PRIMITIVE GRAPHS WHICH ARE MINIMUM SIZE MATCHING IMMUNE

by

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Abstract

This note addresses an issue concerning graphs with no disconnecting matching. A subclass of such graphs, elements of which have no subgraph with no disconnecting matching, has been investigated by Graham. He has exhibited an operation that combines two such graphs while preserving the definitional property. This operation thusly defines an infinite family of these graphs. Farley and Proskurowski have defined an infinite family of graphs with no disconnecting matching which also have the minimum size among all such graphs. The two families of graphs are closely related through the construction operations.

1. Introduction

A graph G=(V,E) consists of a set V of vertices and a (multi-)set E of edges where each edge in E is an unordered pair (u,v), u,v V. Edge (u,v) is said to be incident to u and v; given edge (u,v), u and v are said to be adjacent to each other. The number of nodes |V| is called the order of G, and the number of edges |E| its size. An independent edge set (also called a matching) has no two edges incident to the same vertex. A set of vertices and/or edges of a connected graph G disconnects the graph if the removal of all the elements of the set (including the removal of all edges incident to a removed vertex) results in a graph with at least two connected components. A graph is said to be immune to a given class of vertex and/or edge removals if and only if its connectivity is not changed by such removals. A graph is thus matching immune if and only if it is immune to the removal of any set of edges that constitutes a matching.

Graham [2] investigated a subclass of matching immune graphs which have no matching immune subgraphs. He gave a construction procedure defining an infinite class of those primitive graphs. This procedure also preserves the minimum size for matching immune graphs, unless applied to two graphs of even order.

2. Construction operations

A family of graphs with minimum size among all matching immune graphs has been defined by Farley and Proskurowski [1]. The graphs have been called abc-graphs since they are derived from the trivial graph of one vertex by any sequence of the following three operations:

(a) to a vertex of G connect three new edges and two new vertices to form a triangle;

(b) replace an edge of G by four new edges and two new vertices, forming a four-cycle with the two end vertices of the replaced edge;

(c) if G is of odd order, then add a new vertex connected by two new edges to any two vertices of G (this adds one new vertex and changes the parity of G´s order; thus, this operation can be applied only once).

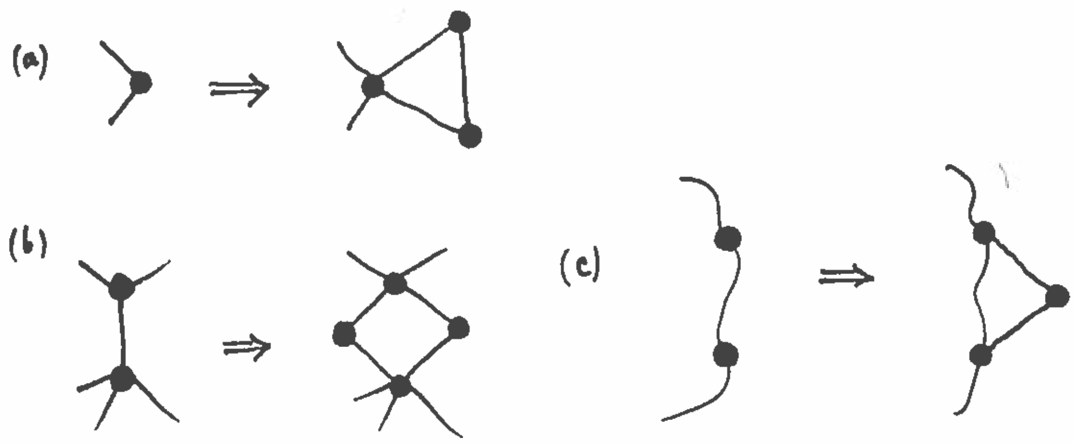

Figure 1 Augmentation operations (a), (b) amd (c).

It should be obvious that any one of the three operations preserves immunity to disconnecting matchings. The size of an abc-graph of order n is [3(n-1)/2], which is minimum for a matching immune graph. ([x] denotes the smallest integer not smaller than x.) Although neither (a) nor (c) preserve the defining property for primitive graphs (since G is a subgraph of the new graph and does not have a disconnecting matching), operation (b) results in a primitive graph if applied to a primitive graph.

The construction proposed by Graham is based on combining two graphs, say G and H. The end-vertices of an edge (x,y) of G and of the path x'-z-y' in H (where z is a vertex of degree 2 in H) are identified ("collapsed") and the three edges (x,y), (x',z), and (y',z) removed to form a new graph K. If G and H are primitive and c is regular in G, then K is primitive as well. Thus, this operation preserves immunity to disconnecting matching. (An edge (x,y) of a matching immune graph G is called <u>regular</u> if G-(x,y) has one disconnecting matching no edge of which is incident with x, and another no edge of which is incident with y.)

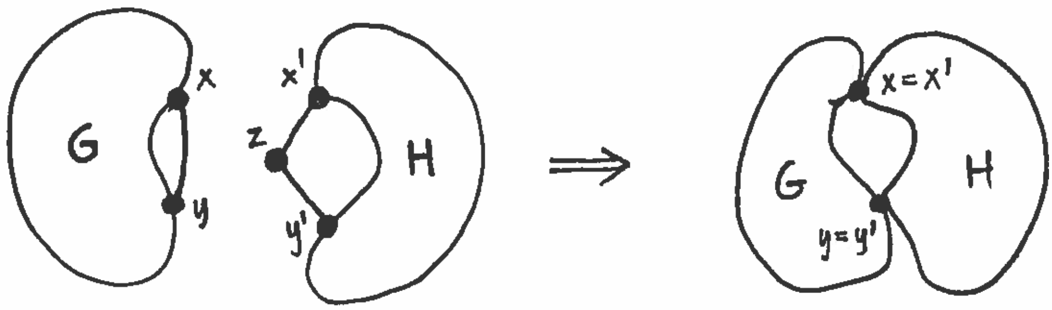

Figure 2 Primitivity preserving combination of two graphs.

Simple manipulation of the minimum size formula shows that if graphs G and H are minimum size matching immune not both of even order, then K is also minimum size:

$$[3(|G|-1)/2] + [3(|H|-1)/2] - 3 = [3(|G|+|H|-4)/2] = [3(|K|-1)/2]$$

When both G and H have even order, then K has more edges than the minimum for its order.

Farley and Proskurowski [1] have conjectured that the class of all minimum size matching immune graphs is the same as the class of abc-graphs. We will show that Graham's construction procedure is equivalent to a number of applications of the augmentation operations ((a), (b) or (c)) ,lending evidential support to our previous conjecture.

Augmentation operations (a) and (b) have obvious inverse reduction operations. These reductions, however, are not necessarily unique since they may be applied to an element (i.e., edge or vertex) of a graph in different (though isomorphic) ways. We will now show that for a given element of an abc-graph of odd order we can always chose a sequence of reductions not involving this element, reducing the graph to a triangle containing this element.

Lemma 1 A given abc-graph G of odd order can be constructed from a triangle by a sequence of operations (a) and (b) not creating a chosen element (vertex or edge) of G.

Proof (by mathemetical induction on the number r of augmentation operations necessary to construct G from a triangle.)

(i) Trivially true for $r=0$.

(ii) Let us assume the thesis true for some $r \geq 0$. Consider a graph G of order $2r+5$, thusly requiring $r+1$ augmentation operations. Let x be a vertex or an edge of G. If G can be reduced by an application of a reduction operation not deleting x, then the thesis follows from the inductive assumption since x is an element of the resultant graph G' of order $2r+3$. Otherwise, we have to consider two cases:

(a) The reduction operation is the inverse of (a). Then G consists of a triangle t containing x connected by an isthmus z to an abc-graph G'. Applying the inductive assumption to G' and z, we find a sequence s of augmentation operations not affecting z which construct G' from a triangle t' containing z. Now, we can perform augmentation operations according to s on the graph $t \cup t'$ resulting from applying operation (a) to z as a part of t. This gives the required sequence of operations constructing G.

(b) The reduction operation is the inverse of (b). Then G consists of a four-cycle connected by two of its non-adjacent vertices, u and v, to an abc-graph G´. Applying the inductive assumption to G´U(u,v) and (u,v) as the fixed element, we obtain a sequence s of augmentation operations constructing G´U(u,v) from a triangle (u,v,w) for some w. Now, let z be the vertex of degree 2 of the four-cycle, not adjacent to x. Consider the triangle consisting of (u,v) and the u,v-path p containing x. Applying the augmentation operation (b) to (u,v) in this triangle yields the four-cycle (u,z,v,w) and p. Performing the augmentation operations according to s results in G, which completes the proof. []

Applying Graham´s operation to two abc-graphs G and H to get a graph K we have to consider the order of the combined graphs. Let us assume that the operation collapsed vertices x and x´, and y and y´ of G and H, respectively, and that the removed edges were (x,y), (x´,z), and (y´,z) (see Figure 2). If G has odd order, then by Lemma 1, G-{(x,y)} can be constructed from two adjacent edges (closing a triangle with (x,y)) by a sequence s of augmentation operations. Graph K can thus be constructed by first constructing H and then applying s to the adjacent edges (x´,z) and (y´,z) of H. Since H is an abc-graph, so is K. If H has odd order, then H-{(x´,z),(y´,z)} can be constructed from the edge (x´,y´) closing the triangle (x´,y´,z) by a sequence s´ of augmentation operations, also by Lemma 1. Applying s´ to (x,y) after G is constructed results in K and proves that K is an abc-graph in this case as well. We have thus proved the following statement.

Theorem 1 Graham's combination operation preserves the property of being minimum size matching immune, unless applied to two graphs of even order.

At the conclusion of his paper, Graham poses a number of open questions which can either be answered, or at least commented upon in the light of the exhibited relation between primitive graphs and abc-graphs.

1. Since primitive graphs are matching immune, the minimum size-to-order ratio is 3/2.

2. Augmentation operation (b) produces minimum size primitive graphs of odd order only. Correctness of the Farley-Proskurowski conjecture would imply that these are all the minimum size of odd order. Whether primitive graphs of even order exist would still remain open. Obviously, abc-graphs of even order are not primitive, since they have matching immune subgraphs. This suggests that there may be no primitive graphs of even order.

References

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[2] R.L.Graham, On primitive graphs and optimal vertex assignments, Annals of the New York Academy of Sciences 175, 1(1970), 170-186.