# Immunity to Subgraph Failures in Communication Networks

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**Abstract**

A network is *immune* to a set of failures $F$ if message transfer can be completed in the presence of $F$. In this paper, we describe networks immune to certain classes of *templated failures*. A templated failure is characterized by (possibly more than one) connected graph, a template; elements of a given network that belong to a subgraph isomorphic with a template can fail together as a dependent failure. We discuss templates that can be described as graphs induced by a single node and a subset of its neighbors. We first deal with immunity to single templated failures. We then consider networks immune to sets of isolated failures, wherein no two adjacent elements of the network may fail. We describe minimal classes of graphs immune to sets of isolated failures characterized by a $K_{1,1}$ template.

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# Introduction

Communication networks provide means for dissemination of information among a set of *sites* by the transmission of *messages* through *calls* placed over *lines* interconnecting the sites. The sites may be processors in a parallel computer, computers in a distributed system, or hosts in a long-distance network. The design specification of a communication network consists of two aspects, topological and operational. A network's topological specification represents the physical, relatively static elements of the network: sites, lines, and any relevant properties of those elements (*e.g.*, length, delay, capacity). A network's operational aspect describes its dynamic behavior through routing tables and other features of the calling process.

We will model the topology of communication networks by means of *combinatorial graphs*: communication sites and communication lines of a network will be represented by *nodes* and *edges* of the corresponding graph. The properties of network topology that are of interest include the interrelation of the *order* (number of nodes), the *size* (number of edges), and the *diameter* (*i.e.*, the greatest length of a shortest path between a pair of nodes) of the graph representing the network. We will consider these properties while investigating graphs that remain connected when elements (nodes and edges) of some predefined *subgraphs* of the graphs are removed. These graphs represent networks that remain operational in the presence of site and line failures, in the sense that message transfer between operational sites can still be completed.

Failures of single elements, both alone and in isolated sets, have been treated extensively in the literature ([1,2,3,4,6,7]). In this paper, we initiate study of subgraph failures and obtain some preliminary results dealing with both types of subgraph failures: single subgraph failures and sets of isolated subgraph failures.

We consider *templated failures*, whereby a single failure is characterized by the failure of all elements of a subgraph of given network; this subgraph is isomorphic to one of the set of *templates*, connected graphs given by a set of nodes and edges which fail together. A motivation for this model is provided by *dependencies* of network failures, as encountered in practice, can fail in isolated $K_{1,1}$ failures. We call such pairs of nodes *safe* pairs. The twin and safe pairs of nodes can be used in an iterative construction of graphs immune to isolated $K_{1,1}$ failures, specified as follows:

**Algorithm Safe**:
**Set** $G$ to $C_4$;
**repeat**
  **choose one**
    add a new node $w$ adjacent to
      a twin or to a safe pair of nodes of $G$,
    **or** add a $K_{i,j}$ to a pair of twin nodes of $G$;
  call this new graph $G$.
**end** of the algorithm.

The size-to-order ratio of the resulting graphs ranges from 2 (for the pure Twin graphs), through $\frac{7}{4}$ for graphs constructed from Twin graphs by connecting nodes of each twin pair by a path including two new nodes ($K_{1,1}$), to $\frac{5}{3}$ for graphs using exclusively $K_{1,2}$ in the construction. We conjecture that the asymptotic size-to-order ratio of $\frac{5}{3}$ is the best possible for networks immune to isolated $K_{1,1}$ failures. The algorithm producing graphs with the conjectured minimum size-to-order ratio is as follows:

**Algorithm K12**:
**Set** $G$ to $C_4$;
**repeat** add a $K_{1,2}$ to a pair of twin nodes of $G$;
    call this new graph $G$.
**end** of the algorithm.

# Communication Protocols

We model communication in a network of given topology as being attained by a *calling* process, whereby a message is forwarded toward its destination along the network's links according to some *protocol*. Since a site that would be called in an undisturbed network may have failed in the afflicted network, the protocol for each site must at least have an alternate call for every possible receiver. We will be able to provide immunity to isolated due to electrical connections transmitting a disturbance (*e.g.*, power surge) or the geographical proximity (as in the case of an earthquake). A more general model of dependency of failures might incorporate the *possibility*, but not the necessity, of failure for these elements (*cf.* [5]).

Rather than dealing with the problem in all its generality, we will restrict our attention to templates consisting of a node and its $k$ *neighbors* (*i.e.*, adjacent nodes). Such graphs are called *stars* and denoted $K_{1,k}$. The node and its neighbors are called the *center* and the *tips* of the star, respectively.

# Immunity to Single Star Failures

When considering failures of star subgraphs, we assume that a failure at a node causes the removal from service of both the node *and its neighbors.* (In other words, rather than talking about failures of lines incident to the node, we assume failures of lines at a distance not greater than 2.) We now discuss the relationship of the size of a graph immune to a single failure (*i.e.*, a graph which is not disconnected by a star removal) to the diameter and the order of the afflicted graph. We fix the order of the original graph to be $n$. We will now consider several classes of graphs that represent different components of this relationship.

The graph of smallest size for given order which remains connected after an arbitrary star failure is the cycle $C_n$ (see Figure 1). Such a graph has $n$ edges and diameter of $n/2$. After failure, the afflicted graph is a path with $n-3$ nodes and diameter of $n-4$.

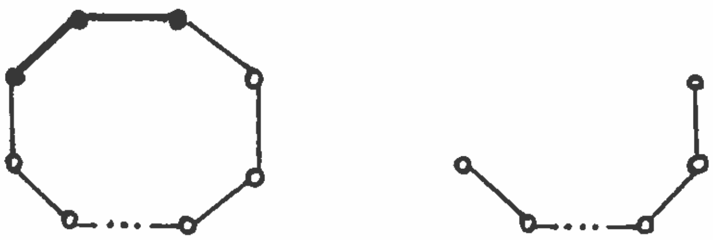

Figure 1: The cycle $C_n$ before and after a star failure

A 'stretched double star' consists of two stars $K_{1,n/2-1}$ with tip nodes connected in a one-to-one manner (see Figure 2). It has $\frac{3}{2}n$ edges and diameter 3. After failure, the afflicted graph has either $n-3$ remaining nodes and diameter of 4, (when a tip node fails) or $n/2$ remaining nodes and diameter $d = 2$ (when a center fails).

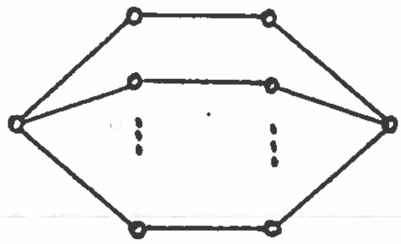

Figure 2: A stretched double star

These two classes of graphs seem to be at opposite extremes of the trade-off scale between number of remaining operational sites and diameter of the remaining network. Somewhere between is the 'squared double star', being two stars $K_{1,\sqrt{n}}$ with their tips connected through $\sqrt{n}$-paths. This graph has approximately $n+\sqrt{n}$ edges and diameter $d = \sqrt{n}$ (see Figure 3). After failure, the squared double star will have either $n-\sqrt{n}$ or $n-3$ nodes and the diameter will be $2\sqrt{n}$.

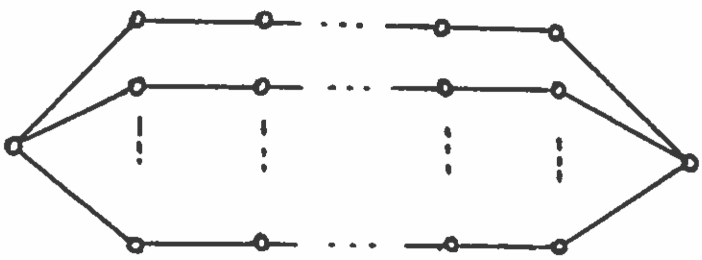

Figure 3: A squared double star

Of course, a complete graph on $n$ nodes is immune to a single templated failure, but since all sites fail when one site fails, immunity is rather vacuous. In general, failure of a high degree node makes a large number of

# Isolated Star Failures

We have introduced the notion of dependent element failure through a set of templates determining a configuration of elements in a network that will always fail together. Now we will study the problem of immunity to a set of *isolated* templated failures, where any occurrence of a single dependent failure in a network is separated (in terms of node adjacency) from any other such failure. A set of templated failures in a network is *isolated* if no two instances of failed subgraphs have adjacent nodes.

A network is immune to isolated failures if no set of such failures disconnects two operational nodes of the network. The study of such networks was begun a few years ago by Farley [3] and Farley and Proskurowski [4], where isolated failures of single elements were considered. The object of those investigations was to find graphs with the minimum size-to-order ratio over all classes of graphs immune to such failures. The obtained results identified the class of minimum-size graphs immune to isolated node and line failures to be *two-trees* (having a size-to-order ratio of 2), and provided an iterative construction algorithm for an infinite class of minimum-size graphs immune to isolated line failures (having a size-to-order ratio of $\frac{3}{2}$). In this section, we will develop designs for networks immune to isolated $K_{1,1}$ failures (having the conjectured minimum size-to-order ratio of $\frac{5}{3}$).

In graphs immune to $K_{1,1}$ failures, the minimum order of a separator is two (a *separator* is a subset of nodes the removal of which would disconnect the graph). Since we investigate graphs of small size-to-order ratio, we turn our attention to properties of two-node separators in $K_{1,1}$ immune graphs. If a two-node separator induces an edge then the graph could be disconnected by the $K_{1,1}$ failure involving the nodes of the separator. Thus, we will consider only two-node separators consisting of non-adjacent nodes.

Two nodes with identical neighborhoods are called a pair of *twins*. A minimal separator consisting of a pair of twins is immune to isolated $K_{1,1}$ failures in the following sense.

**Lemma 1** *Only one node of a pair of twins can be rendered inoperative by a set of isolated $K_{1,1}$ failures.*

Proof: If one node $x$ of a pair of twins $\{x, y\}$ fails, then a node $z$ in their common neighborhood must also fail. Thus $y$ cannot fail in a set of isolated failures. ■

The above lemma provides one idea for the construction of graphs immune to isolated $K_{1,1}$ failures. To bootstrap the associated construction algorithm we start with the simplest graph containing a pair of twins (actually, two pairs), the cycle $C_4$. We can then add nodes in a 'safe' manner by connecting each new vertex to a pair of twins, as suggested by the lemma above.

**Algorithm Twin**:
**Set** $G$ to $C_4$;
**repeat** add a new node $w$ adjacent to a pair of twin nodes of $G$;
call this new graph $G$.
**end** of the algorithm.

We will call all graphs obtained by execution of Algorithm Twin *Twin graphs*. Below we present two lemmas about Twin graphs which follow from induction on the number of iterations performed by Algorithm Twin (or, equivalently, the order of the graph).

**Lemma 2** *Every node of a Twin graph has degree 2 or is a member of a pair of twin nodes.*

**Lemma 3** *Any minimal separator of a Twin graph consists of a pair of twin nodes.*

**Theorem 4** *No Twin graph can be disconnected by a set of isolated $K_{1,1}$ failures.*

Proof: By the Lemmas above. ■

We have thus succeeded in designing a class of networks immune to isolated $K_{1,1}$ failures, the Twin graphs. Their size-to-order ratio tends asymptotically to 2, since every new node requires two new edges. Actually, nodes inoperative, thereby eliminating a significant portion of the graph and trivially simplifying the task of keeping the afflicted graph connected. This seems to point out that in this model of single templated failures, it would be useful to bound the maximum node degree. The cycle mentioned above is a graph with maximum degree 2. It might be of interest to consider star failures in graphs with maximum node degree 3. The standard operation of replacing every node of degree higher than 3 by a cycle of degree 3 nodes transforms the graphs above, a 'stretched double star' and 'squared double star', into 'cylinder' graphs having diameter before failure of approximately $n/8$ and $\frac{3}{2}\sqrt{n}$, respectively (see Figure 4). The maximum degree places a bound on the number of nodes made inoperative by a star failure; in cylinder graphs, at least $n-4$ nodes remain operational after failure. In the cylinder graph derived from a stretched double star, the diameter is unaffected by a star failure, while in the graph derived from a squared double star, a star failure causes the diameter to increase by $\frac{\sqrt{n}}{2}$.

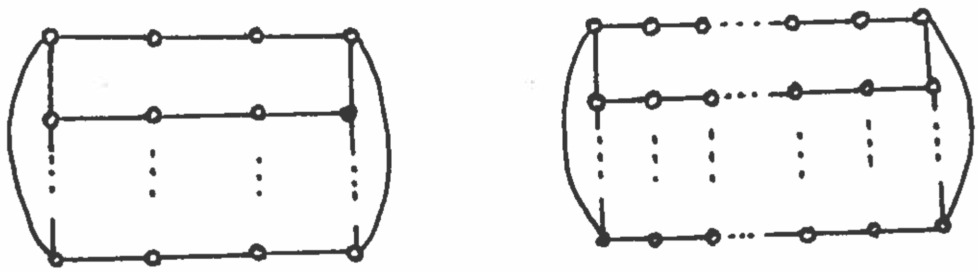

Figure :.4: Cylinder graphs derived from stretched and squared double star graphs

Another example of a graph with bounded node degree is the $k$-dimensional cube. Both its node degree and diameter are bounded by $k$, the logarithm of the graph's order. The diameter is not increased by a single star failure.

one can say more about two-node separators in a graph immune to isolated $K_{1,1}$ failures. (We will denote the set of nodes adjacent to a given node $x$, the neighborhood of $x$, by $\Gamma(x)$.)

**Lemma 5** *Let $\{x, y\}$ be separator in a graph $G$ immune to isolated $K_{1,1}$ failures. If the sets $\Gamma(x) - \Gamma(y)$ and $\Gamma(y) - \Gamma(x)$ are non-empty, then their union induces in $G$ a complete bipartite graph $K_{i,j}$ with the partitions coinciding with the two sets.*

Proof: Assuming the contrary situation, let $u \in \Gamma(x) - \Gamma(y)$ and $v \in \Gamma(y) - \Gamma(x)$ be non-adjacent. Then the failure of the four nodes, *i.e.*, $\{u, x, v, y\}$ is an instance of two isolated $K_{1,1}$ failures that disconnect $G$. ∎

It follows from Lemma 5 that given a pair of twin nodes in an isolated $K_{1,1}$ failure immune graph $G$, the node sets of a $K_{i,j}$ can be made adjacent to each of the twins, respectively, resulting in a new graph $G + K_{i,j}$ that is immune to isolated $K_{1,1}$ failures (see Figure 5).

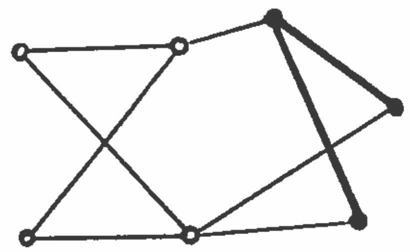

Figure 5: Adding $K_{1,2}$ to a pair of twin nodes.

Notice that adding nodes in $K_{i,j}$ clusters becomes 'size intensive' for $i \geq 2$ (the size-to-order ratio is at least 2). We will thus restrict the bipartite graphs to stars, $K_{1,i}$. We observe that any two of the tips of the $K_{1,i}$ in the construction described above constitute a pair of twin nodes. The nodes of the original twin pair no longer have identical neighborhoods, but they preserve the property of both not becoming inoperative in the presence of isolated $K_{1,1}$ failures. The pair consisting of the center of the new $K_{1,i}$ and the other node adjacent to its $i$ tips is also a pair of nodes not both of which

$K_{1,1}$ failures based on two *calling lists* (routing tables), which for each site include *preferred* and *alternate* call addresses for all other sites as intended receivers.

Thus, to complete our immune network design, it remains to determine preferred and alternate calls for each ordered pair of nodes being the current sender and the ultimate recipient of a message, respectively. We define *preferred*$(x, y)$ and *alternate*$(x, y)$ to denote the preferred and alternate neighbors for the forwarding of a message by $x$ towards $y$. These preferred and alternate routings must guarantee completion of a message transfer between operating sites in the presence of isolated $K_{1,1}$ failures.

The definition of a sufficient communication protocol is straightforward. Given reception or initiation of a message for site $y$ as receiver, site $x$ will first attempt to call *preferred*$(x, y)$. If this cannot be completed, then site $x$ will call *alternate*$(x, y)$. We call this calling process *protocol P*. We assume that the failure of a neighboring site can be determined by lower-level protocols. An attempt to send a message to a failed node should cause the calling process to abort. We realize this behavior by setting, for neighbors $x, y$, *preferred*($x$ and $y$) to $y$ and leaving *alternate*$(x, y)$ undefined.

The routing tables of a Twin graph $G$ can be established in the process of its construction by Algorithm Twin. This happens in two phases. In the first phase, a number of common neighbors are added to a pair of twin nodes in the original $C_4$. When a second pair of twin nodes is used to append a new node, the initial values of routing tables can be established. The second phase follows in iterations of the construction algorithm's loop after more than one pair of twin nodes has been used in the execution of the algorithm. The values of *preferred*$(x, y)$ and *alternate*$(x, y)$ for different pairs of $x$ and $y$ are determined upon the inclusion of each new node $w$.

In the following, we will assume that all adjacent nodes $x, y$ have undefined values of *alternate*$(x, y)$ and *alternate*$(y, x)$, with *preferred*$(x, y) = y$ and *preferred*$(y, x) = x$.

Let a Twin graph $G$ be the graph produced by the first phase. $G$ consists of a complete bipartite graph $K_{2,i}$ $(i \geq 2)$ and a node $w$. Let $\{u, v\}$ be the smaller independent set of nodes, and $w$ be the new node added to a (twin)

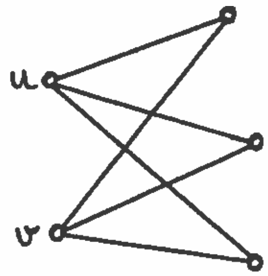


Figure 6: The initial configuration of a Twin graph.

pair $\{x, y\}$ from the other independent set (see Figure 6). We specify:

*preferred*$(u, v)=x$, and *alternate*$(u, v)=y$,

*preferred*$(v, u)=x$, and *alternate*$(v, u)=y$,

*preferred*$(u, w)=x$, and *alternate*$(u, w)=y$,

*preferred*$(v, w)=x$, and *alternate*$(v, w)=y$,

*preferred*$(w, z)=x$, and *alternate*$(w, z)=y$ for all nodes $z \notin \{x, y\}$.

Finally, for all nodes $s, t$ from the other bipartition (the neighbors of $\{u, v\}$), we set

*preferred*$(s, t) = u$ and *alternate*$(s, t) = v$.

Assume now that a Twin graph $G$ is under construction in the second phase. Let $\{x, y\}$ be the pair of twin nodes to which the new node $w$ is made adjacent. For all nodes $z \notin \{x, y\}$ we set

*preferred*$(w, z)=x$, and *alternate*$(w, z)=y$.

For those nodes $z$ that are not neighbors of $x$, we set

*preferred*$(z, w)=$*preferred*$(z, x)$, *alternate*$(z, w)=$*alternate*$(z, x)$,

and for all those that are neighbors of $x$ and $y$), we set

*preferred*$(z, w)=x$ and *alternate*$(z, w)=y$.

**Theorem 6** *Communication protocol P using the routing tables as defined above realizes immunity to isolated $K_{1,1}$ failures in Twin graphs.*

Proof: For every pair of non-adjacent nodes $s$ and $t$, nodes *preferred*$(s,t)$ and *alternate*$(s,t)$ constitute a twin pair. Thus, by Lemma 1, one of the two sites must be operational when $K_{1,1}$ failures are isolated. Also for such nodes, the distance from *preferred*$(s,t)$ and *alternate*$(s,t)$ to $t$ is less than the distance from $s$ to $t$. Thus, there will be no cyclic calling behavior and the message transfer will eventually succeed if $t$ is operational. ∎

Similarly, routing tables can be determined during the construction process for *Safe* and *K12* graphs. There, however, a list of alternate call addresses is needed (rather than a single address).

# Conclusions

In an attempt to generalize our previous results on immunity to isolated element failures, we investigated templated failures and defined families of graphs that remain connected after removal of isolated pairs of adjacent nodes. We conjectured that one of the presented classes is the class of extremal graphs with this property.

Our future research in this area will include investigation of other restricted classes of templated failure immune graphs. We will also consider application of these topologies in reliable completion of other communication tasks such as broadcasting and gossiping.

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