-regular trees and paths by appropriate splitting of some vertices of degree 2. First, we state relevant properties of the associated tree of the interior graph of a mop.

<u>Lemma 6</u> Given the interior graph G of a mop H and its associated tree T. The following properties hold.

(a) A node of degree 3 in T corresponds to an internal triangle in G.

(b) Two adjacent edges in T correspond to adjacent edges in G.

(c) All edges of a star in G correspond to edges of a path in T.

(d) To a path in T corresponds a subgraph of G (not necessarily induced) which is a caterpillar.

(e) A node of degree 2 in T determines a cut-vertex in G.

<u>Proof</u>

(a) A node of degree 3 in T corresponds to a node with no external neighbors in the 3-regular semidual tree of the original mop. Thus, in the original mop, it corresponds to a (triangular) face with no edges on the exterior face.

(b) The common end-node of two adjacent edges in T corresponds to two sides of the triangle in H.

(c) Follows from the definition of the semidual graph of H.

(d) Nodes of degree 2 in T, extended to nodes of degree 3 in the semidual graph of H correspond to a path of triangles in H (a 2-path, see Beineke and Pippert [1]). The edges of the path P correspond to edges of H shared by the adjacent faces (triangles) in the 2-path. These edges form a caterpillar (see Hedetniemi [4] and Proskurowski [5]).

(e) A node u of degree 2 in T determines an edge, c, on the exterior face of H (see Figure 1). That edge, together with the two edges of G, say a and b, corresponding to the edges of T incident with u form a triangle. The vertex of H incident to both a and b is a cut-vertex of H-{c}, and therefore also of G. []

Lemma 7 Let G be the interior graph of a mop. Then every 2-connected component of G is a mop, and the remaining connected components of G are caterpillars.

Proof Let T be the associated tree of the interior graph G of a mop. By Lemma 6, every node of degree 3 in T corresponds to an internal triangle in G, and every node of degree 2 in T determines a cut-vertex in G. Thus, splitting nodes of degree 2 adjacent to at least one node of degree 3 partitions T into subtrees associated with 2-connected components of G and with caterpillars of G. []

3. Sufficient conditions for interior graphs of mops

Not every collection of mops and caterpillars is the interior graph of a mop. In this section, we develop concepts allowing us to state the sufficient conditions for such a collection to be the interior graph of some mop.

A nontrivial block of a grah G is a maximal 2-connected subgraph of G containing more than one edge. By Lemma 7, all nontrivial blocks of an interior graph G of a mop are mops.

Let B be a nontrivial block of a graph G. B is saturated if each of its vertices is a cut-vertex of G. For a vertex u of B, the attached set is the set of vertices of G that can be reached from u by a path not including any other vertex of B.

Theorem 2 If G is the interior graph of a mop then for every cut-vertex v of G the number of saturated blocks containing v is at most 2.

Proof Let us assume that G is the interior graph of a mop and that v is one of its cut-vertices. Let T be the associated tree of G. By Lemma 6(c), the edges of T which correspond to E, the set of edges of G incident with v, form a path P in T. Let Ei ($1 \leq i \leq k$) denote the edges of E in the block Bi of G incident with v. Let the sets {Ei} be ordered according to the clockwise order of blocks Bi around v (see Figure 4). The edges of T which correspond to Ei form a subpath Pi of P. In T, the internal nodes of Pi have degree 3, and the end-nodes degree at most 2. Thus, P can be decomposed into P1,...,Pk, which share their corresponding end-nodes. For every block Bi ($2 \leq i \leq k-1$) there is the corresponding subtree Ti of T for which Pi is a subgraph. If Bi is an edge then Ti=Pi. If Bi is a block with ki>2 vertices, then Ti has ki leaves (nodes of degree at most 2 in T), 2 of which are the end-nodes of Pi. Only ki-2 of these leaf nodes may be shared by the subtrees of T corresponding to attached sets of vertices of Bi. Hence, out of ki-1 vertices of Bi other than v, at least one vertex does not have an attached set and thus is not a cut-vertex. Therefore, Bi ($2 \leq i \leq k-1$) is not saturated and there are at most 2 saturated blocks containing v. []

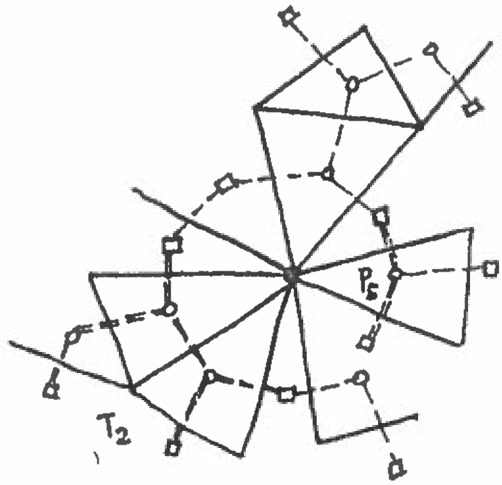

Figure 4 The anatomy of the interior graph of a mop.

Theorem 2 gives a necessary local condition for a graph to be the interior graph of a mop. Figure 5 shows a graph that satisfies the theorem but will be shown not to be the interior graph of any mop. A more global property of a graph necessary for its being the interior graph of a mop is based on the relative location of saturated cut-vertices along the hamiltonian cycle of any mop block of the graph. A cut-vertex v of a graph G is called saturated in a block B if and only if there are exactly two other than B blocks of G containing v which are saturated. (Notice that this implies that B itself is not saturated. In a saturated mop block of G, every exterior edge of that block is necessary to interact with attached sets of the vertices of the block. Moreover, only one of the edges incident with a given cut-vertex can be used to interact with all the other blocks containing that vertex.) A saturated vertex of a mop block needs both of its incident exterior edges of that mop to interact with the two adjacent saturated blocks. Therefore, saturated vertices of a mop must be distributed relatively sparsely along its Hamiltonian cycle or else they put conflicting demands on the incident edges of the cycle. The following procedure determines feasibility of location of saturated vertices in a given mop block of a graph.

<u>Algorithm 1</u> Feasibility checking

Input: A mop block B of a candidate G for the interior graph of a mop.

Output: Labeling of B′s edges indicating feasibility of B as a block of the interior graph G of a mop.

Method:

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{1.} With each vertex v of B associate an integer k(v),
0≤k(v)≤2, indicating how many edges of B incident with v
are needed to interact with other blocks of G containing
v. These values are:
k(v):=0 if v is not a cut-vertex;
      1 if v is not saturated in B;
      2 if v is saturated in B;
{2.} With each exterior edge e of B associate an integer
m(e), 0≤m(e)≤2, using values of k(v) in the following
manner:
{initialize} for each edge e of B do m(e):=0;
{iterate   } while for no edge e m(e)>1 and one of
            the following operations can be applied do
                  for each vertex v of B s.t. k(v)=2 do
                    k(v):=0;
                    for each exterior edge e incident
                            with v do m(e):=m(e)+1;
                  for each vertex v of B s.t. k(v)=1 and
                only one edge e incident with v has m(e)=0
              do k(v):=0; m(e):=1;
{exception} for each vertex v of B s.t. k(v)=1 do
                {the following 1-1 correspondence
                    always exists}
                choose one exterior edge e incident
                  to v, not chosen for another vertex;
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                 k(v):=0; m(e):=m(e)+1;
{check      } if for no edge e m(e)>1
                              then B feasible
                              else B not feasible
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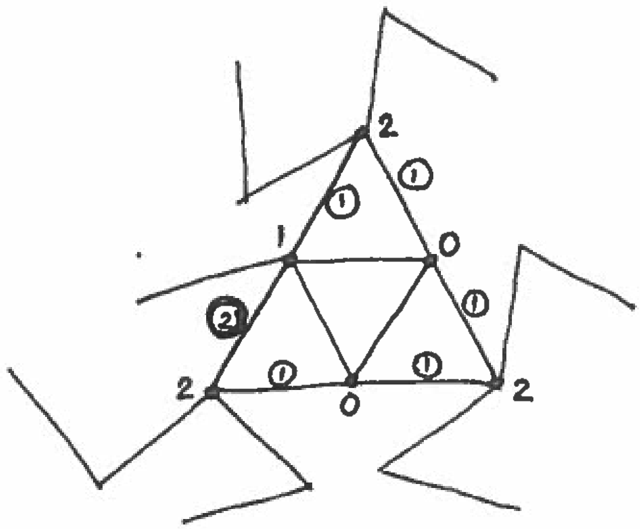

Figure 5 Result of applying Algorithm 1 to a graph.

Theorem 3 A graph G is the interior graph of a mop only if G is a connected collection of mops and caterpillars and every mop block B of G is feasible.

Proof In the case of a successful termination of Algorithm 1 ("B feasible"), the values of all vertex labels are distributed into the edge labels, so that in block B, k(v)=0 for all vertices v and m(e)≤1 for all exterior edges e. We will show that a mop block B of the interior graph G of some mop is feasible by defining a labeling process inverse to that of Algorithm 1. After initial labeling of all exterior edges e of B by m′(e)=1, we will distribute those values into vertex labels, k′(v), based on inspection of the tree T associated with G. We initialize values of k′(v) to 0 for every vertex v of B. For every node u of degree 2 in T which corresponds to a cut-vertex v of B (see

Lemma 6(e) and Figure 1) we do the following. The value associated with the edge a of B incident with v and corresponding to an edge of T incident with u (edge a in Figure 1) is added to the label of vertex v, and the label of a decremented, $k'(v),m'(a):=k'(v)+1,m'(a)-1$. Notice that there may be at most two such nodes u corresponding to the same vertex v. There are exactly two such nodes when v is saturated in B; it then ends up with the label $k'(v)=2$. If v is not saturated in B, then $k'(v)\leq 2$ if v is a cut-vertex, and $k'(v)=0$ otherwise. The demands of cut-vertices of B represented by labeling k′ are at least as severe as those represented by labeling k and still will be classified by Algorithm 1 as feasible. Thus, B is feasible. []

Feasibility of a mop block B of a purported interior graph G may not be enough for the graph to be the interior graph of a mop (see graph in Figure 6). The actual "edge requirements" of a cut-vertex v in such a block (in the sense of Algorithm 1) may be equal to 2 even if v is not saturated. This is because the path in the associated tree of G corresponding to a star in G centered in v may "pass through" B and thus contain two (rather than one) edges corresponding to exterior edges of B incident with v. In such a case, the label $k'(v)$ in the inverse of the feasibility checking algorithm (proof of Theorem 3) will take value 2. We observe that this applies to <u>all</u> <u>but</u> <u>two</u> mop blocks of G containing v.

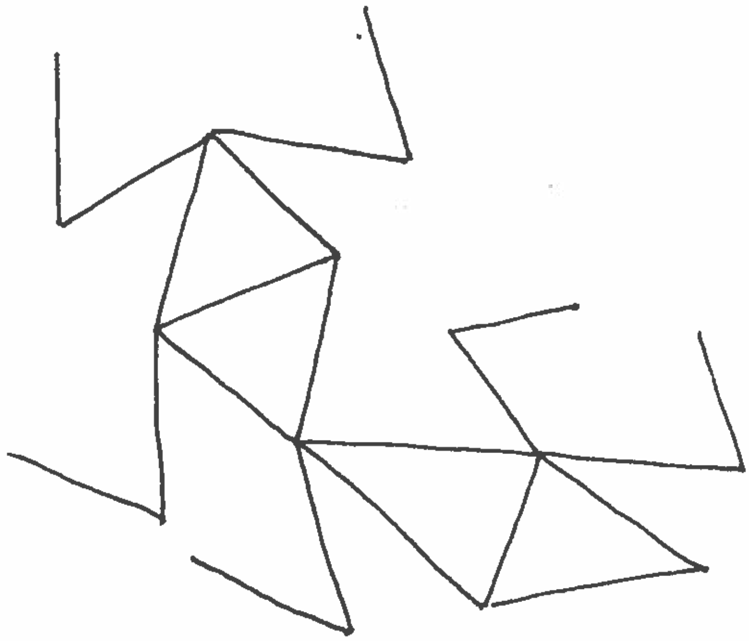

Figure 6 A graph G with feasible mop blocks and no feasible edge requirement

Let us define edge requirement function k(v,B) for all vertices v of a mop block B in a graph G to be a labeling of vertices with integers 0, 1, and 2 subject to following constraints.

k(v,B)= 0 if v is not a cut-vertex

2 if v is saturated in B

1 or 2 otherwise.

For a cut-vertex v of a mop block B, we extend the edge requirement function to all non-mop blocks B′ containing v, by defining k(v,B′)=1 if B′ is saturated, and k(v,B′)=2 otherwise.

Let us call an edge requirement function k for vertices of a graph G which consists of mop and caterpillar blocks feasible iff

(i) for every mop block B of G, B is found feasible by

Algorithm 1 when vertices of B are initially labeled with values k(v,B);

(ii) for every cut-vertex v, k(v,B)=1 in at most two blocks B of G containing v.

The above discussion and Theorem 3 allow us to state the necessary and sufficient condition for a graph to be the interior graph of a mop.

<u>Theorem 4</u> A graph G is the interior graph of a mop if and only if G is a connected collection of mops and caterpillars and has a feasible edge requirement function.

<u>Proof</u> (Necessity follows from the proof of Theorem 3.) To prove sufficiency, we will show that if every nontrivial (i.e., mop) block of the graph G is feasible according to the Algorithm 1 then, given feasible edge requirement k, G is the interior graph of some mop. To this end, we show that there exists a tree T of maximum degree 3 such that edges of G are in a one-to-one correspondance with edges of T, and every node of degree 3 in T corresponds to a triangle in G. Guided by the properties spelled out in Lemma 6 we will be able to find a mop H for which G is the interior graph. For every block B of G there is a tree associated with it (when B is a caterpillar, then this tree is a path). We now combine these trees into a tree T associated with G in the following manner.

It is sufficient to consider only cut-vertices of G which are incident with at least one mop block. For each such vertex v, we order linearly the blocks of G containing v, B1,...,Bm, so that no block B for which k(B,v)=1 has both preceding and succeding blocks. For blocks Bi, $1<i<m$, there are two unique leaf nodes of the corresponding associated tree Ti with pendant edges corresponding to exterior edges of Bi incident with v. (For an edge Bi, the two nodes are end-nodes of the corresponding edge Ti.) There are similar single nodes in T1 and Tm with pendant edges corresponding to edges of B1 and Bm into which the values of the label k(v,Bi) have been distributed by Algorithm 1 applied to the initial labeling k(u,Bi) (i=1 and i=m). (For an edge Bi, there either is a unique node, when k(v,Bi)=2, or two end-nodes, when k(v,Bi)=1.) We finally construct the tree T as a union of all trees Ti, $1 \le i \le m$, where the leaf nodes described above are pairwise identified: one leaf of Ti is identified with a leaf of T(i-1), and the other leaf of Ti is identified with a leaf of T(i+1). Since this newly constructed tree T has degree at most 3, it is associated with the interior graph of some mop, by Lemma 2. By our construction, G is this interior graph. []

4. Complexity of finding feasible edge requirement

In the preceding section we have shown that the existence of a feasible edge requirement function is a necessary and sufficient condition for a given collection of mop and caterpillar blocks to be the interior graph of some mop. We now briefly discuss the complexity of checking whether such a function exists.

Of primary importance in finding a feasible edge requirement function is the fact that the blocks of the candidate graph are connected in a tree-like fashion, i.e., removal of anyone but a pendant block disconnects some of the remaining blocks. This leads to a situation in which, once a tentative labeling of vertices of a block (assignment of function values) is made, it can "spread" independently into the subtrees of blocks. Let us define an <u>interval</u> of cut vertices of a mop block to be a maximal path spanned on such vertices along the Hamiltonian cycle of the mop, either separated from other vertices on the cycle by non-cut vertices, or containing all the vertices of the (saturated) block. Additional simplification of the tentative labeling process follows from the independence of labeling different intervals of cut-vertices of a mop block. We will now consider a process of tentative labeling of a single connected component of intervals of cut vertices, possibly sharing vertices with some non-mop blocks, c.f. Figure 7.

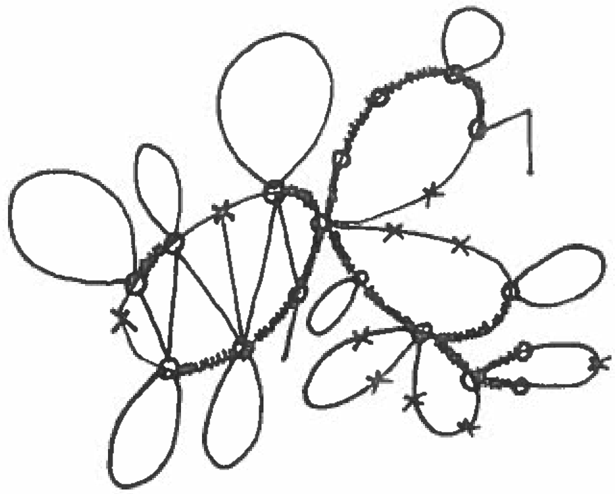

Figure 7 Collection of connected intervals of cut vertices (o - cut vertices, x - non-cut vertices).

Because of the availability of exterior edges incident with vertices of an interval (at most one more than the number of vertices in the interval), no vertex v of a saturated B mop can have assigned value k(v,B)=2, and for any other interval, only one vertex can have assigned value 2. Similarly, no vertex v can have k(v,B)=1 for more than two blocks B incident with it. One needs also consider edge requirement of such vertices in non-mop blocks, but those are uniquely determined, see the definition.

Since an interval identifies uniquely the mop block to which it belongs, in the following we will implicitely make use of this identification. Below, we present an algorithm assigning values of the edge requirement function to vertices of connected intervals of cut vertices. The structure of the interval adjacencies is tree-like, and thus the algorithm can be implemented efficiently utilizing, for instance, the depth first search of the tree.

Algorithm 2 Finding feasible edge requirement

Input: A connected component of intervals of cut vertices

Output: A feasible edge requirement, if one exists.

Method:
```
 { initialize }
 for every non-mop block B and every vertex v
   shared by B and an interval do
   ifB is saturated then k(v,B):=1 else k(v,B):=2;
 for every vertex v of a saturated block B do k(v,B):=1;
 for every interval i with a saturated vertex v do
   begin let B be the block of the interval i; k(v,B):=2;
   for every vertex u=v of i do k(u,B):=1 end;
 { enforce the labeling }
 while there is a vertex v with two blocks B´ such that k(v,B)=1
   and contained block B in which it has not been labeled
 do begin k(v,B):=2;
       for every vertex u=v in i do k(u,B):=1 end
       end;
 { if the labeling does not violate the constraints, it
   now can be extended to a feasible edge requrement   }
 repeat
   while there is a vertex v in interval i which has no value
      assigned in its block B and for two blocks B´ k(v,B´)=1
   do begin k(v,B):=2;
                 for every vertex u=v in i do k(u,B):=1 end
          end;
   choose a feasible labeling of vertices of any interval
   still not labeled
 until all vertices are labeled.
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The correctness of the above algorithm follows from the fact that, due to the forcible labeling in the while loop, a vertex v for which no edge requirement has been set in block B is in non more than one block B´ in which k(v,B´)=1. Thus, we can always assign k(v,B) to 1 without violating the constraints.

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