Graph reductions, and techniques for finding minimal forbidden minors

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Abstract

Knowing that a class of graphs has a finite set of minimal forbidden minors is one thing, knowing what they are is another. We present an account of techniques used to find small sets of minimal forbidden minors for few classes of graphs with treewidth at most 3.

1 Introduction

A finite representation of an infinite class of objects constitues a very attractive tool and an elegant result. For graphs, there have been a number of forbidden substructure characterizations, the most famous being the Planar Graphs Theorem of Kuratowski:

> A graph is planar if and only if it does not contain a subgraph homeomorphic to either K_5 or $K_{3,3}$.

Wilf [21] gives an introduction to the notion of obstructions in his column that introduces work of Robertson and Seymour. There, one can find Kuratowski's theorem stated in terms of minors (Thomassen [19] attributes it to Harary and Tutte):

> A graph is planar if and only if it does not contain a minor isomorphic to either K_5 or $K_{3,3}$.

As a reminder, we state few basic definitions. A graph H is a *minor* of a graph G if contracting some edges of a subgraph of G ('minor-taking') would give a graph isomorphic to H (we will consider only *simple* graphs, without multiple edges). For a class $\mathcal{C}$ of graphs closed under minor-taking, F is a *minimal forbidden minor* if it is not in $\mathcal{C}$, but every minor of F is in $\mathcal{C}$.

For a fixed positive integer k, the complete graph on k vertices, K_k is a k-tree and every k-tree with $n > k$ vertices can be constructed from a k-tree with $n-1$ vertices by adding to it a vertex adjacent to all vertices of a subgraph isomorphic to K_k. A graph that can be embedded in a k-tree is called *partial k-tree*, or alternatively, it is said to have *treewidth* at most k.

In the study of graphs with bounded treewidth (partial k-trees), there are obvious characterizations for $k = 1, 2$ by forbidden subgraphs homeomorphic to K_3 and K_4, respectively. Although K_5 is likewise forbidden for $k = 3$, the set of minimal forbidden minors (*obstructions*) for partial 3-trees is obviously larger. Before discussing the tools used in the discoveries of that set, we briefly present approaches to determining obstruction sets for some smaller graph classes.

For *partial 1-trees* (forests), the completeness of $\{K_3\}$ as the set of obstructions follows directly from the definition (the acyclicity of the graphs).

Biconnected *partial 2-trees* (simple series-parallel graphs) can be recognized by iterating the series vertex reduction (see below). A proof of $\{K_4\}$ as their obstruction set follows from considering the reduction rule: A minimal forbidden minor must be cubic and biconnected, and end vertices of any edge must have a common neighbor. K_4 is the only graph that fits this description.

K_4 is also a minimal forbidden minor for the *outerplanar graphs*. By the definition, so is the graph $K_{2,3}$. To see that these two graphs constitute the set of minimal forbidden minors for outerplanar graphs, consider any plane embedding of a series-parallel non-outerplanar graph. It must have an interior vertex and at least one vertex between its attachments to the cycle constituting the boundary of the exterior mesh in that embedding, in each direction around that cycle. Such a graph has a subgraph homeomorphic to $K_{2,3}$ and thus has $K_{2,3}$ as a minor. (Thomassen [19] traces this characterization to Chartrand and Harary.)

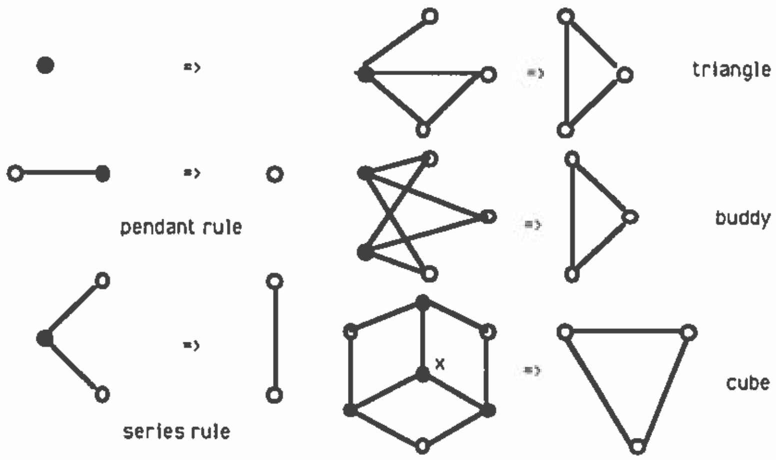

Figure 1: Reduction rules for recognition of partial 3-trees

In this note, we intend to illustrate the concept of graph reduction in the search for a complete set of obstructions. For this purpose, we give a short survey of the different approaches that lead to the discovery of the set of minimal forbidden minors for partial 3-trees. We will start with a presentation of one of the tools (vertex reduction system), then describe the search for obstructions, and conclude with a brief description of continued efforts to find a general (constructible) algorithm paradigm for obstruction sets.

2 Vertex reduction rules for partial k-trees

Recognition of forests by checking the irreducible result of repeated 'pruning of leaves' (removal of degree 1 vertices) and discarding isolated vertices has been taken for granted for a long time. Recognition of partial 2-trees by, in addition, contracting an edge incident with a degree 2 vertex has been proposed by Wald and Colbourn [20]. Arnborg and Proskurowski [2] give a complete set of confluent vertex reduction rules for partial 3-trees. (This means that a graph is a partial 3-tree if and only if any sequence of applications of these rules reduces it to the empty graph; if one does, so does any other.) There, the three reduction rules mentioned above are augmented by three more reductions of degree 3 vertices (see Figure 1). These mimic pruning 3-leaves (degree 3 vertices) of an embedding 3-tree, but also indicate that not all degree 3 vertices in a partial 3-tree are such 3-leaves.

Independently, the same set of reduction rules was derived by Kajitani *et al.* [11] who discovered the necessity of certain configurations in 3-connected partial 3-trees following a very similar line of reasoning.

A fairly natural implementation of these rules leads to an $\mathcal{O}(n \log n)$ algorithm; Matousek and Thomas [15] noticed that the rules can be modified to yield a linear algorithm.

As these reduction rules constitute an important tool in the investigations of 'small' properties of partial k-trees, the following result of Lagergren [13] is quite discouraging.

> There is no complete set of confluent vertex reduction rules that reduce a graph to the empty graph if and only if the graph is a partial k-tree, for $k > 3$.

Yet, it turns out that there exist more general graph reduction systems that decide membership in classes of partial k-trees. Namely, Arnborg *et al.* show in [1] that this is the case for any subclass of partial k-trees (fixed k) definable by the *Monadic Second Order Logic (MSOL)* (*cf.*, for instance, Courcelle [7]).

> For any class of graphs of bounded treewidth that can be described by an expression in the *MSOL* formalism there is a finite terminating graph rewriting system with the following property: Repeated applications of the rewrite rules lead to an irreducible graph that is a member of a finite accepting set of graphs if and only if the original graph is a member of the graph class in question.

Such a graph rewriting system can be implemented as a linear time (although space intensive) algorithm. As usual, however, constructing such a system might be a difficult task, even though the existence proof (of the above result) is constructive. More importantly for the subject of this note, such a system gives little insight into construction of the set of forbidden minors. Yet, it might provide some computational help, see section 4.

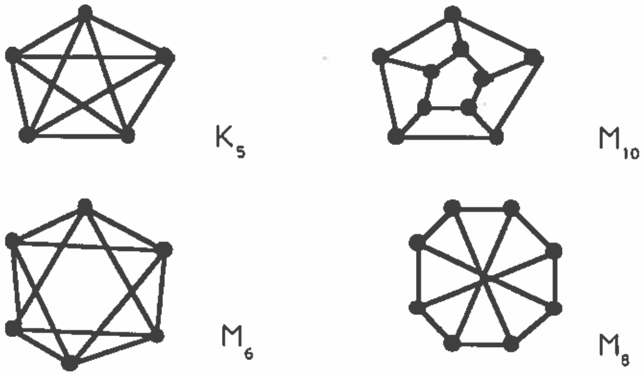

Figure 2: The set of minimal forbidden minors for partial 3-trees

3 Minimal forbidden minors for partial 3-trees

The sets of minimal forbidden minors for *partial 3-trees* and for *planar partial 3-trees* have been discussed independently, even though the former – in conjunction with the minors form of the Planar Graph Theorem – implies the latter.

El Mallah and Colbourn [10] state the following characterization of planar partial 3-trees.

> A planar graph is a partial 3-tree if and only if it does not have a minor isomorphic to either M_6 or M_{10} in Figure 2.

In their proofs, they exploit the duality between Δ-Y and Y-Δ reductions (replacing triangle K_3 by star $K_{1,3}$ and *vice-versa*, respectively) and properties of geometric duals of the graphs defined with help of these reductions. The proof of a similar result presented by Dai and Sato [9] is based on Tutte's characterization of planar 3-connected graphs. Both papers rely on the presence of the lefthand-sides of vertex reduction rules used in recognition of partial 3-trees in edge-contracted planar graphs.

Arnborg, Proskurowski and Corneil characterize the class of partial 3-trees in [4].

A graph is a partial 3-tree if and only if it does not have a minor isomorphic to any of the graphs in Figure 2.

Their proof depends very heavily on the small complete set of confluent reduction rules for this class of graphs (see above). The minimality of the investigated minors implies 3-connectivity, and the reduction rules (and the fact that they are vertex-reducing) imply that any vertex degree can be only 3 or 4. Investigation of cases of possible neighborhood configurations of contracted or extracted edge in any minimal forbidden minor (those 'configurations' must admit vertex reductions) completes their proof.

Theirs was just one of several independent investigations that ended with similar results. The approach based on the same set of reduction rules for partial 3-trees was used by Borie, Parker and Tovey [6].

Satyanarayana and Tung [17] do not use the reduction rules in their proofs, but they rediscover (in fact) the properties of 3-connected components of minimal forbidden minors implied by those reductions. The flow of their proofs follows a similar path of discovering cubic minimal forbidden minors, then 4-regular such graphs, and then showing that minimum vertex degree 3 implies 3-regularity of a minimal forbidden minor.

A recent paper by Satyanarayana and Politoff [16] gives an alternative proof of the minimal forbidden minors for partial 3-trees. In their discussion of *quasi 4-connected graphs* (that have no 3-vertex separators except for those that separate several degree 3 vertices) they find that only few graphs are 'responsible' for this property. Namely, a non-planar quasi 4-connected graph has a K_5 minor or is a 'small graph'. A planar quasi 4-connected graph has M_6 as a minor or is some other 'small graph'. These 'small graphs' are M_8 and M_{10} from Figure 2, and some partial 3-trees. Since no partial 3-tree, except for some small ones with only trivial 3-separators, is quasi 4-connected, every large enough quasi 4-connected graph has a minor from the set of minimal forbidden minors for partial 3-trees. Analysis of those small partial 3-trees and the 'small graphs' in their lemmata implies the desired characterization of partial 3-trees. Although tediously relying on case analysis, the proofs are somewhat shorter than case analyses in the previously published proofs.

4 Other tools for finding forbidden minors

Graph reduction rules are of some help in constructing the obstruction set, but they are by no means the only tool available. The computational power of modern computers and the skill of their programmers can go a long way in searching for minimal forbidden minors, especially among subclasses of partial k-trees, for small values of k. An example of such a result is the set of 110 minimal forbidden minors for graphs with pathwidth 2 constructed by Kinnersley [12]. Similar result concerning acyclic such minors for $k = 2, 3$ and 4 is presented by Takahashi *et al.* [18].

Another approach, yet to be implemented, is to construct the obstruction set using raw computational power for searching a finite list of graphs among which all such graphs are guaranteed to be found. Arnborg *et al.* [5] describe the translation process of a *MSOL* formula defining a subclass of bounded treewidth graphs into a *tree automaton*. The number of states in the resulting automaton can be used to determine a bound on the number of vertices in a minimal forbidden minor for that class.

Using an encoding of tree decompositions of width k, Lagergren and Arnborg [14] find a finite congruence in a *graph algebra* that defines the class of partial k-trees. Subsequently, they describe how to obtain the set of irreducibles that contains the obstruction set by a procedure similar to the construction algorithm for the corresponding graph reduction system of [1] (*cf.* Section 2).

Given a *graph grammar* defining a class of graphs, Courcelle and Proskurowski [8] use formal linquistic tools to derive another graph grammar that generates a finite superset of minimal forbidden minors for the original class. Graph grammars, upon which we will not elaborate here, provide another way of defining classes of graphs with interesting algorithmic properties.

Using a finite characterization, a class of graphs with bounded treewidth can be defined by:

(i) a set of minimal forbidden minors,

(ii) an *MSOL* description,

(iii) a graph reduction system.

These have been shown equivalent by Lagergren and Arnborg [14]. While we have some developed understanding of the conceptual relationships between these description methods, constructive proofs of their equivalence remain still an important research topic. This is due in part to the potentially gigantic size of any such finite characterization.

We have chosen a small example of the class of partial 3-trees to illustrate some of the notions used in discovering the corresponding obstruction set. Vertex reduction rules used in this process can not be directly generalized for graphs with larger treewidth, but more general graph reduction systems might bring some assistance in the search for minimal forbidden minors bounding the set of candidate graphs.

Disclaimer: It is not within the scope of this note to present a complete historical and methodological survey of this extensive and exciting area. The author readily accepts the blame for omissions of references to and timing of any independent work.

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