Characterization and Complexity of Domination-type Problems in Graphs

Jan Arne Telle

CIS-TR-93-14
May 1993

Department of Computer and Information Science
University of Oregon

Characterization and Complexity of Domination-type Problems in Graphs*

Jan Arne Telle
Department of Computer and Information Science
University of Oregon, Eugene, Oregon 97403, USA

Abstract

Many graph parameters are the optimal value of an objective function over selected subsets S of vertices with some constraint on how many selected neighbors vertices in S, and vertices not in S, can have. Classic examples are minimum dominating set and maximum independent set. We give a characterization of these graph parameters that unifies their definitions, facilitates their common algorithmic treatment and allows for their uniform complexity classification. We investigate the computational complexity of problems admitting this characterization, identify classes of NP-complete problems and classes of problems solvable in polynomial time. We distinguish vertex subset properties admitting the characterization which have the feature that any such set in a graph has the same size.

1 Introduction

If every vertex in a selected subset S of vertices of a graph has zero selected neighbors then S is an independent set, and similarly if every vertex not in S has at least one selected neighbor then S is a dominating set. This suggests a common characterization of independent sets and dominating sets based on the constraints imposed on the number of selected neighbors the vertices in S, and vertices not in S, can have. As we show in this paper, a large collection of vertex subset properties found in the literature, including variants of efficiency, packing and irredundance, admit such a characterization. Many graph parameters are the optimal value of an objective function over subsets of vertices admitting the characterization. From the standard

*This work supported by NSF grant CCR9213439. Author's email address: telle@cs.uoregon.edu

definitions of these parameters it is not obvious that they are related as described here. This characterization thus facilitates the common algorithmic treatment of the problems computing these parameters.

The current bibliography [15] of papers related to the general topic of domination in graphs, maintained by S.Hedetniemi and R.Laskar, has about 400 entries covering a large variety of domination-type parameters. We are mainly interested in the algorithmic theory of domination in graphs. A paper in this field will typically introduce a new domination-type parameter, contrast it with related domination parameters and give computational complexity results. As an example, Bernard, Hedetniemi and Jacobs [2] define the efficiency of a graph, show the NP-completeness of computing the efficiency of a bipartite graph and give a linear-time algorithm computing the efficiency of a tree. Telle and Proskurowski [19] extend this latter result by giving a linear-time algorithm computing the efficiency of a partial k-tree, for fixed k. Upon the introduction of a slight variation of the efficiency parameter this work would then usually be repeated. In contrast, the characterization we propose here facilitates the common algorithmic treatment of these parameters and allows for their uniform complexity classification. These parameters oftentimes arise from various fields, traditionally seen as separate, with the confusing effect that naming conventions are not standardized. As an example, Biggs [3] and later Kratochvil [17] consider Perfect Codes in graphs (as a generalization of error-correcting codes), while Bange, Barkauskas, Slater [1] study Efficient Dominating Sets in graphs (a variant of domination), and Fellows, Hoover [10] investigate what they call Perfect Domination. In fact, they are all studying the exact same vertex subset property, designated in our characterization as $[\rho_1, \sigma_0]$-sets.

In the next section, we present our characterization and show that many of the graph parameters found in the literature admit such a characterization. In section 3, we investigate the computational complexity of the problems computing these parameters. We identify several classes of both NP-complete problems and of problems solvable by a greedy algorithm. The NP-completeness results identify properties with interesting features, such as when both maximum and minimum versions are NP-complete (independent dominating sets), or when merely deciding if a graph has a vertex subset with the property is NP-complete (perfect codes). We show a class of problems which are NP-complete even when restricted to planar bipartite graphs of maximum degree three. A natural by-product of these results is the introduction of several new domination-type parameters in graphs. In section 4 we give a theorem describing those vertex subset properties admitting the characterization which have the feature that any such set in a graph has the same size. In the last section we show a refinement of the characterization allowing for the definition of maximal and minimal versions of the same vertex subset properties, and also of vertex subset properties related to irredundant sets.

2 Characterization of Domination-type Problems

We use standard graph terminology [4]. For a vertex $v \in V(G)$ of a graph G, let $N_G(v) = \{u : (u,v) \in E(G)\}$ be the set of neighbors of v and $deg_G(v) = |N_G(v)|$. For $S \subseteq V(G)$ let $G[S]$ denote the graph induced in G by S and let the symbols σ and ρ denote membership in S and membership in $V(G) - S$, respectively.

Definition 1 Given a graph G and a set $S \subseteq V(G)$ of *selected* vertices

- The *state* of a vertex $v \in V(G)$ is

$$state_S(v) \stackrel{df}{=} \begin{cases} \rho_i & \text{if } v \notin S \text{ and } |N_G(v) \cap S| = i \\ \sigma_i & \text{if } v \in S \text{ and } |N_G(v) \cap S| = i \end{cases}$$

- Define syntactic abbreviations

$$\rho_{\leq i} \equiv \rho_0, \rho_1, \ldots, \rho_i \qquad \sigma_{\leq i} \equiv \sigma_0, \sigma_1, \ldots, \sigma_i \qquad \rho_{\geq i} \equiv \rho_i, \rho_{i+1}, \ldots \qquad \sigma_{\geq i} \equiv \sigma_i, \sigma_{i+1}, \ldots$$

The latter two abbreviations each represent an infinite set of states. Mnemonically, σ represents a vertex selected for S and ρ a vertex rejected from S, with the subscript indicating the number of neighbors the vertex has in S. A variety of vertex subset properties can be defined by allowing only a specific set L as *legal* states of vertices. For example, S is a dominating set if state ρ_0 is not allowed for any vertex, giving the legal states $L = \{\rho_{\geq 1}, \sigma_{\geq 0}\}$. Optimization problems over these sets often maximize or minimize the size of the set of vertices with states in a given $M \subseteq L$. For the minimum dominating set problem, $M = \{\sigma_{\geq 0}\}$.

Definition 2 Given sets M and L of vertex states and a graph G

- $S \subseteq V(G)$ is an $[L]$-set if $\forall v \in V(G) : state_S(v) \in L$
- $minM[L]$ (or $maxM[L]$) is the problem minimizing (or maximizing) $|\{v : state_S(v) \in M\}|$ over all $[L]$-sets S
- $min[L]$ (or $max[L]$) is shorthand for $minM[L]$ (or $maxM[L]$) when M consists of all σ-states in L, in effect optimizing the size of the selected set of vertices.
- $minM[L](G)$ (or $maxM[L](G)$) is the corresponding parameter for G

Our notation	Standard terminology
$[\rho_{\geq 0}, \sigma_0]$-set	Independent set
$[\rho_{\geq 1}, \sigma_{\geq 0}]$-set	Dominating set
$[\rho_{\leq 1}, \sigma_0]$-set	Strong Stable set or 2-Packing
$[\rho_1, \sigma_0]$-set	Efficient Dominating set or Perfect Code
$[\rho_{\geq 1}, \sigma_0]$-set	Independent Dominating set
$[\rho_1, \sigma_{\geq 0}]$-set	Perfect Dominating set
$[\rho_{\geq 1}, \sigma_{\geq 1}]$-set	Total Dominating set
$[\rho_1, \sigma_1]$-set	Total Perfect Dominating set
$[\rho_{\leq 1}, \sigma_{\geq 0}]$-set	Nearly Perfect set
$[\rho_{\leq 1}, \sigma_{\leq 1}]$-set	Total Nearly Perfect set
$[\rho_1, \sigma_{\leq 1}]$-set	Weakly Perfect Dominating set
$[\rho_{\geq 0}, \sigma_{\leq k-1}]$-set	k-dependent set
$[\rho_{\geq k}, \sigma_{\geq 0}]$-set	k-dominating set
$[\rho_{\geq 0}, \sigma_k]$-set	Set inducing k-regular subgraph
$max\{\rho_1\}[\rho_{\geq 0}, \sigma_{\geq 0}]$	Efficiency problem
$min\{\sigma_{\geq 0}\}[\rho_{\geq 1}, \sigma_{\geq 0}]$	Domination problem

Table 1: Some classical vertex subset properties and graph parameters

Thus, a dominating set is a $[\rho_{\geq 1}, \sigma_{\geq 0}]$-set, with the square brackets implying the set notation. Table 1 shows some of the classical vertex subset properties [12, 5, 10, 1, 2, 6] and a few graph parameters as expressed using our characterization. Note that properties traditionally defined using closed neighborhoods are easily captured by the characterization. For vertex weighted versions of these parameters optimize the sum of the weights of vertices with state in M, the cardinality version corresponding to unity weights. For directed graphs define $N_G(v)$ as $\{u : \langle u, v\rangle \in \text{Arcs}(G)\}$ to obtain directed versions of these domination-like properties and parameters. In the last section of this paper, we give an extension of this characterization to encompass also parameters related to irredundant vertex subsets, and also to maximal and minimal versions of the vertex subsets given here.

Table 1 can be used as a quick reference guide to the exact definitions of the various properties and parameters represented. The characterization may also be useful when introducing new parameters. In another paper [19], we give practical algorithms on partial k-trees (graphs of treewidth bounded by k) solving any problem admitting the given characterization. A measure of the complexity of the resulting algorithm solving a problem with legal states L is the set A_L (a superset of L) of states needed for algorithmic purposes and its syntactic size $|A_L|$. Suffice it to say that any non-parameterized problem derived from Table 1 has $|A_L| \leq 4$.

Theorem 1 *[19] For any problem admitting the given characterization having legal states L there is an algorithm which takes a graph G with n vertices and a width k tree-decomposition of G as input, and gives a solution for G in* $\mathcal{O}(n|A_L|^{2k})$ *steps.*

The next section focuses on the computational complexity on general graphs of the problems admitting the characterization.

3 Complexity Results

We would like to classify the computational complexity of the problems admitting the given characterization. We are mainly interested in classifying problems as NP-complete or solvable in polynomial time.

From Table 1 we note that subset properties characterizable by two syntactic states (using the abbreviations) in which vertices have zero, one, at least zero, or at least one selected neighbors, attract the most interest. Similarly, the objective functions most studied involve minimizing or maximizing the cardinality of the set of selected vertices, and for each entry in Table 1, except Nearly Perfect Sets, there is at least one NP-complete problem related to such a parameter. For Independent Dominating Sets, it is well known that both minimizing and maximizing the size of such a set is an NP-hard problem. We give some vertex subset properties not found in Table 1, in which vertices have zero, one, more than zero, or more than one selected neighbors, sharing this feature of Independent Dominating sets.

Theorem 2 *The decision problems* $opt[\rho_{\geq 1}, \sigma_1]$, $opt[\rho_{\geq 1}, \sigma_{\leq 1}]$, $opt[\rho_{\geq 2}, \sigma_0]$ *and* $opt[\rho_{\geq 2}, \sigma_1]$ *with opt replaced by max or min are all NP-complete.*

To our knowledge, $max[\rho_{\geq 1}, \sigma_1]$ is the only one of these parameters that has been studied previously[13]. The theorem will follow from Lemmas 1-4.

Lemma 1 *The decision problems* $min[\rho_{\geq 1}, \sigma_1]$ *and* $max[\rho_{\geq 1}, \sigma_1]$ *are both NP-complete.*

Proof: Clearly, both problems are in NP. The problems $min[\rho_{\geq 1}, \sigma_0]$ (minimum independent dominating set) and $max[\rho_{\geq 1}, \sigma_0]$ (maximum independent dominating set) are known to be NP-complete. For any graph G we describe a graph G', such that G has a $[\rho_{\geq 1}, \sigma_0]$-set S if and only if G' has a $[\rho_{\geq 1}, \sigma_1]$-set S' such that $2|S| = |S'|$. Let $G'(V) = \{u_0, u_1, u_2, u_3 : u \in V\}$. Fix an orientation EO of $G(E)$ and let $G(E') = \{\{(u_0, u_1), (u_1, u_2), (u_2, u_3) : u \in G(V)\} \cup \{(u_0, v_1), (u_1, v_1), (u_1, v_2), (u_1, v_3), (u_2, v_0), (u_2, v_1), (u_2, v_2), (u_3, v_2) : \langle u, v \rangle \in EO\}\}$. See Figure 1.

For one direction of the proof, let S be $[\rho_{\geq 1}, \sigma_0]$ in G. Then $S' = \{u_1, u_2 : u \in S\}$ has twice the cardinality and is $[\rho_{\geq 1}, \sigma_1]$ in G'. This since $(u_1, u_2) \in G'(E)$ and if $v \in N(u)$ any v_i is adjacent to either u_1 or u_2.

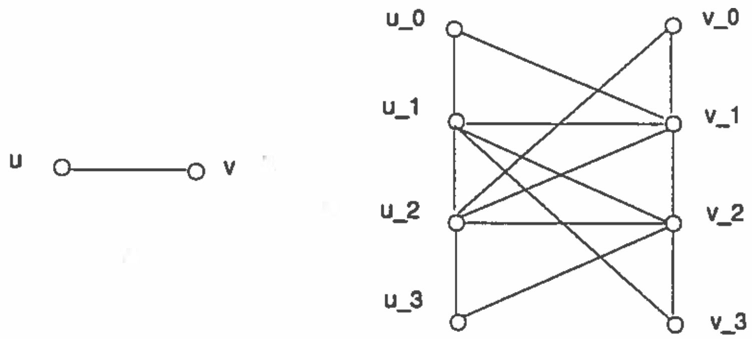

Figure 1: Given G on the left, the reduction for Lemma 1 constructs G' on the right

For the other direction of the proof, we first show that a $[\rho_{\geq 1}, \sigma_1]$-set S' in G' is of the form $S' = \{u_1, u_2 : u \in S\}$ for some $S \subseteq G(V)$. Suppose some $u_0, u_1 \in S'$. Then, regardless of the orientation EO, $N(\{u_0, u_1\}) \cap S' = \{u_0, u_1\}$, and since $u_3 \notin N(\{u_0, u_1\})$ and $N(u_3) - N(\{u_0, u_1\}) = \emptyset$, we must have $states_{S'}(u_3) \in \{\rho_0, \sigma_0\}$, but then S' is not $[\rho_{\geq 1}, \sigma_1]$. Similarly, $u_2, u_3 \in S' \Rightarrow states_{S'}(u_0) \in \{\rho_0, \sigma_0\}$. In the following, assume $\langle u, v \rangle \in EO$. Then $(u_1, v_1 \in S') \vee (u_1, v_3 \in S') \Rightarrow states_{S'}(u_3) \in \{\rho_0, \sigma_0\}$, and $(u_2, v_2 \in S') \vee (u_2, v_0 \in S') \Rightarrow states_{S'}(u_0) \in \{\rho_0, \sigma_0\}$, and $(u_1, v_2 \in S') \vee (u_3, v_2 \in S') \Rightarrow states_{S'}(v_0) \in \{\rho_0, \sigma_0\}$, and $(u_2, v_1 \in S') \vee (u_0, v_1 \in S') \Rightarrow states_{S'}(v_3) \in \{\rho_0, \sigma_0\}$. We have shown that if S' is a $[\rho_{\geq 1}, \sigma_1]$ set of G' then $S' = \{w_1, w_2 : w \in S\}$ for some $S \subseteq G(V)$. But then it is easy to see that S must be a $[\rho_{\geq 1}, \sigma_0]$-set in G and $2|S| = |S'|$.
The transformation is easily done in polynomial time, so we conclude that both $min[\rho_{\geq 1}, \sigma_1]$ and $max[\rho_{\geq 1}, \sigma_1]$ are NP-complete. □

Lemma 2 *The decision problems $min[\rho_{\geq 1}, \sigma_{\leq 1}]$ and $max[\rho_{\geq 1}, \sigma_{\leq 1}]$ are both NP-complete.*

Proof: Maximum problem by reduction from maximum independent set (proof omitted) and minimum problem follows from Corollary 3.

Lemma 3 *The decision problems $min[\rho_{\geq 2}, \sigma_0]$ and $max[\rho_{\geq 2}, \sigma_0]$ are both NP-complete.*

Proof: Clearly, both problems are in NP. As in Lemma 1 we reduce from $min[\rho_{\geq 1}, \sigma_0]$ and $max[\rho_{\geq 1}, \sigma_0]$ by, for a given graph G, constructing a graph G' such that G has a $[\rho_{\geq 1}, \sigma_0]$-set S if and only if G' has a $[\rho_{\geq 2}, \sigma_0]$-set S' such that $2|S| = |S'|$.
Let $G'(V) = \{u_0, u_1, u_2 : u \in G(V)\}$ and $G'(E) = \{\{(u_0, u_1), (u_1, u_2) : u \in V\} \cup \{(u_i, v_j) : 0 \leq i, j \leq 2 \wedge (u, v) \in G(E)\}\}$. See Figure 2.
For one direction of the proof, let S be $[\rho_{\geq 1}, \sigma_0]$ in G. Then $S' = \{u_0, u_2 : u \in S\}$ has twice the cardinality and is $[\rho_{\geq 2}, \sigma_0]$ in G'. This since $(u_0, u_2) \notin G'(E)$ so any vertex in S' has state 3_0 and a vertex $u_i \notin S'$ has $|N_{G'}(u_i) \cap S'| = |\{v_0, v_2 : v \in N_G(u) \cap S\}| \geq 2$.

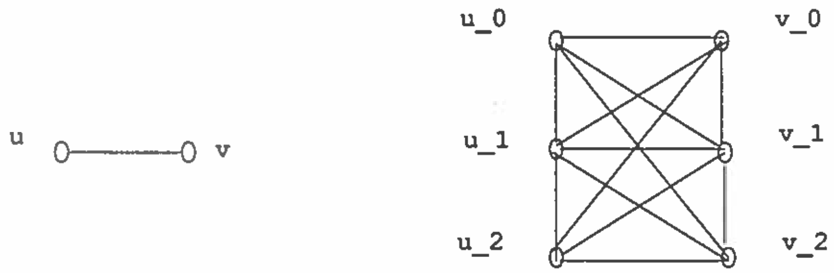

Figure 2: Given G on the left, the reduction for Lemma 3 constructs G' on the right

For the other direction of the proof, we first show that any $[\rho_{\geq 2}, \sigma_0]$-set S' in G' is of the form $S' = \{u_0, u_2 : u \in S\}$ for some $S \subseteq G(V)$. Suppose some $u_1 \in S'$. Then $states_{S'}(u_1) = 3_0$ so that $N(u_1) \cap S' = \emptyset$. But $N(u_0) - N(u_1) = u_1$ so that $N(u_0) \cap S' = u_1$. Hence $states_{S'}(u_0) \in \{1, 3_1\}$ so S' is not a $[\rho_{\geq 2}, \sigma_0]$-set. Now suppose $u_0 \in S'$. Again $N(u_0) \cap S' = \emptyset$, and since $N(u_2) - N(u_0) = \emptyset$ we must have $u_2 \in S'$. Similarly one shows $u_2 \in S' \Rightarrow u_0 \in S'$. So $S' = \{u_0, u_2 : u \in S\}$ for some $S \subseteq G(V)$ as claimed. But then it is easy to see that S must be a $[\rho_{\geq 1}, \sigma_0]$-set in G and $2|S| = |S'|$.
The transformation is easily done in polynomial time, so we conclude that both $min[\rho_{\geq 2}, \sigma_0]$ and $max[\rho_{\geq 2}, \sigma_0]$ are NP-complete. □

Lemma 4 *The decision problems $[\rho_{\geq 2}, \sigma_1]$ and $max[\rho_{\geq 2}, \sigma_1]$ are both NP-complete.*

Proof: Clearly, both problems are in NP. The problems $min[\rho_{\geq 2}, \sigma_0]$ and $max[\rho_{\geq 2}, \sigma_0]$ were shown NP-complete in Lemma 3. For any given graph G we describe a graph G' such that G has a $[\rho_{\geq 2}, \sigma_0]$-set S if and only if G' has a $[\rho_{\geq 2}, \sigma_1]$-set S' such that $2|S| = |S'|$. Let $G'(V) = \{u_0, u_1, u_2, u_3 : u \in G(V)\}$ and let $G'(E) = \{\{(u_0, u_1), (u_0, u_2), (u_1, u_2), (u_1,$ $u \in G(V)\} \cup \{(u_0, v_2), (u_1, v_1), (u_1, v_2), (u_2, v_3) : (u, v) = (v, u) \in G(E)\}\}$. See Figure 3.
For one direction of the proof, let S be $[\rho_{\geq 2}, \sigma_0]$ in G. Then $S' = \{u_1, u_2 : u \in S\}$ is $[\rho_{\geq 2}, \sigma_1]$ in G' and $2|S| = |S'|$. This since $(u_1, u_2) \in G'(E)$ and for $v \in N(u)$ any v_i is adjacent to either u_1 or u_2.
For the other direction of the proof, we first show that any $[\rho_{\geq 2}, \sigma_1]$-set S' in G' is of the form $S' = \{u_1, u_2 : u \in S\}$ for some $S \subseteq G(V)$. Suppose some $u_0, u_1 \in S'$. Then $N(\{u_0, u_1\}) \cap S' = \{u_0, u_1\}$ and since $u_3 \in N(u_1)$ shares all its neighbors with u_1 we have $states_{S'}(u_3) = \rho_1$ so that S' is not $[\rho_{\geq 2}, \sigma_1]$. Similarly, $(u_0, u_2 \in S') \vee (u_0, v_2 \in S') \Rightarrow states_{S'}(u_3) = \rho_1$, and $(u_3, u_2 \in S') \vee (u_3, u_1 \in S') \vee (u_3, v_2 \in S') \Rightarrow states_{S'}(u_0) = \rho_1$. Hence, if S' is $[\rho_{\geq 2}, \sigma_0]$ it cannot contain any vertex u_0 or u_3. In the following, assume $(u, v) \in G(E)$. As above, we have $u_1, v_1 \in S' \Rightarrow states_{S'}(u_0) = \rho_1$. Suppose $u_1, v_2 \in S'$. Then $N(\{u_1, v_2\}) \cap S' = \{u_1, v_2\}$ and since $N(u_2) - N(\{u_1, v_2\})$ contains

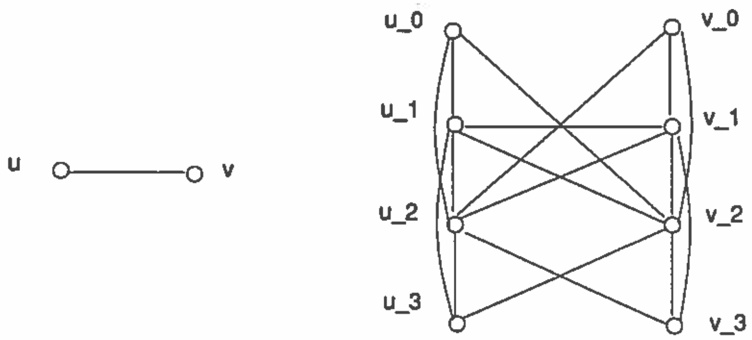

Figure 3: Given G on the left, the reduction for Lemma 4 constructs G' on the right

only vertices of type w_0 or w_3, neither of which can be in S' we get $state_{S'}(u_2) = \rho_1$ so that S' is not $[\rho_{\geq 2}, \sigma_1]$. Similarly, $u_2, v_1 \in S' \Rightarrow state_{G_{x'}}(v_2) = \rho_1$.
We have shown that if S' is a $[\rho_{\geq 2}, \sigma_1]$ set of G' then $S' = \{u_1, u_2 : u \in S\}$ for some $S \subseteq G(V)$. But then it is easy to see that S must be a $[\rho_{\geq 2}, \sigma_0]$-set in G and $2|S| = |S'|$.
The transformation is easily done in polynomial time, so we conclude that both $min[\rho_{\geq 2}, \sigma_1]$ and $max[\rho_{\geq 2}, \sigma_1]$ are NP-complete. □

Theorem 1 follows from Lemmas 1-4.

For certain subset properties, such as Perfect codes, it is well known that merely deciding if a graph has *any* such set is an NP-complete problem. We give a general theorem identifying a large class of such properties. The proof of this theorem involves a reduction from the NP-complete problem Exact 3-Cover (X3C) and is a generalization of a reduction used in [17].

Definition X3C
Instance: Set U and $T \subseteq \binom{U}{3}$.
Question: $\exists T' \subseteq T$, where T' a partition of U?

Theorem 3 *For any set of legal states L containing a finite positive number of both ρ-states and σ-states, and with $\rho_0 \notin L$, the problem of deciding if a graph has an $[L]$-set is NP-complete.*

Proof: Let $L = \{\rho_{p_1}, \rho_{p_2}, ..., \rho_{p_m}, \sigma_{q_1}, \sigma_{q_2}, ..., \sigma_{q_n}\}$, where $n, m \geq 1$ and p_i, q_i non-negative integers satisfying $0 < p_1 < p_2 < ... < p_m$ and $q_1 < q_2 < ... < q_n$.
We reduce the NP-complete problem X3C to the problem of deciding if a graph has an $[L]$-set. The theorem will follow since, in polynomial time, it is easy to verify that some $S \subseteq V(G)$ is an $[L]$-set and also easy to compute the reduction we are about to describe.

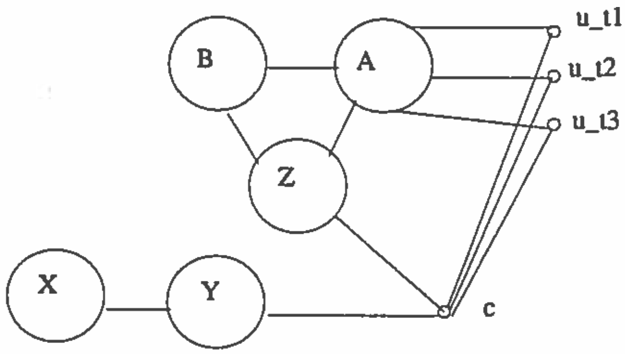

Figure 4: A rough sketch of the components of G_t where the absence of a line between two components reflects the absence of an edge in G_t connecting any two vertices from those two components.

Given an instance of X3C we want a graph G such that G has an $[L]$-set $S \subseteq V(G)$ if and only if $\exists T' \subseteq T$, a partition of U. For $t = \{u_{t1}, u_{t2}, u_{t3}\} \in T$ we will construct a graph G_t where $V(G_t) = W_t \dot\cup \{u_{t1}, u_{t2}, u_{t3}\}$ with the property:
Any $S_t \subseteq V(G_t)$ for which $states_S(w) \in L, \forall w \in W_t$ has either
(i) $states_S(u_{t1}) = states_S(u_{t2}) = states_S(u_{t3}) = \rho_0$ or
(ii) $states_S(u_{t1}) = states_S(u_{t2}) = states_S(u_{t3}) = \rho_{p_m}$.
Moreover, sets of type (i) and sets of type (ii) should exist for G_t.

Claim1: $G = \cup_{t \in T} G_t$ has $[L]$-set $S \subseteq V(G) \Leftrightarrow \exists T' \subseteq T$, a partition of U.
($\Leftarrow$:) Note the parts G_t of the graph G share only the vertices representing U. For each $t \in T'$ choose a set $S_t \subseteq V(G_t)$ of type (ii) for G_t. For each $t \notin T'$ choose a set $S_t \subseteq V(G_t)$ of type (i) for G_t. Let $S = \cup_{t \in T} S_t$.
($\Rightarrow$:) For any $[L]$-set S of G we must have $S \cap V(G_t)$ be either a set of type (i) or a set of type (ii) for G_t. This since only G_t contains the vertices W_t. Since $\rho_0 \notin L$, and since a vertex $u \notin S$ can have at most p_m neighbors in S, we must have that $T' = \{t : V(G_t) \cap S$ is a set of type (ii) for $G_t\}$ is a partition of U.

Construction of G_t: Let $V(G_t) = A \dot\cup B \dot\cup X \dot\cup Y \dot\cup Z \dot\cup \{c\} \dot\cup \{u_{t1}, u_{t2}, u_{t3}\}$. See Figure 4 for a rough sketch of how these components are connected together. As a preview, we mention that $\{A \cup Y\}$ will be a selected set of type (ii) and $\{B \cup Y\}$ a selected set of type (i) for G_t. X and Y will be such that a selected set cannot contain any vertex from X but must contain all vertices from Y. The vertex c will be connected to enough vertices of Y so that none of its other neighbors, namely $Z \cup \{u_{t1}, u_{t2}, u_{t3}\}$, can be selected. The vertices Z will ensure that either all or none of the neighbors of u_{tk} are selected.
Let $A = A^1 \dot\cup ... \dot\cup A^{p_m}$ and $B = B^1 \dot\cup ... \dot\cup B^{p_m}$ with $A^i = \{a^i_1, ..., a^i_{q_1+1}\}$ and $B^i = \{b^i_1, ..., b^i_{q_1+1}\}$, and let $G[A^i]$, $G[B^i], \forall i$ be complete graphs on $q_1 + 1$ vertices,

with no other edges between As or between Bs. Edges connecting vertices of A with vertices of B are restricted to $(a_k^i, b_k^j), \forall i, j, k$. Edges incident with $\{u_{t1}, u_{t2}, u_{t3}\}$ in G_t are restricted to $(a_1^i, u_{tk}), \forall i, k$.
Let $\beta = \max\{p_m, q_n\} > 0$ and $\alpha = \lceil \frac{\beta}{(p_1(q_n+1))} \rceil > 0$.
Let $Y = Y^1 \dot\cup ... \dot\cup Y^{p_1\alpha}$ and $G[Y^i], \forall i$, a complete graph on $q_n + 1$ vertices.
Let $X = \{x_1, x_2, ..., x_{(q_n+1)(\beta+1)\alpha}\}$ with $G[X]$ containing no edges.
We add edges connecting X-vertices with Y-vertices such that each vertex of X gets p_1 neighbors in Y and each vertex of Y gets $\beta + 1$ neighbors in X. This can be done since $|X| = \alpha(q_n + 1)(\beta + 1)$ and $|Y| = \alpha(q_n + 1)p_1$.
The vertex c is connected to p_m vertices of Y, note $|Y| \geq p_m > 0$, and c is also connected to every vertex of $Z \dot\cup \{u_{t1}, u_{t2}, u_{t3}\}$.
It remains to describe the vertices and edges contributed by Z. Let $Z = Z^1 \dot\cup Z^2 \dot\cup Z^3 \dot\cup \{z'\}$ with $Z^k = \{z_1^k, ..., z_{p_m}^k\}$ for $k \in \{1, 2, 3\}$.
The vertex $z_i^1, \forall i$, is connected to a_1^i and to b_1^i and also has $p_1 - 1$ neighbors in Y.
The vertex $z_i^2, \forall i$, is connected to a_1^i and to b_1^i and also has $p_m - 1$ neighbors in Y.
The vertex $z_i^3, \forall i$, is connected to a_1^i and to b_1^i and also has $p_1 - 1$ neighbors in Y.
The vertex z' is connected to $\{a_1^1, ..., a_1^{p_m}, b_1^1, ..., b_1^{p_m}\}$.
This completes the description of G_t.

Claim2: $A \cup Y$ is a set of type (ii) and $B \cup Y$ is a set of type (i) for G_t.
Proof of claim: We consider $A \cup Y$ first. $G[A \cup Y]$ is a collection of p_m copies of K_{q_1+1} for the As and $p_1\alpha$ copies of K_{q_n+1} for the Ys, so $state_{A\cup Y}(a) = \sigma_{q_1}, \forall a \in A$ and $state_{A\cup Y}(y) = \sigma_{q_m}, \forall y \in Y$. Moreover, $\forall x \in X$ we have $N(x) \subseteq Y$ and $|N(x)| = p_1$ so $state_{A\cup Y}(x) = \rho_{p_1}$. For the vertex c we have $N(c) \subseteq \{Y \cup Z \cup \{u_{t1}, u_{t2}, u_{t3}\}\}$ and $|N(c) \cap Y| = p_m$, so $state_{A\cup Y}(c) = \rho_{p_m}$. The vertices $z \in Z^1 \cup Z^3$ have $|N(z) \cap \{A \cup Y\}| = p_1$, so $state_{A\cup Y}(z) = \rho_{p_1}$. Similarly, $\forall z \in Z^2$ we have $|N(z) \cap \{A \cup Y\}| = p_m$, so $state_{A\cup Y}(z) = \rho_{p_m}$. The vertex z' has $N(z') \subseteq A \cup B$ and $|N(z) \cap A| = p_m$, so $state_{A\cup Y}(z') = \rho_{p_m}$.
So far, the argument for $B \cup Y$ being a set of type (i) can be obtained from the above by replacing B for A and vice-versa.
Since $\forall b \in B$, $N(b) \subseteq A \cup Z$ and $|N(b) \cap A| = p_m$, we have $state_{A\cup Y}(b) = \rho_{p_m}$. Similarly, $\forall a \in A$ we have $N(a) \subseteq B \cup Z \cup \{u_{t1}, u_{t2}, u_{t3}\}$ and $|N(a) \cap B| = p_m$, so $state_{B\cup Y}(a) = \rho_{p_m}$.
What remains is the argument for the vertices $\{u_{t1}, u_{t2}, u_{t3}\}$. We have for $k \in \{1, 2, 3\}$, $N(u_{tk}) = \{a_1^1, ..., a_1^{p_m}\}$, so $state_{A\cup Y}(u_{tk}) = \rho_{p_m}$ and $state_{B\cup Y}(u_{tk}) = \rho_{p_0}$ so that $A \cup Y$ is a set of type (ii) and $B \cup Y$ is a set of type (i), completing the proof of the claim.

Claim3: For any $S_t \subseteq V(G_t)$ with $states_{S_t}(w) \in L, \forall w \in V(G_t) - \{u_{t1}, u_{t2}, u_{t3}\}$ we have $Y \subseteq S_t$ and also $\{Z \cup \{u_{t1}, u_{t2}, u_{t3}\}\} \cap S_t = \emptyset$.
Proof of claim: $\forall y \in Y$ we have $|N(y) \cap X| > \beta$, so $\exists x \in N(y) : x \notin S_t$. But then $states_{S_t}(x) = \rho_{p_1}$ and $y \in S_t$. This means that $\forall y \in Y$ we have $states_{S_t}(y) = \sigma_{q_n}$ and

since $\exists y \in Y : c \in N(y) - Y$ we also have $c \notin S_t$. But then since $|N(c) \cap Y| = p_m$ we have $states_{S_t}(c) = \rho_{p_m}$ and $\{Z \cup \{u_{t1}, u_{t2}, u_{t3}\}\} \cap S_t = \{N(c) - Y\} \cap S_t = \emptyset$, completing the proof of the claim.

Claim4: For any $S_t \subseteq V(G_t)$ with $states_{S_t}(w) \in L, \forall w \in V(G_t) - \{u_{t1}, u_{t2}, u_{t3}\}$ we have either $a_1^i \in S_t, 1 \leq i \leq p_m$ or $a_1^i \notin S_t, 1 \leq i \leq p_m$.
Proof of claim: From Claim3 we have $Z \cap S_t = \emptyset$ and $Y \subseteq S_t$. In particular, $states_{S_t}(z_i^1) \in \{\rho_{p_1}, \rho_{p_1+1}\}$, similarly $states_{S_t}(z_i^2) \in \{\rho_{p_m-1}, \rho_{p_m}\}$ and $states_{S_t}(z_i^3) \in \{\rho_{p_1}, \rho_{p_1+1}\}$. In turn, we consider the two cases $a_1^1 \in S_t$ and $a_1^1 \notin S_t$.
$a_1^1 \in S_t$ gives $states_{S_t}(z_i^2) = \rho_{p_m}, \forall i$, so $b_1^i \notin S_t, \forall i$. This in turn gives $states_{S_t}(z_i^3) = \rho_{p_1}$ so $a_1^i \in S_t, \forall i$, completing the first case.
$a_1^1 \notin S_t$ gives $states_{S_t}(z_i^1) = \rho_{p_1}, \forall i$, so $b_1^i \in S_t, \forall i$. This in turn gives $states_{S_t}(z') = \rho_{p_m}$ so $a_1^i \notin S_t, \forall i$, completing the proof of the claim.

Each of $\{u_{t1}, u_{t2}, u_{t3}\}$ is adjacent to exactly $\{a_1^1, ..., a_1^{p_m}\}$ and by Claim3 cannot be in S_t. Hence, Claim4 actually shows that any $S_t \subseteq V(G_t)$ such that $states_{S_t}(w) \in L, \forall w \in V(G_t) - \{u_{t1}, u_{t2}, u_{t3}\}$ has either
(i) $states_{S_t}(u_{t1}) = states_{S_t}(u_{t2}) = states_{S_t}(u_{t3}) = \rho_0$
or (ii) $states_{S_t}(u_{t1}) = states_{S_t}(u_{t2}) = states_{S_t}(u_{t3}) = \rho_{p_m}$.
Thus G_t has the claimed properties and the theorem follows. □.

We have the following corollary:

Corollary 1 *Any decision problems of the form $max[L]$, $min[L]$, or $maxL[P]$ where $L \subseteq P$ contains a finite positive number of both ρ-states and σ-states, and with $\rho_0 \notin L$, is NP-complete.*

Proof: For a given graph G, we have $max[L]$ defined, $min[L]$ defined and $maxL[P](G) = |V(G)|$ if and only if G has an $[L]$-set, and the latter problem was just shown to be NP-complete. □

As our next theorem shows, some of these decision problems are NP-complete even for very restrictive classes of graphs. The reduction used is a simple special case of the one just given, and uses the NP-complete problem Planar 3-Dimensional Matching (P3DM). A similar reduction is used in [10] to show the NP-completeness of finding Perfect Codes ($[\rho_1, \sigma_0]$-sets) in planar graphs of maximum degree three.

Definition 3DM
Instance: Sets U_1, U_2, U_3 with $U = U_1 \cup U_2 \cup U_3$ and $T \subseteq U_1 \times U_2 \times U_3$.
Question: $\exists T' \subseteq T$, where T' a partition of U?

With an instance I of 3DM, we associate the bipartite graph G_I where $V(G_I) = U \cup T$ and $E(G_I) = \{(u, t) : u \in U \wedge u \in t \in T\}$. In [8] it is shown that the Planar 3DM problem, 3DM restricted to instances where G_I is planar, is still NP-complete.

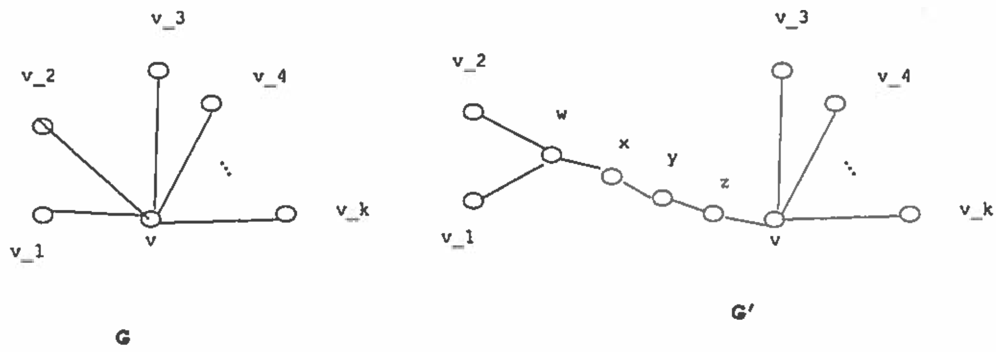

Figure 5: Transformation of G to G' used in Theorem

Theorem 4 *The problem of deciding if a planar bipartite graph of maximum degree three has any $[\rho_1, \sigma_1]$-set (Total Perfect Dominating Set) is NP-complete.*

Proof: Given an instance I of P3DM, we construct a graph G having a $[\rho_1, \sigma_1]$-set if and only if $\exists T' \subseteq T$, a partition of U. Let G be the graph G_I augmented by adding, for each $t \in T$, the vertices a_t and b_t, and edges connecting a_t to both t and b_t. Since this reduction does not distinguish between the sets U_1, U_2, U_3, the instance I can be viewed as an instance of X3C, and the argument that G has a $[\rho_1, \sigma_1]$-set if and only if $\exists T' \subseteq T$, a partition of U, is left out since it is in easy analogy with the argument used for the previous theorem.

Note that G_I and G are both planar bipartite graphs. We next show an easy transformation of a graph G having a vertex of degree larger than three to a graph G' with the following properties:

(i) if G planar and bipartite then G' planar and bipartite,

(ii) $\Sigma_{\{v:deg_G(v)\geq 4\}} deg_G(v) > \Sigma_{\{v:deg_{G'}(v)\geq 4\}} deg_{G'}(v)$

(iii) G has a $[\rho_1, \sigma_1]$-set if and only if G' has a $[\rho_1, \sigma_1]$-set.

Hence, applying such a polytime transformation repeatedly, starting with G, until the resulting graph has no vertices of degree larger than three, yields a graph proving the theorem.

We define the transformation by describing the resulting graph G'. Let v be a distinguished vertex of G with $N_G(v) = \{v_1, v_2, ..., v_k\}$ and $k \geq 4$. Let G' have vertices $V(G') = V(G) \dot\cup \{w, x, y, z\}$ and edges $E(G') = E(G) - \{(v_1, v), (v_2, v)\} \dot\cup \{(v_1, w), (v_2, w), (w, x), (x, y), (y, z), (z, v)\}$. See Figure 5. Note the transformation is local, with changes only to the neighborhoods of v_1, v_2 and v.

We prove the stated properties of the transformation:

(i) Planarity is obviously preserved. If A, B is an appropriate bipartition of $V(G)$ then w.l.o.g. we must have $v \in A$, $N(v) \subseteq B$ so that $A \cup \{w, y\}$ and $B \cup \{x, z\}$ forms an appropriate bipartition of $V(G')$. (ii) The new vertices all have degree less than

4, whereas the degree of v decreases to $k-1$. (iii) Let S and S' be $[\rho_1, \sigma_1]$-sets in G and G', respectively. Note that $\{w, x, y, z, v\}$ induces a 5-path in G' so there are 4 possibilities for $\{w, x, y, z, v\} \cap S'$, namely $\{y, z\}, \{w, z, v\}, \{w, x, v\}$ and $\{x, y\}$. We similarly split the possibilities for choice of S into 4 classes, namely
$|\{v_1, v_2\} \cap S| = 1 \wedge v \notin S \wedge |\{v_3, ..., v_k\} \cap S| = 0$,
$|\{v_1, v_2\} \cap S| = 1 \wedge v \in S \wedge |\{v_3, ..., v_k\} \cap S| = 0$,
$|\{v_1, v_2\} \cap S| = 0 \wedge v \in S \wedge |\{v_3, ..., v_k\} \cap S| = 1$,
$|\{v_1, v_2\} \cap S| = 0 \wedge v \notin S \wedge |\{v_3, ..., v_k\} \cap S| = 1$.
It is easy to check that the 4 possibilities for choice of S' have, in the order given, characterizations in terms of effect on v and $N(v)$ which are identical to those just given for S, and indeed property (iii) holds. □

To our knowledge, the complexity of this problem, finding Total Perfect Dominating Sets in graphs, was previously not known [5]. We have the following corollary.

Corollary 2 *The decision problems $min[\rho_1, \sigma_1]$, $max[\rho_1, \sigma_1]$ and $max\{\rho_1, \sigma_1\}[L]$ with $\{\rho_1, \sigma_1\} \subseteq L$ are NP-complete even when restricted to planar bipartite graphs of maximum degree three.*

A problem encompassed by this corollary is $max\{\rho_1, \sigma_1\}[\rho_{\geq 0}, \sigma_{\geq 0}]$, which we call Total Efficiency. This problem arises in communication networks, if we assume that a communication round has two time-disjoint phases, sends and receives, and that a processor receives a message whenever it has a single sending neighbor. The maximum number of processing elements that can receive a message in one communication round is the Total Efficiency of the graph underlying the network topology.

A result of Kratochvíl [17] shows the NP-completeness of deciding if a planar 3-regular graph has a $[\rho_1, \sigma_0]$-set (perfect code). This is a strong result with the following implications for problems admitting our characterization.

Corollary 3 *Any decision problem of the form designated by a) or b) below is NP-complete on planar 3-regular graphs.*
a) $max\{\rho_1, \sigma_0\}[L]$ with $\{\rho_1, \sigma_0\} \subseteq L$
b) $min[L]$ with $\rho_0 \notin L$ and $\{\rho_1, \sigma_0\} \subseteq L$

Proof: Let G be a planar 3-regular graph. If a problem satisfies case a) then clearly G has a $[\rho_1, \sigma_0]$-set if and only if the corresponding parameter of G has the value $|V|$. If a problem B satisfies case b) we show that G has a perfect code if and only if the corresponding parameter of G has the value $|G(V)|/4$. Since every vertex of G has degree 3, a perfect code of G has size $|G(V)|/4$ and is clearly a dominating set. Moreover, a dominating set of G which is not a perfect code will have more than $|G(V)|/4$ vertices. A set achieving the minimum value of G asked for by problem B is a dominating set since ρ_0 is not legal and it could be a perfect code since ρ_1 and σ_0 are legal. The corollary follows. □

An example of a problem encompassed by this corollary is $min[\rho_{\geq 1}, \sigma_0, \sigma_{\geq 2}]$, the problem of finding a minimum size dominating set inducing a subgraph with no vertices of degree one.
We are currently trying to resolve the complexity status of the maximization version of this problem, equivalent to finding a maximum induced subgraph with no vertices of degree one, $max[\rho_{\geq 0}, \sigma_0, \sigma_{\geq 2}]$.

We now turn to problems with an easy solution algorithm.

Theorem 5 *The problem $max[L]$ is solvable by a greedy algorithm if $\sigma_{\geq k}$ is the only σ-state in L and either (i), (ii), (iii), (iv) or (v) holds*
(i) $\{\rho_0, \rho_1, ..., \rho_{k-1}\} \subseteq L$
(ii) $\{\rho_0, \rho_1, ..., \rho_{k-1}\} \cap L = \emptyset$
(iii) $\rho_0 \notin L$ and $\{\rho_1, \rho_2, ..., \rho_{k-1}\} \subseteq L$
(iv) $\rho_{\geq h}$ is the only ρ-state in L, for some h
(v) ρ_0 and $\rho_{\geq h}$ are the only ρ-states in L, for some h

Proof: For each of the five cases we describe a greedy algorithm which takes a graph G as input and gives $max[L](G)$ as output. The algorithms use data structures $B\sigma$,$B\rho$ of type *set*.

Algorithm-i(G)
$B\sigma$, $B\rho := V(G)$, $\emptyset$;
while ($\exists v \in B\sigma : |N(v) \cap B\sigma| < k$) do $B\sigma$, $B\rho := B\sigma \setminus \{v\}$, $B\rho \cup \{v\}$;
output($B\sigma$);

Algorithm-ii(G),Algorithm-iii(G),Algorithm-iv(G)
Algorithm-i with the output-statement replaced by
if ($\exists v \in B\rho : state_{B\sigma}(v) \notin L$) then output($\nexists[L]$-set) else output($B\sigma$);

Algorithm-v(G)
$B\sigma$, $B\rho := V(G)$, $\emptyset$;
while (I: $\exists v \in B\sigma : |N(v) \cap B\sigma| < k$) or (II: $\exists w \in B\rho : |N(w) \cap B\sigma| < h$) do
  Case I : $B\sigma$, $B\rho := B\sigma \setminus \{v\}$, $B\rho \cup \{v\}$;
  Case II: $B\sigma$, $B\rho := B\sigma \setminus \{N(w) \cap B\sigma\}$, $B\rho \cup \{N(w) \cap B\sigma\} \setminus \{w\}$;
output($B\sigma$);

Claim: For any of the above algorithms, we have the loop invariant: "A vertex $v \notin B\sigma$ cannot be a member of any $[L]$-set of G."

Proof of claim: The loop invariant is true initially since $B\sigma = V(G)$. Let $B\sigma$ and $B\sigma'$ be the values before and after an execution of the loop and let S be an $[L]$-set of G. From the loop invariant we have $S \subseteq B\sigma$ and show that $S \subseteq B\sigma'$
Case I (all five algorithms): $B\sigma \setminus B\sigma' = \{v\}$ and $v \in B\sigma : |N(v) \cap B\sigma| < k$. Since $\sigma_{\geq k}$ is the only σ-state in L, $S \subseteq B\sigma$ cannot contain v.
Case II (Algorithm-v only): $v \in B\sigma \setminus B\sigma'$ and $\exists w : v \in N(w)$ where $w \in B\rho$ and $|N(w) \cap B\sigma| < h$. When a vertex is added to $B\rho$ it is also removed from the non-growing set $B\sigma$ so that $B\rho \cap B\sigma = \emptyset$ and in particular $w \notin S$. Since ρ_0 and $\rho_{\geq h}$ are the only ρ-states in L for Algorithm-v, $states_S(w) = \rho_0$ so that $N(w) \cap S = \emptyset$. Since $v \in N(w)$ this completes the proof of the claim.

At termination of Algorithm-i, Algorithm-ii, Algorithm-iii and Algorithm-iv all vertices in $B\sigma = S$ have at least k neighbors in S and all vertices not in S (in $B\rho$) have less than k neighbors in S, since $B\sigma$ never grew during execution. Hence, when $\{\rho_0, \rho_1, ..., \rho_{k-1}\} \subseteq L$ we have S an $[L]$-set, indeed a maximum-size $[L]$-set by the claim, so Algorithm-i is correct. When $\rho_0 \notin L$ and $\{\rho_1, \rho_2, ..., \rho_{k-1}\} \subseteq L$ then if $\exists v : states_S(v) = \rho_0$ there cannot be any $[L]$-set in G but if such a vertex does not exist S must be a maximum-size $[L]$-set, so Algorithm-ii is correct. Similarly, Algorithm-iii and Algorithm-iv are easily seen to be correct.
At termination of Algorithm-v all vertices in $B\sigma = S$ have at least k neighbors in S and all vertices not in S have either at least h neighbors in S (these vertices are in $B\rho$) or no neighbors in S. Since ρ_0 and $\rho_{\geq h}$ are both in L for Algorithm-v we have S an $[L]$-set, indeed a maximum-size $[L]$-set by the claim, so Algorithm-v is correct. □

4 A graph-theoretic result

As another example of application of our characterization, we consider a generalization of perfect codes ($[\rho_1, \sigma_0]$-sets) and extend a result that holds for perfect codes to this generalization.

Lemma 5 *For $p \geq 1, q \geq 0$ if both A and B are $[\rho_p, \sigma_q]$-sets of a graph G then $|A| = |B|$.*

Proof: Let $X_I = A \cap B$, $X_A = A - X_I$ and $X_B = B - X_I$. Since A, B are $[\rho_p, \sigma_q]$-sets, both $G[A]$ and $G[B]$ are q-regular, so we have $\forall v \in X_I : |N(v) \cap X_A| = |N(v) \cap X_B|$. Let $\sum_{v \in X_I} |N(v) \cap X_A| = r = \sum_{v \in X_I} |N(v) \cap X_B|$. Consider the subgraph $F = (A \cup B, \{(u,v) \in E : (u \in X_A \wedge v \in B) \vee (u \in A \wedge v \in X_B)\})$. Since A, B are $[\rho_p, \sigma_q]$-sets, $\forall v \in X_A \cup X_B : deg_F(v) = p$. Let $H = F[X_A \cup X_B]$. Then H is bipartite, so that $\sum_{v \in X_A} deg_H(v) = \sum_{v \in X_B} deg_H(v)$. We then have

$$p|X_A| = \sum_{v \in X_A} deg_F(v) = \sum_{v \in X_A} deg_H(v) + r = \sum_{v \in X_B} deg_H(v) + r = \sum_{v \in X_B} deg_F(v) = p|X_B|$$

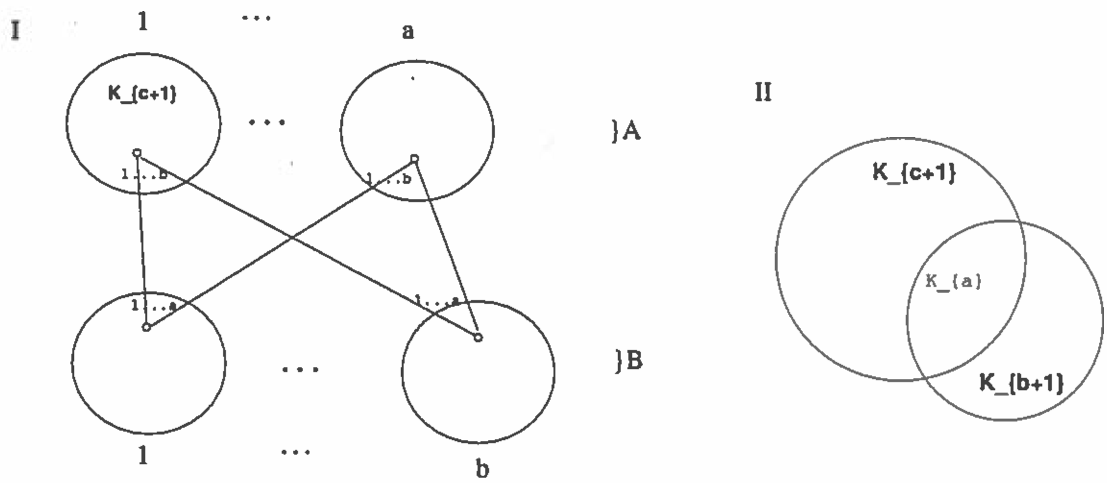

Figure 6: I) A graph having $[\rho_a, \rho_b, \sigma_c]$-sets A and B. II) A graph having two $[\rho_a, \sigma_b, \sigma_c]$-sets of size $b+1$ and $c+1$ ($a \leq b+1$)

and since $p > 0$, $|A| = |B|$. □

Theorem 6 *For a set of vertex states L, the statement "For any graph G, all $[L]$-sets have the same size" is true if and only if (i) or (ii) holds*
(i) $L = \{\rho_p, \sigma_q\}$ for some $p \geq 1, q \geq 0$
(ii) L has either no ρ-states or no σ-states

Proof: If L has no ρ-states then the only possible $[L]$-set is $S = V(G)$ and if L has no σ-states then the only possible $[L]$-set is $S = \emptyset$. One direction of the proof then follows from Lemma 5. For the other direction of the proof, consider sets L not of type (i) or (ii), and construct graphs with two $[L]$-sets of different sizes. First note that if $\rho_0 \in L$ then $S = \emptyset$ is an $[L]$-set and it is easy to construct a graph with some larger $[L]$-set. The remaining cases are covered by two arguments, depending on whether there is more than one legal state for selected vertices, or more than one legal state for non-selected vertices. In both cases, we construct a graph G with appropriate sets A and B (each set inducing a collection of complete graphs) of different sizes.
Case 1: Suppose $\{\rho_a, \rho_b, \sigma_c\} \subseteq L$ where $a < b$. Then let $G = (A \cup B, E)$ where A induces a copies of K_{c+1} and B induces b copies of K_{c+1}, clearly both c-regular. The remaining edges form a perfect matching between each pair of K_{c+1}'s, one from each of A and B. See Figure 6-I. Thus a vertex in A has b neighbors in B and a vertex in B has a neighbors in A. $|A| = a(c+1) < b(c+1) = |B|$ since $a < b$.
Case 2: Suppose $\{\rho_a, \sigma_b, \sigma_c\} \subseteq L$ where $b < c$. If $a \leq b+1$ let $G = (A \cup B, E)$ such that A and B induce K_{b+1} and K_{c+1}, respectively, and $A \cap B$ induces K_a, this accounting for all the edges. See Figure 6-II.

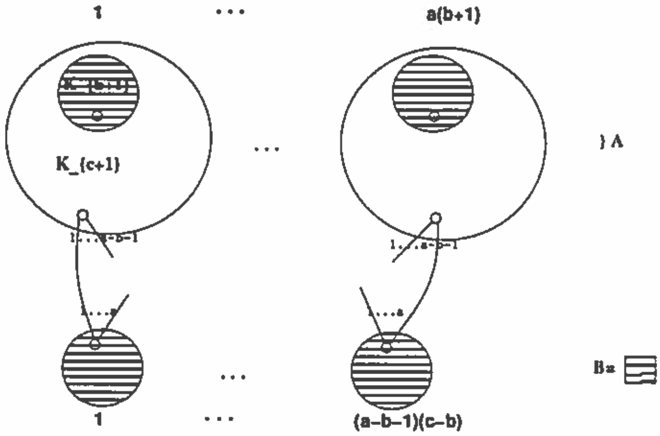

Figure 7: A graph having $[\rho_a, \sigma_b, \sigma_c]$-sets A and B $(a > b+1)$

If $a > b+1$ we use the graph depicted in Figure 7. As before, A induces the K_{c+1}s and B induces the K_{b+1}s (shaded in the figure). The remaining edges are between $A - B$ and $B - A$ and can be added in any way such that each vertex of $A - B$ gets $a - b - 1$ additional edges and each vertex of $B - A$ gets a additional edges. Thus, the bipartite graph between $A - B$ and $B - A$ must have $(c + 1 - (b + 1))(a - b - 1)a(b + 1) = (b+1)a(a - b - 1)(c - b)$ edges, counting from $A - B$ or $B - A$ respectively, and since $a - b - 1 < a$ we have $|A| > |B|$. □

5 A refined characterization

We give a refinement of the characterization useful for describing maximal and minimal vertex subsets with a given property.

Definition 3 Given a set L of vertex states and a graph G

- $S \subseteq V(G)$ is a maximal (minimal) $[L]$-set if there is no vertex $v \in V(G) - S$ $(v \in S)$ such that $S \cup \{v\}$ $(S - \{v\})$ is an $[L]$-set.

Parameters related to irredundant sets in graphs will also be expressible using the refinement. Irredundant sets require some vertices to have at least one neighbor with a given state, motivating the definition of a *refined* vertex state as the juxtaposition (denoted by $\cdot$) of the state of the vertex with the state of one of its neighbors.

Definition 4 Given a graph G and a selected set of vertices $S \subseteq V(G)$

- The set of *refined* vertex states of $v \in V(G)$ is

  $rstates_S(v) = \{states_S(v)\} \cup \{states_S(v) \cdot states_S(w) : w \in N(v)\}$

Our notation	Standard terminology
$[\rho_{\geq 0}, \sigma_0, \sigma_{\geq 1} \cdot \rho_1]$-set	Irredundant set (closed-closed)
$[\rho_{\geq 0}, \sigma_0, \sigma_{\geq 1} \cdot \rho_1, \sigma_{\geq 1} \cdot \sigma_1]$-set	closed-open Irredundant set
$[\rho_{\geq 0}, \sigma_{\geq 0} \cdot \rho_1, \sigma_{\geq 0} \cdot \sigma_1]$-set	open-open Irredundant set
$[\rho_{\geq 0}, \sigma_{\geq 0} \cdot \rho_1]$-set	open-closed Irredundant set
$[\rho_{\geq 0}, \sigma_{\leq k-1}, \sigma_{\geq k} \cdot \rho_k]$-set	k-Irredundant set
$[\rho_{\geq 1}, \sigma_0, \sigma_{\geq 1} \cdot \rho_1]$-set	Minimal Dominating set
$[\rho_{\leq 1} \cdot \rho_1, \sigma_{\geq 0}]$-set	Maximal Nearly Perfect set
$max\{\sigma_{\geq 0}\}[\rho_{\geq 0}, \sigma_0, \sigma_{\geq 1} \cdot \rho_1]$	Upper Irredundance parameter
$max\{\sigma_{\geq 0}\}[\rho_{\geq 1}, \sigma_0, \sigma_{\geq 1} \cdot \rho_1]$	Upper Dominating parameter

Table 2: Some vertex subset properties and graph parameters defined using refined states

- For sets R, M of refined states, S is an $[R]$-set if $\forall v \in V(G) : rstates_S(v) \cap R \neq \emptyset$ and $minM[R]$ (or $maxM[R]$) is the parameter minimizing (or maximizing) $|\{v : rstates_S(v) \cap M \neq \emptyset\}|$ over all $[R]$-sets S

Abbreviations like $\sigma_{\geq 1} \cdot \rho_1$ stand for $\sigma_1 \cdot \rho_1, \sigma_2 \cdot \rho_1, \ldots$, in analogy with earlier definitions. For example, irredundant sets have legal refined states $R = \{\rho_{\geq 0}, \sigma_0, \sigma_{\geq 1} \cdot \rho_1\}$, meaning that $states_S(v) \in \{\sigma_1, \sigma_2, \ldots\} \Rightarrow \exists w \in N_G(v) : states_S(w) = \rho_1$ (a selected vertex having at least one selected neighbor must also have a private non-selected neighbor.) Table 2 gives examples of vertex subset properties and graph parameters [7, 9, 11, 16] admitting a characterization using refined states. The discriminating term closed-closed for irredundant sets arises from the definition of an irredundant set S as one for which $\forall v \in S$ the union of the closed neighborhoods of vertices in $S - \{v\}$ is strictly smaller than the union of the closed neighborhoods of vertices in S.

Given a set of (non-refined) vertex states L we give a general procedure constructing sets of refined vertex states $Lmax$ and $Lmin$ such that the $[Lmax]$-sets are exactly the maximal $[L]$-sets and the $[Lmin]$-sets are exactly the minimal $[L]$-sets. Define

$Amax = \{\rho_i : \rho_i \in L \wedge \sigma_i \notin L\}$
$Amin = \{\sigma_i : \sigma_i \in L \wedge \rho_i \notin L\}$
$Bmax = \{\rho_i : \rho_i \in L \wedge \rho_{i+1} \notin L\} \cup \{\sigma_i : \sigma_i \in L \wedge \sigma_{i+1} \notin L\}$
$Bmin = \{\rho_i : \rho_i \in L \wedge \rho_{i-1} \notin L\} \cup \{\sigma_i : \sigma_i \in L \wedge \sigma_{i-1} \notin L\}$

A subset of vertices S is a maximal (similarly, minimal) $[L]$-set if and only if S is an $[L]$-set and $\forall v \in V(G) - S$ ($\forall v \in S$) either $states_S(v) \in Amax$ ($\in Amin$) or $\exists u \in N_G(v) : states_S(u) \in Bmax$ ($\in Bmin$). Let $L\rho$ and $L\sigma$ be the sets of ρ-states and σ-states in L, respectively, so that $L = L\rho \dot\cup L\sigma$. We then have

$Lmax \stackrel{df}{=} Amax \cup L\sigma \cup \{a \cdot b : a \in L\rho - Amax \wedge b \in Bmax\}$

$Lmin \stackrel{df}{=} Amin \cup L\rho \cup \{a \cdot b : a \in L\sigma - Amin \wedge b \in Bmin\}$

and it is easy to see that maximal $[L]$-sets are exactly the $[Lmax]$-sets and that minimal $[L]$-sets are exactly the $[Lmin]$-sets. Table 2 shows the resulting characterizations for minimal dominating sets and maximal nearly perfect sets. Note that maximal $[L]$-sets (similarly, minimal $[L]$-sets) are exactly the $[L]$-sets themselves if $Amax$ ($Amin$) contains every ρ-state (every σ-state) in L or if $Bmax = L$ ($Bmin = L$) and the graph G has no isolated vertices.

Acknowledgements: Thanks to my advisor Andrzej Proskurowski for many ideas.

References

[1] D.W.Bange, A.E.Barkauskas and P.J.Slater, Efficient dominating sets in graphs, in: Ringeisen and Roberts, eds., *Applications of Discrete Mathematics*, SIAM, (1988).

[2] P.J.Bernhard, S.T.Hedetniemi and D.P.Jacobs, Efficient sets in graphs, to appear *Discrete Applied Mathematics*.

[3] N.Biggs, Perfect codes in graphs, *J.Comb. Theory, Series B 15(1973)*

[4] J.A.Bondy and U.S.R.Murty, *Graph theory with applications*, 1976.

[5] E.J.Cockayne, B.L.Hartnell, S.T.Hedetniemi and R.Laskar, Perfect domination in graphs, manuscript (1992), to appear in Special issue of JCISS.

[6] E.J.Cockayne and S.T.Hedetniemi, Towards a theory of domination in graphs, *Networks*, 7 (1977), 211-219.

[7] E.J.Cockayne, S.T.Hedetniemi and D.J.Miller, Properties of hereditary hypergraphs and middle graphs, *Can.Math.Bull.* 21.4 pp.461-468 (1978)

[8] M.E.Dyer and A.M.Frieze, Planar 3DM is NP-complete, *Journal of Algorithms*, vol.7, pp.174-184, (1983)

[9] A.Farley and A.Proskurowski, Computing the maximum order of an open irredundant set in a tree, *Congr.Numer.* 41 (1984) 219-228.

[10] M.Fellows and M.Hoover, Perfect domination, *Australian J. Combinatorics* 3 (1991), 141-150.

[11] J.F.Finck and M.S.Jacobson, On n-domination, n-dependence and forbidden subgraphs, in *Graph Theory with Applications to Algorithms and Computer Science*, Wiley (1984) pp. 301-312.

[12] M.Garey and D.Johnson, *Computers and Intractability*, Freeman, San Fransisco, 1979.

[13] S.T.Hedetniemi, personal communication.

[14] S.T.Hedetniemi and R.Laskar, Recent results and open problems in domination theory, *Applications of Discrete Mathematics* (SIAM, 1988).

[15] S.T.Hedetniemi and R.Laskar, Bibliography on domination in graphs and some basic definitions of domination parameters, *Discrete Mathematics* 86 (1990).

[16] M.S.Jacobson, K.Peters and D.R.Fall, On n-irredundance and n-domination, *Ars Combinatoria* 29B (1990) pp. 151-160.

[17] J.Kratochvíl, Perfect codes in general graphs, monograph, Academia Praha (1991).

[18] J.A.Telle, Characterization of domination-type parameters in graphs, to appear in *Proceedings 24th SouthEastern Conference on Combinatorics Graph Theory and Computing, Congressus Numerantium* (1993)

[19] J.A.Telle and A.Proskurowski, Practical algorithms on partial k-trees with an application to domination-type problems, University of Oregon Technical Report CIS-TR-93-04, to appear in *Proceedings WADS'93, LNCS (Workshop on Algorithms and Data Structures).*

[20] J.A.Telle and A.Proskurowski, Efficient sets in partial k-trees, *Discrete Applied Mathematics 40* (1993)