Plane embeddings of 2-trees and biconnected partial 2-trees

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Abstract

We consider different plane embeddings of partial 2-trees, count them and give an efficient algorithm constructing a minimum cardinality cover of faces by vertices. These tasks are facilitated by a unique tree representation of plane embeddings of 2-trees.

1 Introduction

1.1 Motivation

Partial 2-trees constitute a nontrivial class of planar graphs that includes outerplanar graphs and series-parallel graphs. They admit efficient algorithms solving many inherently hard problems on general graphs ([7,1]). This property of algorithmic tractability follows from the tree-like structure of partial 2-trees and 2-trees, which are graphs imbedding partial 2-trees. We propose a tree representation of 2-trees that is very useful in algorithmic treatment of these graphs.

The central algorithmic problem considered here is connected with plane embeddings of 2-trees and partial 2-trees. We first consider the problem of enumerating plane embeddings of partial 2-trees. (The problem for outerplanar graphs has been solved in [10].) Solution of the problem is facilitated by the fact that the union of minimal separators of any 2-tree has a very distinct structure. This fact implies a one-to-one correspondence between certain subgraphs of partial 2-trees (outerplanar subgraphs pivotal for plane embedding) and the corresponding subgraphs of the imbedding 2-trees. This intermediate result complements the study of interior graphs of maximal outerplanar graphs in [4].

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We then consider restricted covers of faces of a plane graph by vertices. This notion has been introduced in [8] and investigated in [9]. In the case of maximal outerplanar graphs, a tree representation of a plane embedding was used both to count the number of different embeddings (see [10]) and to construct an embedding admitting a perfect vertex cover of all graph's faces. The proposed tree representation of 2-trees is crucial for constructing perfect *FIVC* for 2-trees and partial 2-trees.

1.2 Definitions

We will deal with simple, loopless combinatorial graphs. An edge is *incident* with its *end vertices* which are mutually *adjacent*. A *simple path* between two vertices u and v is a sequence of edges such that each of their end vertices (other than u or v) is incident with exactly two neighboring edges. If $u = v$, we have a *simple cycle*. A graph is *connected* if there is a path between any two of its vertices. In a connected graph, a subset S of vertices is a *separator* if its removal disconnects the graph. A tree is a connected acyclic graph. A graph G is *outerplanar* if there is an embedding of G in the plane such that all vertices lie on the boundary of the infinite region of the plane (the outer face). Thus, an outerplanar graph has no subgraph homeomorphic to (a subdivision of) the complete bipartite graph $K_{2,3}$. The set of *boundary cycles* consists of the boundaries of the faces, regions of the plane in a plane embedding. We identify a plane embedding of a graph with the set of its boundary cycles. A subgraph of G *induced* by a subset S of the vertices of G consists of S and all edges of G with both end vertices in S.

A *2-tree* is either the complete graph on 3 vertices (the *triangle* K_3) or a graph with $n > 3$ vertices obtained from a 2-tree G on n–1 vertices by adding a new vertex adjacent exactly to both end vertices of an edge of G. (An alternative definition involves construction of a 2-tree by a sum of two smaller 2-trees that have an edge in common.) Every minimal separator of a 2-tree consists of the end vertices of an edge [6].

A *partial 2-tree* is a subgraph of a 2-tree (it can be imbedded in a 2-tree) with the same set of vertices. The class of partial 2-trees is identical with a slight generalization of series-parallel graphs, graphs with *treewidth* at most 2. It is well known that a graph is a partial 2-tree if and only if it contains no homeomorph of K_4. For emphasis, we will call 2-trees *full*. Also, we will distinguish between an *embedding* of a planar graph in the plane and an *imbedding* of a partial 2-tree in a full 2-tree.

We will use the following classification of edges in a full or partial 2-tree H. An edge $e = (a, b)$ is called *exterior* if $\{a, b\}$ is not a separator of H, otherwise it is called *interior*. An edge $e = (a, b)$ is called *strongly interior* if the graph $H - \{a, b\}$ has more than two connected components and *weakly interior* otherwise. A strongly interior edge $e = (a, b)$ is *terminal* if and only if at most one of the graphs $G_i = H - \bigoplus_{j \neq i} C_j$ is not outerplanar (here, C_i's are the connected components of $H - \{a, b\}$.)

Lemma 1 *Every non-outerplanar 2-tree H has a terminal strongly interior edge.*

Proof: If a 2-tree H is not outerplanar, it has a strongly interior edge since it contains a homeomorph of $K_{2,3}$. Assume that there is no terminal such edge. Then, there is a strongly interior edge e that separates H into at least two non-outerplanar components: C, that has the maximum size over all strongly interior edges and the corresponding components, and D. The latter has a strongly interior edge f separating H into connected components, one of which is non-outerplanar and properly includes C, thus contradicting the definition of C as maximum size. (This argument should give an intuition about the name of the terminal strongly interior edge.) ∎

Outerplanar 2-trees are known as *maximal outerplanar* graphs (*mops*), which are also identical with all triangulations of polygons.

2 Structure of 2-trees and their separators

2.1 Interior graphs of 2-trees

Hedetniemi *et al.* [4] define the interior graph of a maximal outerplanar graph (*mop*, for short) as the union of its interior edges. They completely characterize the interior graphs of mops and show that such a graph is a connected union of mops and caterpillars. We obtain a similar result for partial 2-trees.

Lemma 2 *Any tree is the interior graph of some 2-tree.*

Proof: (by induction on the number of vertices.) By inspection, the lemma is true for $n = 2$ and $n = 3$ vertices. For $n \geq 3$, consider a tree T with $n + 1$ vertices. Unless $T = K_{1,n}$ ("a star") we can split T into smaller trees T_1 and T_2 by removing an edge e, so that $|T_i| + 1 \leq n$ $(i = 1, 2)$. By the inductive hypothesis, each of the trees T_i augmented by e is the interior graph of a 2-tree G_i $(i = 1, 2)$. A 2-tree G obtained from G_1 and G_2 by identifying the copies of e in each of them has T as its interior graph. As for the remaining case, $K_{1,n}$ is the interior graph of the 2-tree with a universal vertex and n remaining vertices inducing a path ("a wheel without one external edge"). ∎

Theorem 1 *A connected partial 2-tree H is the interior graph of some 2-tree if and only if it has no induced cycles of length greater than 3.*

Proof: (sufficiency) Any such H has biconnected components, H_i, that are either edges or 2-trees. A 2-tree component H_i is the interior graph of a 2-tree G_i obtained from H_i by adding a triangle (a vertex adjacent to both end vertices of an edge) to each exterior edge. An edge component H_i is the interior graph of a 2-tree G_i obtained by adding two triangles to H_i. Consider the union of such graphs G_i (identifying the corresponding copies of articulation vertices of H). For each articulation point v of G, choose from each connected component of $G-v$ an edge e of H incident to v and connect the other end vertices of these edges by a path. This results in a 2-tree that has H as its interior graph.

(necessity) Let a partial 2-tree H be the interior graph of a 2-tree G. As a chordal graph, G has no induced cordless cycles other than triangles. Removing all vertices of degree 2 from G does not introduce any induced cycles. Since those vertices are incident to all exterior edges of G, their removal results in the interior graph, H, also without induced cycles of length greater than 3. ∎

The necessity result of [4] that the interior graph of a mop is a connected union of mops and caterpillars is an immediate corollary of Theorem 1 (since caterpillars are the only acyclic interior graphs of mops). However, the outerplanarity constitutes a nontrivial hinder for the sufficiency of this condition.

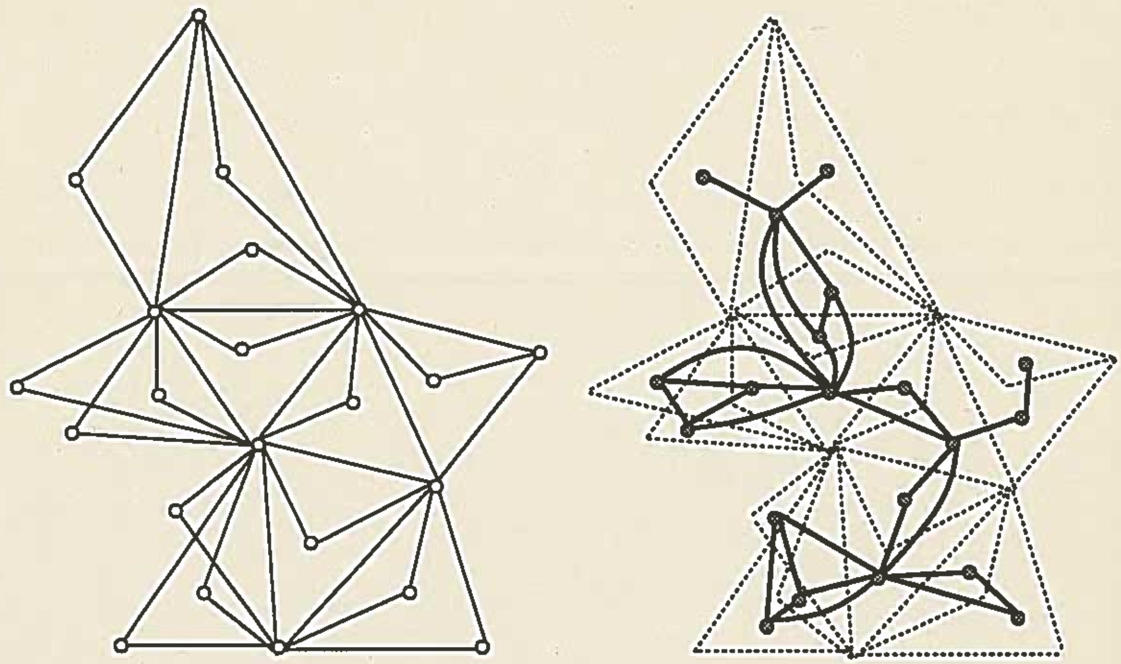
2.2 Tree representation of 2-trees

We will represent a plane embedding of a 2-tree by a rooted, ordered tree with sibling nodes (children of the same parent) partitioned into two sets. Towards this goal, we first define a unique *associated graph* $D(G)$ of a given 2-tree G. $D(G)$ is the intersection graph of triangles of G over the set of edges. Thus, *nodes* of $D(G)$ correspond to triangles of G, and *branches* of $D(G)$ correspond to edges of G that are in at least two triangles (an example can be found in Fig. 1b).

It can be easily verified that each node v of $D(G)$ is in at most three maximal cliques. Furthermore, there is at least one node in $D(G)$ that belongs to exactly one maximal clique. We will call such a node (and the corresponding triangle of G) *pendant*.

We will now give an algorithm constructing a graph that represents a plane embedding G_p of G (G_p is assumed to be specified by the set of boundary cycles). Let r be a pendant node in $D(G)$. Let T_r denote a directed tree rooted at r and obtained by the breadth-first traversal of $D(G)$ (Fig. 3b). Consider a node $v \in T_r$. Let T_v denote the maximal subtree of T_r rooted at v. Each node in T_v corresponds to a triangle in G. Let $G(v)$ denote a subgraph of G consisting of vertices and edges in the triangles corresponding to nodes in T_v. Let us define the natural relation of *inclusion* between triangles of a plane embedding: a triangle u is included in a triangle v if at least one of u's vertices is strictly inside the region bordered by v. We partition the children of v into three subsets:

- $In(v)$: nodes with the corresponding triangles included in v,
- $Out(v)$: nodes with the corresponding triangles not related to v,



(a) (b)

Figure 1: G and its associated graph $D(G)$

- $Cov(v)$: nodes with the corresponding triangles including v.

These nodes belong to at most two maximal cliques of $D(G)$ that also contain v. If $v = r$, then these nodes belong to just one clique. Partition further the nodes in $In(v)$ into $In_1(v)$ and $In_2(v)$ depending on to which of the two maximal cliques they belong. Let $Out_1(v)$ and $Out_2(v)$ (respectively $Cov_1(v)$ and $Cov_2(v)$) be a similar partition of $Out(v)$ (respectively $Cov(v)$). Order the nodes in each of the sets $In_1(v)$, $Out_1(v)$, $Cov_1(v)$, $In_2(v)$, $Out_2(v)$, $Cov_2(v)$ according to the relationship of the inclusion in the plane between the corresponding triangles of G (the triangle that includes all the other first). From now on we will treat these sets as ordered lists and $||$ will denote their concatenation; the equality will take under consideration the order of elements.

Note that either $Cov_1(v) = \emptyset$ or $Cov_2(v) = \emptyset$ due to the fact that G_p is planar. In fact, the planarity of G_p forces the set of nodes $U = \{u \in D(G) | Cov(u) \neq \emptyset\}$ to be on a single directed path in T_r. Let v be a node of $D(G)$ with $Cov(v) \neq \emptyset$ that is farthest away from the root r. Assume that $Cov_1(v) = \emptyset$ and $Cov_2(v) \neq \emptyset$ and let z_1 denote the first node in $Cov_2(v)$ (Fig. 2a). Consider the plane embedding (Fig. 2b) where:

- the triangle z_1 is drawn outside of v (i.e., z_1 is removed from $Cov_2(v)$ and added in front of $Out_2(v)$);

- triangles corresponding to nodes in $Out_1(z_1)$ and $Out_2(z_1)$ are drawn inside z_1 in the same order (i.e., $Out_1(z_1)$ and $Out_2(z_1)$ become $In_1(z_1)$ and $In_2(z_1)$, respectively);
- triangles corresponding to nodes in $In_1(z_1)$ and $In_2(z_1)$ are drawn outside z_1 in the same order (i.e., $In_1(z_1)$ and $In_2(z_1)$ become $Out_1(z_1)$ and $Out_2(z_1)$, respectively).

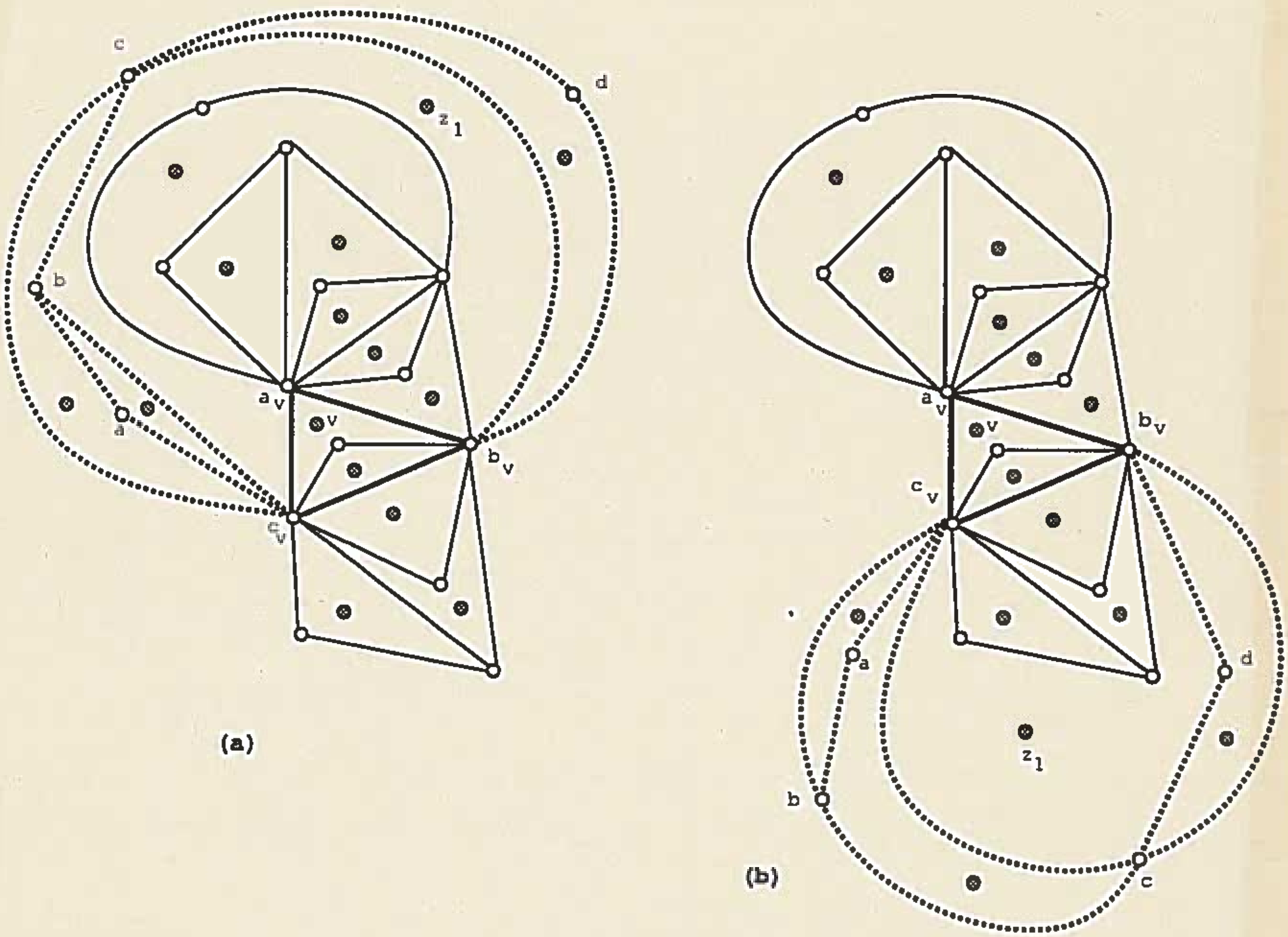


Figure 2: Identical plane embeddings

Although the two drawings are different, they represent the same plane embedding. Since the above transformation can be applied repeatedly, we shall henceforth assume that $U = \emptyset$ (or equivalently, $Cov_1(v) = \emptyset$ and $Cov_2(v) = \emptyset$ for all $v \in T_r$).

The tree T_r together with the ordered partitions $In_1(v)$, $Out_1(v)$, $In_2(v)$, $Out_2(v)$ for every node v in T_r other than r will be called an *in-graph* of T_r *rooted at* r (Fig. 3c). It will be denoted by $\vec{T}_r$. The root r will have all children in one partition $In(r), Out(r)$.

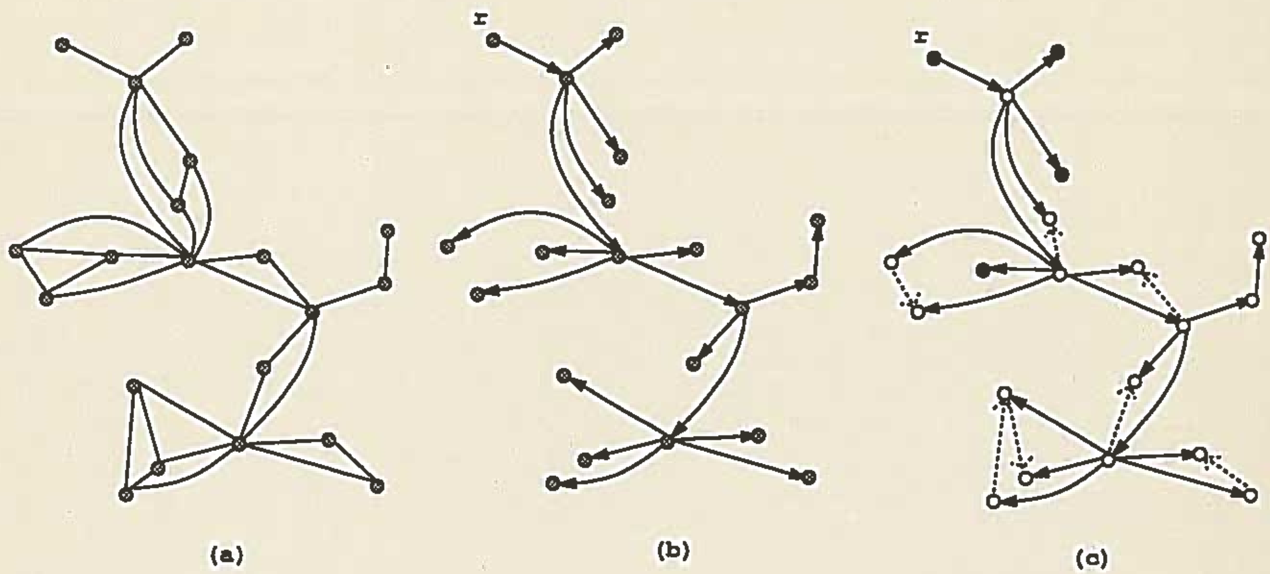


Figure 3: $D(G)$, its breadth-first tree T_r and an in-graph $\vec{T}_r$

Lemma 3 *Every plane embedding G_p of a full 2-tree G can be represented by an in-graph $\vec{T}_r$, where r is a pendant node in $D(G)$.*

Proof. Follows immediately from the above discussion. ▪

When drawing in-graphs as in Fig. 3c, we will use open (respectively bold) circles to indicate nodes of *Out*-subsets (respectively *In*-subsets). The order in the subsets will be indicated by directed dashed paths.

Lemma 4 *Every in-graph $\vec{T}_r$ of G defines a unique plane embedding G_p of G.*

Proof. The embedding associated with $\vec{T}_r$ is obtained in the following manner (Fig. 4):

- Draw the triangle r.
- Traverse the nodes of $\vec{T}_r$ in any parent-first order. When leaving a node v, draw in the nested fashion triangles corresponding to $In_1(v)$ and $In_2(v)$ (respectively

$Out_1(v)$ and $Out_2(v)$) inside (respectively outside) the triangle v. The ordering of triangles is given by the directed paths (first node corresponding to the outermost triangle). It is always possible to place triangles without violating the planarity of the already embedded subgraph. Note that triangles drawn when traversing $In_1(v)$ and $In_2(v)$ (and their successors) will all be inside v. The unique face inside v that contains all three vertices on its boundary will be considered as the one corresponding to v. ∎

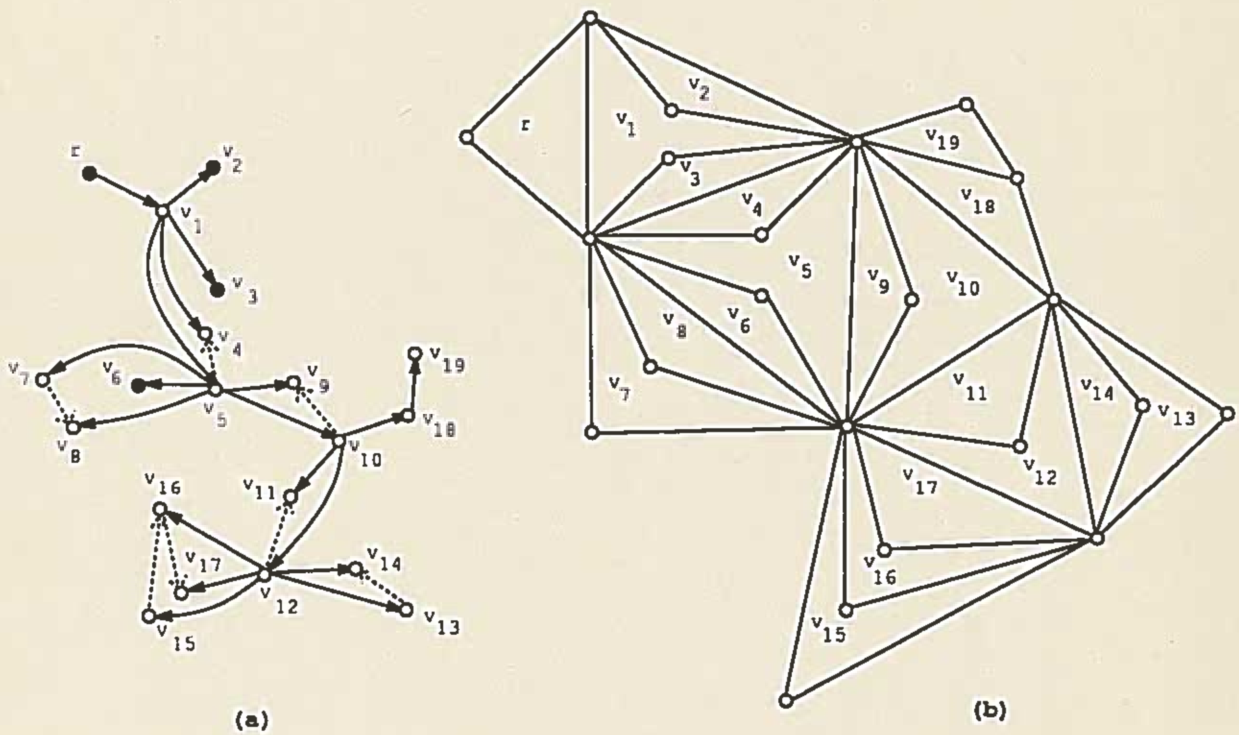


Figure 4: $\vec{T}_r$ and the corresponding embedding G_p

Let r be a pendant node of a pendant clique in $D(G)$. Let $\vec{T}_r$ be an in-graph rooted at r. Let us define a transformation of $G_p(v)$ called informally "turning inside-out" of G_p at v that swaps the In and Out subsets while preserving the order of their elements.

More formally, for a vertex v of T_r,

- $In'_1(v) = Out_1(v)$, $In'_2(v) = Out_2(v)$,
- $Out'_1(v) = In_1(v)$, $Out'_2(v) = In_2(v)$.

For $v = r$, the subscripts are immaterial. Define the in-graph $\phi(\vec{T}_r)$ obtained from $\vec{T}_r$ by turning G_p inside-out at r.

Lemma 5 *Two distinct in-graphs $\vec{T}_r$ and $\vec{T'}_r$ both rooted at the same pendant node r in $D(G)$ represent the same plane embedding of G if and only if $\phi(\vec{T}_r) = \vec{T'}_r$.*

Proof. It can be easily verified that if $\phi(\vec{T}_r) = \vec{T'}_r$, then the respective plane embeddings are identical.

Let In and Out be associated with T_r, and In' and Out' be associated with T'_r. For the implication in the other direction, note that if $In_1(v) = In'_1(v)$, $Out_1(v) = Out'_1(v)$, $In_2(v) = In'_2(v)$, $Out_2(v) = Out'_2(v)$ for all non-root nodes $v \in \vec{T}_r$ then there is nothing to prove. Let therefore v be a node of $\vec{T}_r$ with at least one of the equalities violated. Assume that v is selected such that for all nodes on the path between v and the root r, the above equalities are satisfied. Let $v = \{a_v, b_v, c_v\}$ be the triangle in G corresponding to the node v. Assume without loss of generality that $In_1(v) \neq In'_1(v)$ or $Out_1(v) \neq Out'_1(v)$. Let $In_1(v) = \{x_1, x_2, ..., x_k\}$, $In'_1(v) = \{x'_1, x'_2, ..., x'_{k'}\}$, $Out_1(v) = \{y_1, y_2, ..., y_l\}$, $Out'_1(v) = \{y'_1, y'_2, ..., y'_{l'}\}$. Let $u = \{a_u, b_u, c_u\}$ be the triangle in G corresponding to a node $u \in In_1(v)||Out_1(v) = In'_1(v)||Out'_1(v)$. Assume without loss of generality that $a_u = a_v$ and $c_u = c_v$ for all $u \in In_1(v)||Out_1(v)$.

Assume that $k \geq 1$ and consider the face corresponding to x_k in the plane embedding given by $\vec{T}_r$. This face contains no vertices b_u, $u \in In_1(v)||Out_1(v) \setminus \{x_k\}$. Neither does it contain b_v. Since $\vec{T}'_r$ represents the same embedding, x_k is either $x'_{k'}$ or $y'_{l'}$. By a similar argument, if $l \geq 1$, then y_l is either $y'_{l'}$ or $x'_{k'}$.

Assume that $k \geq 2$ and let x_i, x_{i+1}, $1 \leq i < k$, be a pair of consecutive nodes in $In_1(v)$. Hence, the plane embedding must have a face with both b_{x_i} and $b_{x_{i+1}}$ on its boundary. Consequently, x_i and x_{i+1} must appear next to each other in either $In'_1(v)$ or $Out'_1(v)$. Similar arguments apply if $l \geq 2$ and y_j, y_{j+1}, $j \geq 1$, is a pair of consecutive nodes in $Out_1(v)$.

It follows from the above arguments that either

- $In_1(v) = In'_1(v)$, $Out_1(v) = Out'_1(v)$, or
- $In_1(v) = Out'_1(v)$, $Out_1(v) = In'_1(v)$, and $In_1(v)||Out_1(v) \neq \emptyset$.

Assume that v is not the root and $In_1(v) = Out'_1(v)$ and $Out_1(v) = In'_1$. Assume that $In_1(v) \neq \emptyset$. Since x_1 is in front of $In_1(v)$, no face with b_{x_1} on its boundary can contain a vertex not in $G(v)$. But x_1 is also in front of $Out'_1(v)$ implying that at least one face with b_{x_1} on its boundary must contain a vertex outside $G(v)$, a contradiction. If $In_1(v) = \emptyset$ then $Out_1(v) \neq \emptyset$ leads to a similar contradiction. ■

3 Counting plane embeddings

3.1 Plane embeddings of 2-trees

A *frame* in a 2-tree H is a maximal (with respect to subgraph inclusion) outerplanar subgraph of H that does not contain any strongly interior edge as an interior edge. Any frame is also a mop. Strongly interior edges of a 2-tree partition it into frames (if one allows multiple copies of those edges).

We will first prove that a biconnected partial 2-tree has the same plane embeddings as any full 2-tree that imbedds it (modulo embeddings of its frames). Since the frames are outerplanar and the plane embeddings problem for outerplanar graphs has been solved ([10]), solving the problem for full 2-trees will imply the solution for partial 2-trees.

Lemma 6 *A biconnected partial 2-tree G contains all exterior edges of any full 2-tree imbedding with the same set of vertices.*

Proof: Removal of an exterior edge from a 2-tree introduces an articulation point. If there were an imbedding H of G missing an exterior edge, it would be separable and so would be any partial graph of H. This contradicts biconnectivity of G. ∎

Strongly interior edges of a 2-tree H partition H into maximal outerplanar components (frames) in the following sense: In every non-outerplanar 2-tree, there is a terminal strongly interior edge, say $e = (a, b)$. Add the outerplanar graphs G_i (connected components C_i of $H - \{a, b\}$ augmented by $\{a, b\}$ and the adjacent edges) to the set of frames and remove the corresponding components C_i to obtain a 2-tree H'. Repeat the operation until only one edge remains.

Lemma 7 *Any 2-tree imbedding H of a biconnected partial 2-tree G contains the same set of strongly interior edges.*

Proof: The lemma follows from the uniqueness of the set of exterior edges (Lemma 6). If (a, b) is a strongly interior edge in an imbedding H of G, then the removal of $\{a, b\}$ disconnects G into more than two components. Since $H - \{a, b\}$ consists of at least three connected components, G contains three disjoint paths between a and b. A 2-tree imbedding of G in which any two of the three paths are connected by a path not using a or b, would have a subgraph homeomorphic to K_4. This would contradict the absence of such a subgraph in partial (and thus also in full) 2-trees. Thus, $\{a, b\}$ is a separator in any 2-tree imbedding of G. ∎

3.2 Counting plane embeddings of 2-trees

Using the tree representation, it is a rather straightforward task to count all plane embeddings of 2-trees. Let v denote a node of an in-graph $\vec{T_r}$ of a 2-tree G, $v \neq r$. Let $In_1(v)||Out_1(v) = \{x_1, x_2, \ldots, x_{k_v}\}$, $k_v \geq 0$ and $In_2(v)||Out_2(v) = \{y_1, y_2, \ldots, y_{l_v}\}$, $l_v \geq 0$. There are $k_v!$ permutations of $In_1(v)||Out_1(v)$. Each permutation can be split in k_v+1 ways such that the first i elements, $0 \leq i \leq k_v$, belong to $In_1(v)$ while the remaining $k_v - i$ elements belong to $Out_1(v)$. It follows immediately from Lemma 4 that the number of plane embeddings of G is

$$\pi(G) = \frac{1}{2}\Pi_{v \in \vec{T_r}}(l_v + 1)!(k_v + 1)!$$

The coefficient $\frac{1}{2}$ is due to the fact that plane embeddings obtained by turning inside-out $In(r)$ and $Out(r)$ subsets at the root r are identical.

3.3 Counting plane embeddings of partial 2-trees

For a planar graph G, let $\pi(G)$ be the number of plane embeddings (*i.e.*, embeddings with different sets of boundary cycles). Let $\pi'(G)$ be the number of plane embeddings of G when the outer face containing a specified edge is distinguished. (It is easy to see that this number is independent of the particular edge chosen. When the graph G has at least 2 faces, then $\pi'(G) = 2\pi(G)$ since every edge is in exactly two faces.)

Lemma 8 *Let $e = (a, b)$ be an interior edge of a 2-tree H with l components C_i of $H - \{a, b\}$. Let $H_i = H - \bigoplus_{j \neq i} C_j$. Then*

$$\pi(H) = \frac{1}{2} l! \prod_{1 \leq i \leq l} \pi'(H_i)$$

Proof: We will use the idea of permuting the components H_i to produce all embeddings of H while avoiding duplication by omitting embeddings related in a manner similar to the mapping ϕ of the preceding subsection. The proof will follow by induction on l:

(i) $l = 2$. Assume that H is drawn with a vertical edge e separating C_1 on the left of e from C_2 on the right of e and consider given embeddings of H_1 and H_2. The same embedding of H can be found among plane drawings of H with both C_1 and C_2 on one side of e. On the other hand, every embedding of H_1 and H_2 with the distinguished outer side of e contributes multiplicatively a new set of boundary cycles. Thus, $\pi(H) = \pi'(H_1) \cdot \pi'(H_2) = \frac{1}{2} l! \prod \pi'(H_i)$.

(ii) $l > 2$. Let x be an end vertex of e. For each H_i, choose an arbitrary edge e_i incident with x. An embedding of H is uniquely given by the position of e_l in the ordering of e_i around x and the given embeddings of the H_i's $(1 \leq i < l)$ fixing the 'outer side' of e. Assuming $\pi(H - C_l) = \frac{1}{2}(l-1)! \prod_{1 \leq i < l} \pi'(H_i)$, each embedding of H_l contributes multiplicatively to the number of different sets of boundary cycles and the above observation proves the desired formula, since there are l possible positions for e_l. ∎

Note that the outerplanar case (with the embedding count given in [10] as $\pi(H) = 2^{f-2}$, where $f > 1$ is the number of interior faces in H) is a simple corollary of Lemma 8, since every separating edge of an outerplanar graph gives $l = 2$ and the absence of such an edge gives the base case of $\pi(H) = 1$. Since all mops of a given size have the same number of interior faces, the number of plane embeddings of a mop is completely determined by its size.

Lemma 9 *Given a biconnected partial 2-tree G, the number of plane embeddings is the same for every 2-tree H imbedding G.*

Proof: By Lemma 3, any two 2-tree imbeddings of G differ at most on some subset of weakly interior edges. Yet, the sizes of the corresponding frames are identical. Since the frames of a 2-tree are maximal outerplanar, it follows by Lemma 8 that the number of plane embeddings of H is determined by the size of frames of H and their interactions through strongly interior edges of H. These are identical for all imbeddings of G. ∎

¿From these lemmas follows immediately a formula counting the number of plane embeddings for a partial 2-tree with minimal separators that induce edges.

Theorem 2 *Let $\{a, b\}$ be a separator of a biconnected partial 2-tree G with l components C_i of $G - \{a, b\}$. Let $G_i = G - \bigoplus_{j \neq i} C_j$. Then*
(i) If (a, b) is an edge of G, then $\pi(G) = \frac{1}{2} l! \prod_{1 \leq i \leq l} \pi'(G_i)$.
(ii) Otherwise, $\pi(G) = \frac{1}{2}(l-1)! \prod_{1 \leq i \leq l} \pi'(G_i)$.

Proof: (i) Since it is almost identical with the proof of Lemma 8, we omit it.

(ii) If $l = 2$, then either G is outerplanar, or one can find a strongly interior edge as in (i). Let $\{a, b\}$ be a minimal separator of G not inducing an edge. Since we assume that the number of connected components of $G - \{a, b\}$ is at least 3, (a, b) is an edge in any 2-tree imbedding H of G. We notice that in this case, the $l \geq 3$ components can be "permuted" in $\frac{1}{2}(l-1)!$ ways. Each subgraph G_i of G will be defined as $G - \bigoplus_{j \neq i} C_j$ *augmented* by the edge (a, b). (In the previous case of the separator inducing an edge, the edge $e = (a, b)$ acts as an extra component.) ∎

4 Independent Face covers

4.1 Perfect FIVC for 2-trees

Let $G = (V, E)$ be a biconnected planar graph. A subset S of vertices is called a *perfect face-independent vertex cover* (or *perfect FIVC*, for short) of G if there exists a plane embedding G_p of G in which every face has exactly one vertex in S. A set W of cycles of a graph H is called a *perfect vertex-independent face cover* (or *perfect*, for short) if there is a plane embedding H_p of H such that W is a subset of boundary cycles of faces and every vertex of H is in exactly one cycle of W. A perfect *VIFC* in this plane embedding of H is simply a 2-factor of H which consists of facial cycles. In the geometric dual G_p^* of G_p, a perfect *FIVC* of G corresponds to a set of faces of G_p^* which is a perfect *VIFC* of the vertices of G_p^*. The problems of finding perfect *FIVC* and perfect *VIFC* are NP-complete in general, see [3,2]. When restricted to outerplanar graphs, these problems are polynomially solvable, see [8].

In this section, we describe a polynomial time algorithm that, given a full 2-tree G, finds a plane embedding G_p of G that admits a perfect *FIVC*, or decides that no such embedding exists. In fact, the algorithm finds a plane embedding with a minimum cardinality perfect *FIVC*, if a perfect *FIVC* exists. The algorithm follows an approach similar to that of [9]. It processes in a bottom-up manner the breadth-first search tree T_r of the associated graph $D(G)$.

If T_r consists of the root r alone, the problem is trivial. In the following, we assume that T_r contains at least two nodes. Let $v \in T_r$, $v \neq r$. Recall that T_v is the maximal subtree of T_r rooted at v and $G(v)$ is the corresponding subgraph of G. Consider the branch (u, v) of T_r entering v ($(u, v) \notin T_v$). Assume that the corresponding edge in G (and in $G(v)$) is (a_v, b_v).

Define the following minimum cardinality covers among all plane embeddings of $G(v)$.

- $I(v)$ = face-independent vertex cover of all but the exterior face.
- $B(v)$ = face-independent vertex cover of all but the face corresponding to v and the exterior face.
- $F(v)$ = face-independent vertex cover of all but the face corresponding to v.
- $L(v)$ = face-independent vertex cover of all faces with the face corresponding to v and the exterior face covered by a_v.
- $R(v)$ = face-independent vertex cover of all faces with the face corresponding to v and the exterior face covered by b_v.

- $E(v)$ = face-independent vertex cover of all faces using neither a_v nor b_v.

Initially, for each pendant node $v = \{a_v, b_v, c_v\}$ of T_r, $v \neq r$, where (a_v, b_v) is a common edge with another triangle, we let the corresponding covers be (cf. Fig. 5):

- $I(v)$ = undefined; it is impossible to cover the face corresponding to v without covering the exterior face.
- $B(v) = \emptyset$.
- $F(v)$ = undefined; it is impossible to cover the exterior face without covering the face corresponding to v.
- $L(v) = \{a_v\}$.
- $R(v) = \{b_v\}$.
- $E(v) = \{c_v\}$.

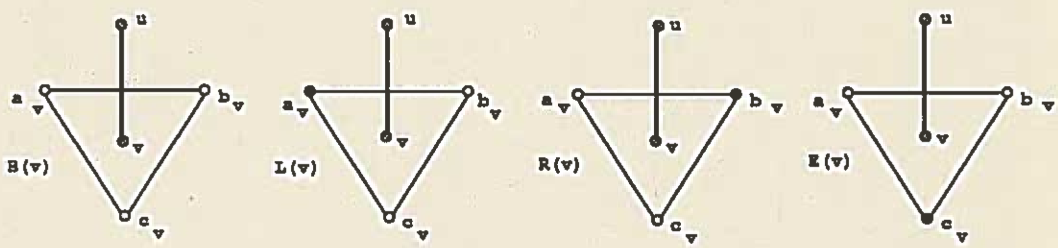


Figure 5: Existing covers of $G(v)$, v a pendant node in T_r

Any node for which the above six covers have been determined is said to be *labeled*. Hence, all pendant nodes of T_r other than r are initially the only labeled nodes. Assume that an unlabeled node v is chosen such that all its children in T_r are labeled. In the remainder of this section, we explain how to determine the six covers for v. Let $In_1(v)||Out_1(v) = \{x_1, x_2, ..., x_k\}$ and $In_2(v)||Out_2(v) = \{y_1, y_2, ..., y_l\}$ denote the children of v in T_v.

4.1.1 FIVC for All but the Exterior Face ($I(v)$)

Suppose that $I(v)$ exists. Let $In_1(v), Out_1(v), In_2(v), Out_2(v)$ denote the ordered partition of children of v such that the corresponding embedding of $G(v)$ admits this $I(v)$.

Either $In_1(v) \neq \emptyset$ or $In_2(v) \neq \emptyset$; otherwise it would be impossible to cover the face corresponding to v by $I(v)$. Suppose that the face corresponding to v is covered by a vertex from the triangle corresponding to a node in T_{v_i}, $v_i \in In_2(v)$. Assume that $In_1(v) \neq \emptyset$ (Fig. 6a). The first node in $In_1(v)$ is a root of a subtree of T_v. None of the vertices of triangles corresponding to nodes in this subtree can cover the face corresponding to v; otherwise this face would be covered twice. Consider the plane embedding of $G(v)$ obtained by adding $In_1(v)$ to the end of $Out_1(v)$, i.e., the embedding with $Out_1'(v) = Out_1(v)||In_1(v)$ and $In_1'(v) = \emptyset$. $I(v)$ is still a *FIVC* of $G(v)$ for all but the exterior face (Fig. 6b).

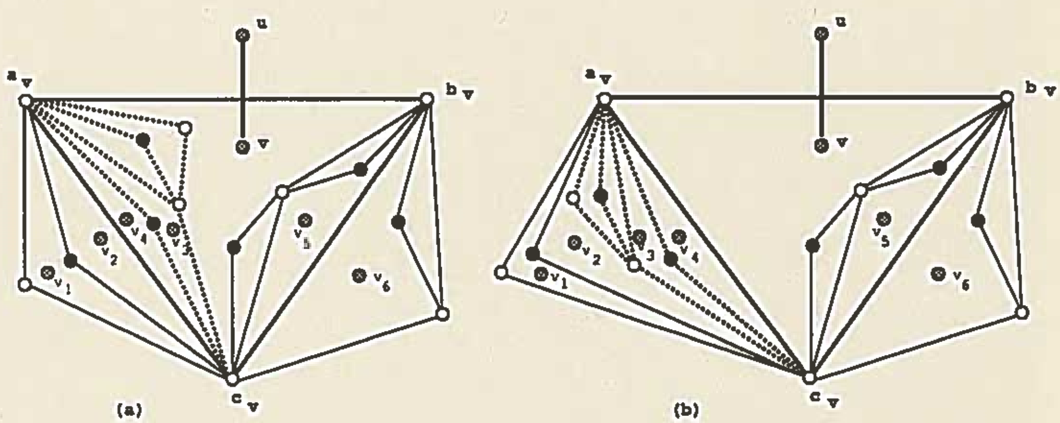


Figure 6: Two different plane embeddings of $G(v)$ admitting the same $I(v)$

We can therefore first assume that $In_1(v) = \emptyset$ and search for the minimum cardinality $I_1(v)$ (or decide that it does not exist) among such embeddings of G. Then, the minimum cardinality $I_2(v)$ for embeddings of G with $In_2(v) = \emptyset$ is found or it is decided that it does not exist. The smallest of these two is the desired $I(v)$. If none of them exists, then neither does $I(v)$.

Our analysis will determine feasible *covering sequences*, i.e., sets of vertex covers of T_x, for every child x of v. Without loss of generality, let us assume that $In_1(v) = \emptyset$. Assume that $Out_2(v) \neq \emptyset$ (Fig. 7a). Consider the embedding of $G(v)$ obtained by adding $Out_2(v)$ to the end of $In_2(v)$, i.e., the embedding with $In_2'(v) = In_2(v)||Out_2(v)$ and $Out_2'(v) = \emptyset$. $I(v)$ is still a *FIVC* of $G(v)$ for all but the exterior face (Fig. 7b). Hence, when looking for the minimum cardinality $I(v)$ with $In_1(v) = \emptyset$, we can assume that $Out_2(v) = \emptyset$.

Suppose therefore that $In_1(v) = Out_2(v) = \emptyset$. Let $Out_1(v) = \{x_1, x_2, ..., x_k\}$ and $In_2(v) = \{y_1, y_2, ..., y_l\}$ with the indicated orders admitting $I(v)$. Then $G(x_1)$ must be covered by either $I(x_1)$ or $B(x_1)$; otherwise the exterior face would be covered. If $G(x_1)$ is covered by $I(x_1)$, then $G(x_2)$ must be covered by either $I(x_2)$ or $B(x_2)$. If $G(x_1)$ is covered by $B(x_1)$, then $G(x_2)$ must be covered by either $F(x_2)$ or $E(x_2)$. If $G(x_2)$ is covered by

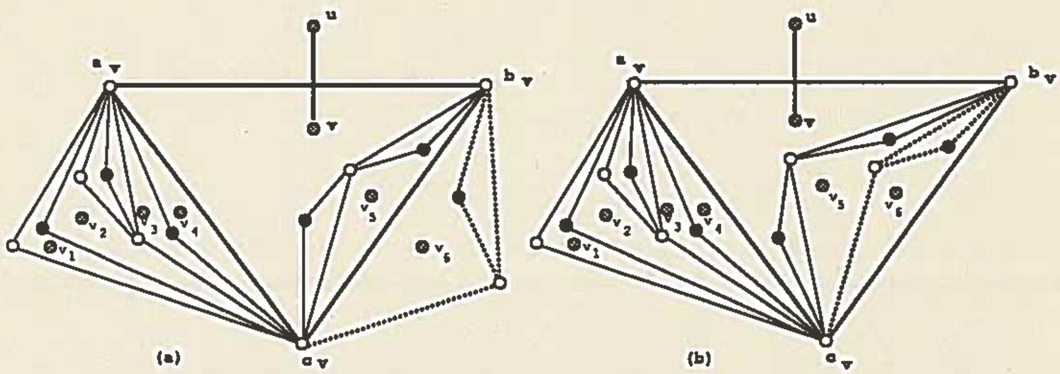


Figure 7: Two different plane embeddings of $G(v)$ admitting the same $I(v)$

$F(x_2)$, then $G(x_3)$ must be covered by either $F(x_3)$ or $E(x_3)$. If $G(x_2)$ is covered by $E(x_2)$, then $G(x_3)$ must be covered by either $I(x_3)$ or $B(x_3)$. Note that $G(x_k)$ must be covered by either $I(x_k)$ or $E(x_k)$. Hence, the covering sequence of $Out_1(v) = \{x_1, x_2, ..., x_k\}$ must be formed as a path in the forest shown in Fig. 8a with leaves being either $I(x_k)$ or $E(x_k)$.

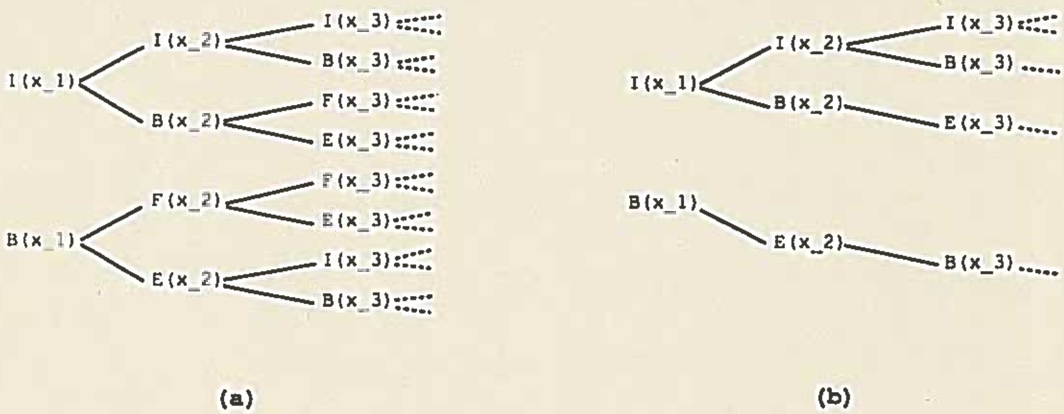


Figure 8: Covering sequences of $Out_1(v) = \{x_1, x_2, ..., x_k\}$ for $I(v)$

Suppose that $I(x_i)$, $3 \leq i \leq k$, is preceded by $E(x_{i-1})$ (Fig. 9a with $i = 3$). Then we can place x_i in front of $Out_1(v)$ without affecting $I(v)$. Hence, we can assume that all $I(x_i)$ covers occur only in the beginning of the sequence $Out_1(v)$ (Fig. 9b).

By turning $G_p(z)$ inside-out at every child z of x_i, the cover $I(x_i)$ becomes $F(x_i)$ (and vice versa). Consequently, we can assume that no x_i, $2 \leq i \leq k-1$, is covered by $F(x_i)$. Otherwise, we could place x_i in front of $Out_1(v)$ and cover it by $I(x_i)$ (Fig. 10 with $i = 3$).

It follows that at least one optimal covering sequence of $Out_1(v)$ in $I(v)$ is a path in the pruned forest shown in Fig. 8b with leaves being either $I(x_k)$ or $E(x_k)$.

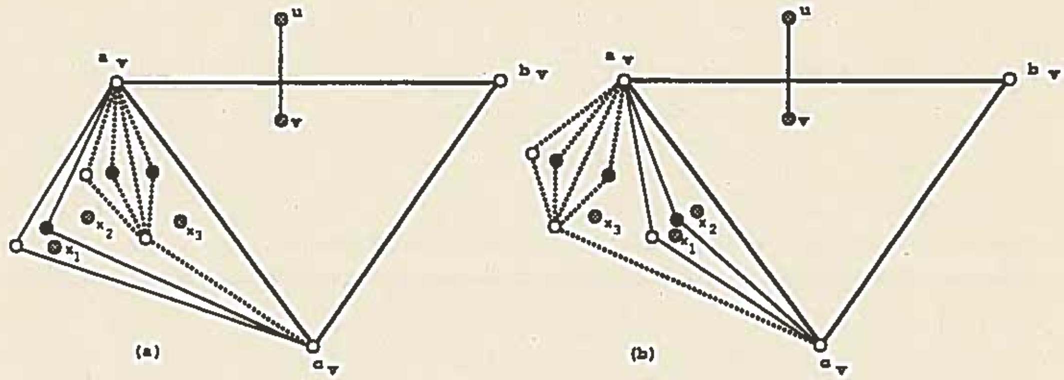


Figure 9: Equivalent covering sequences of $Out_1(v)$ in $I(v)$

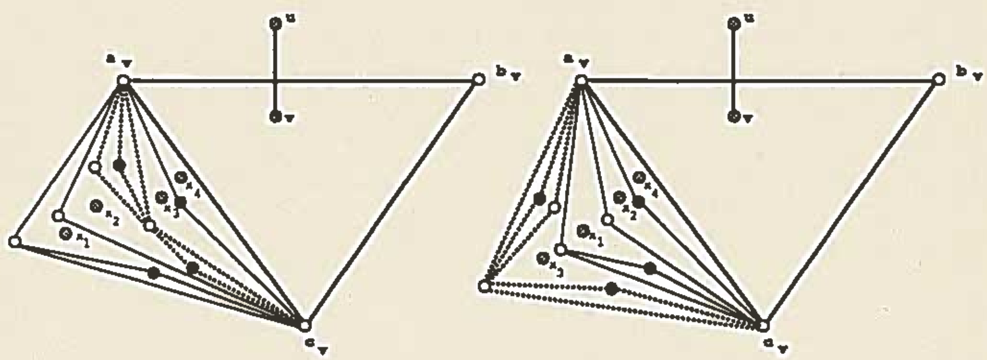


Figure 10: Equivalent covering sequences of $Out_1(v)$ in $I(v)$

Given these restrictions, we can now determine the optimal covering sequence of $Out_1(v)$ in $I(v)$, provided that one exists.

Consider a complete undirected graph K with $x_1, x_2, ..., x_k$ as its vertices. With every edge (x_i, x_j), associate the cost

$$\min\{|I(x_i)| + |I(x_j)|, |B(x_i)| + |E(x_j)|, |E(x_i)| + |B(x_j)|\}$$

For undefined covers, their cardinality is defined to be ∞. This cost gives the minimum cardinality of partial FIVCes of T_{x_i} and T_{x_j} that are compatible if x_i and x_j were to be placed consecutively in a plane embedding of G.

Suppose first that k is even. Solve the minimum cost perfect matching problem on K. End-vertices of edges in this matching with costs determined by the first minimization term are placed in front of $Out_1(v)$ in any order. The remaining end-vertices are then placed pairwise. The order within each such pair (x_i, x_j) depends on whether the edge cost was determined by the second minimization term (x_i precedes x_j) or by the third minimization term (x_j precedes x_i).

If k is odd, at least one of the subgraphs $G(x_m)$, $1 \leq m \leq k$, must be covered by $I(x_m)$. For each choice of x_m, we need to solve the minimum cost perfect matching problem M_m on the complete subgraph of K induced by $Out_1(v) \setminus \{x_m\}$. Select the matching M_m such that its cost together with $|I(x_m)|$ is minimized. Place this x_m in front of $Out_1(v)$. The remaining nodes of $Out_1(v)$ are ordered as in the case k even.

Let us now look at how to cover $In_2(v)$. $G(y_1)$ must be covered by either $F(y_1)$ or $E(y_1)$. If $G(y_1)$ is covered by $F(y_1)$, then $G(y_2)$ must be covered by either $F(y_2)$ or $E(y_2)$. If $G(y_1)$ is covered by $E(y_1)$, then $G(y_2)$ must be covered by either $I(y_2)$ or $B(y_2)$. If $G(y_2)$ is covered by $I(y_2)$, then $G(y_3)$ must be covered by either $I(y_3)$ or $B(y_3)$. If $G(y_2)$ is covered by $B(y_2)$, then $G(y_3)$ must be covered by either $F(y_3)$ or $E(y_3)$. Note that $G(y_l)$ must be covered by either $I(y_l)$ or $E(y_l)$. Hence, the covering sequence of $y_1, y_2, ..., y_l$ must be a path in the forest shown in Fig. 11a with leaves being either $I(y_l)$ or $E(y_l)$.

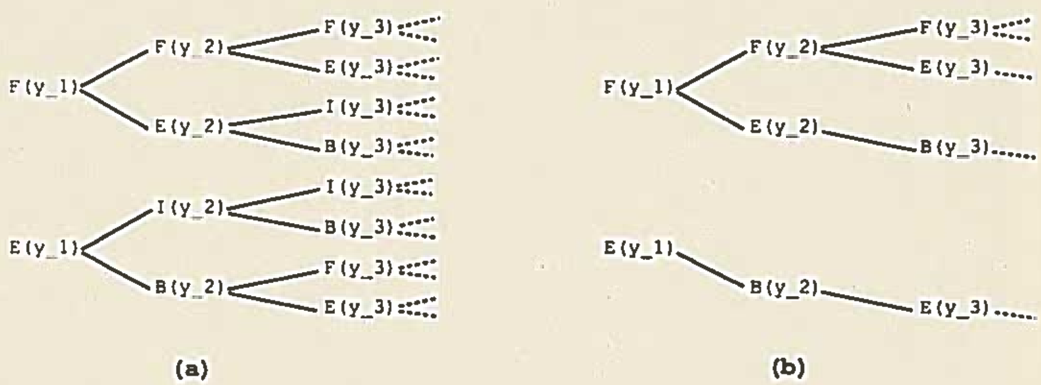


Figure 11: Covering sequences of $In_2(v) = \{y_1, y_2, ..., y_l\}$ for $I(v)$

Suppose that $F(y_i)$, $3 \leq i \leq l-1$, is preceded by $B(y_{i-1})$ (Fig. 12a with $i = 3$). Then we can place $F(y_i)$ in front of $In_2(v)$ without affecting $I(v)$. Hence, we can assume that $F(x_i)$ occurs only in the beginning of the covering sequence $In_2(v)$ (Fig. 12b).

As already mentioned, by turning $I(y_i)$ inside-out, we obtain $F(y_i)$ (and vice versa). Consequently, we can assume that no y_i, $3 \leq i \leq l-1$, is covered by $I(y_i)$. If it were, we could place y_i in front of $In_2(v)$ and cover it by $F(y_i)$ (Fig. 13 with $i = 3$).

It follows that at least one optimal covering sequence of $In_2(v)$ in $I(v)$ is a path in the pruned tree shown in Fig. 11b with leaves being $E(x_l)$.

Given these restrictions, we can now determine the optimal covering sequence of $In_2(v)$ in $I(v)$, provided that one exists.

Consider a complete undirected graph K with $y_1, y_2, ..., y_l$ as its vertices. With every

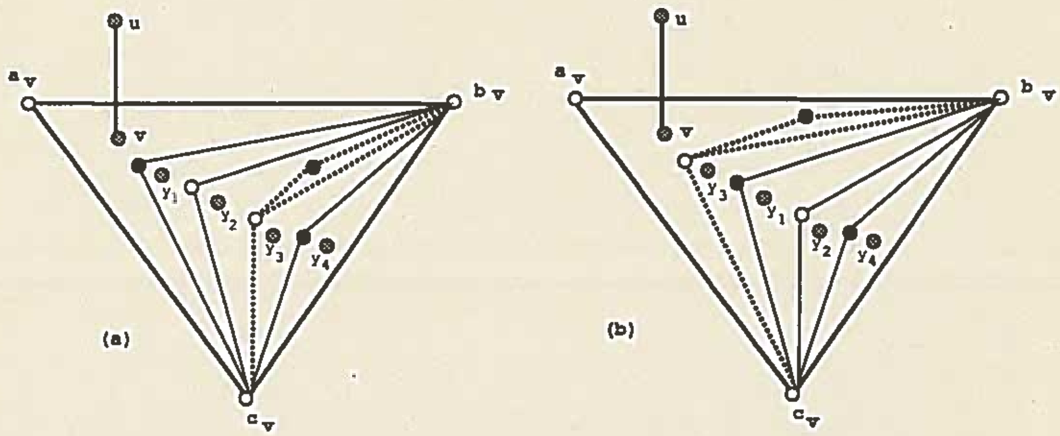


Figure 12: Equivalent covering sequences of $In_2(v)$ in $I(v)$

edge (y_i, y_j) of K we associate the cost

$$\min\{|F(y_i)| + |F(y_j)|, |B(y_i)| + |E(y_j)|, |E(y_i)| + |B(y_j)|\}$$

Suppose first that l is even. Then there is a pair of nodes y_m, y_n, $1 \le m < n \le l$, that have covers $F(y_m)$, $E(y_n)$. For each choice of the pair y_m, y_n, solve the minimum cost perfect matching problem M_{mn} on the complete subgraph of K induced by $In_2(v) \setminus \{y_m, y_n\}$. Select the matching M_{mn} such that its cost together with $\min\{|F(y_m)|+|E(y_n)|, |E(y_m)|+|F(y_n)|\}$ is minimized. End-vertices of edges in M_{mn} with their costs determined by the first minimization term ($|F(y_i)| + |F(y_j)|$) are placed in front of $In_2(v)$ (in any order). If $|F(y_m)| + |E(y_n)| < |E(y_m)| + |F(y_n)|$, then y_m is placed in $In_2(v)$, followed by y_n. If this is not the case, then y_n is followed by y_m. The remaining end-vertices of edges in M_{mn} are then placed pairwise. The order within each pair depends on whether the edge cost was determined by the second minimization term (y_i precedes y_j) or by the third minimization term (y_j precedes y_i).

Suppose now that l is odd. At least one of the subgraphs $G(y_m)$, $1 \le m \le l$, must be covered by $E(y_m)$. For each choice of y_m, we need to solve the minimum cost perfect matching problem M_m on the subgraph of K induced by $In_2(v) \setminus \{y_m\}$. Select the matching M_m such that its cost together with $|E(y_m)|$ is minimized. End-vertices of edges in M_m with their cost determined by the first minimization term are placed in front of $In_2(v)$, followed by y_m. The remaining end-vertices of edges in M_m are then placed pairwise. The order within each pair depends on whether the edge cost was determined by the second minimization term (y_i precedes y_j) or by the third minimization term (y_j precedes y_i).

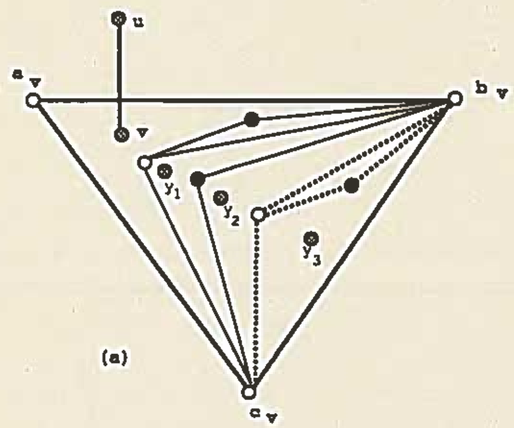


Figure 13: Equivalent covering sequences of $In_2(v)$ in $I(v)$

4.1.2 FIVC for All but the Face Corresponding to v and to the Exterior Face ($B(v)$)

We employ an argument similar to that of the preceding section. Suppose that $B(v)$ exists. Let $In_1(v), Out_1(v), In_2(v), Out_2(v)$ denote the ordered partition of children of v such that the corresponding embedding of $G(v)$ admits this $B(v)$. We can assume that $In_1(v) = In_2(v) = \emptyset$. Suppose, to the contrary, that $In_1(v) \neq \emptyset$ (Fig. 14a). Consider the embedding of $G(v)$ obtained by adding $In_1(v)$ at the front of $Out_1(v)$, i.e., $Out'_1(v) = In_1(v)||Out_1(v)$ and $In'_1(v) = \emptyset$. $B(v)$ is still a *FIVC* of $G(v)$ covering all faces except for the face corresponding to v and to the exterior face (Fig. 14b).

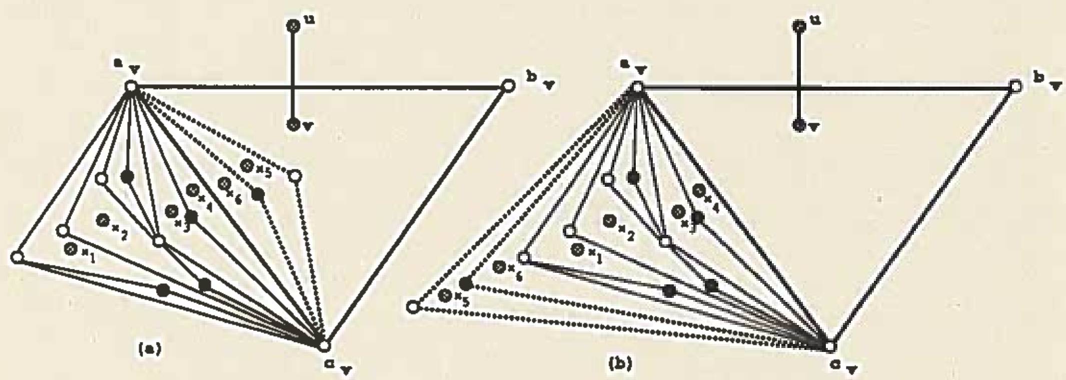


Figure 14: Equivalent covering sequences of $In_1(v)$ and $Out_1(v)$ in $B(v)$

Suppose therefore that $In_1(v) = In_2(v) = \emptyset$. Order of nodes within $Out_1(v)$ and $Out_2(v)$ and the corresponding covering sequences can be determined in the same manner as the order and covering sequence of $Out_1(v)$ for $I(v)$ (Fig. 8).

4.1.3 FIVC for All but the Face Corresponding to v ($F(v)$)

Suppose that $F(v)$ exists. Let $In_1(v), Out_1(v), In_2(v), Out_2(v)$ denote the ordered partition of children of v such that the corresponding embedding of $G(v)$ admits a minimum cardinality $F(v)$. We can then assume that $In_1(v) = In_2(v) = \emptyset$. If this is not the case, we can consider the embedding with $In'_1(v) = \emptyset$, $Out'_1(v) = Out_1(v)||In_1(v)$, $In'_2(v) = \emptyset$, $Out'_2(v) = Out_2(v)||In_2(v)$ (see Fig. 6). $F(v)$ is still a *FIVC* of $G(v)$ for all but the face corresponding to v.

Suppose therefore that $In_1(v) = In_2(v) = \emptyset$. The exterior face is covered either by the covering sequence of $Out_1(v)$ or by the covering sequence of $Out_2(v)$. The minimum sum of the corresponding covers' cardinalities determines the desired *FIVC* as follows.

Assume that the exterior face is covered by a covering sequence of $Out_1(v)$. This covering sequence of $Out_1(v)$ must be the same as the covering sequence of $In_2(v)$ for $I(v)$ (Fig. 11). The covering sequence of $Out_2(v)$ must be the same as the covering sequence of $Out_1(v)$ for $I(v)$ (Fig. 8).

Assume that the exterior face is covered by the covering sequence of $Out_2(v)$. The covering sequence of $Out_1(v)$ must be the same as the covering sequence of $Out_1(v)$ for $I(v)$ (Fig. 8). The covering sequence of $Out_2(v)$ must be the same as the covering sequence of $In_2(v)$ for $I(v)$ (Fig. 11).

4.1.4 FIVC for All Faces Using a_v ($L(v)$)

Suppose that $L(v)$ exists. Let $In_1(v), Out_1(v), In_2(v), Out_2(v)$ denote the ordered partition of children of v such that the corresponding embedding of $G(v)$ admits this $L(v)$.

It can be easily verified that we can assume that $In_1(v) = \emptyset$, $In_2(v) = \emptyset$. The covering sequence of $Out_1(v)$ must be $L(x_1), L(x_2), ..., L(x_k)$. The covering sequence of $Out_2(v)$ can be determined in the same manner as the covering sequence of $In_2(v)$ for $I(v)$ (*cf.* Fig. 11).

4.1.5 FIVC for All Faces Using b_v ($R(v)$)

Suppose that $R(v)$ exists. Let $In_1(v), Out_1(v), In_2(v), Out_2(v)$ denote the ordered partition of children of v such that the corresponding embedding of $G(v)$ admits this $R(v)$.

It can be easily verified that we can assume that $In_1(v) = \emptyset$, $In_2(v) = \emptyset$. The covering sequence of $Out_2(v)$ must be $R(x_1), R(x_2), ..., R(x_k)$. The covering sequence of $Out_1(v)$ can be determined in the same manner as the covering sequence of $In_2(v)$ for $I(v)$ (*cf.* Fig. 11).

4.1.6 FIVC for All Faces Using neither a_v nor b_v ($E(v)$)

Suppose that $E(v)$ exists. Let $In_1(v), Out_1(v), In_2(v), Out_2(v)$ denote the ordered partition of children of v such that the corresponding embedding of $G(v)$ admits this $E(v)$.

Suppose first that the exterior face and the face corresponding to v are covered by c_v. We can then assume that $In_1(v) = In_2(v) = \emptyset$. Let $Out_1(v) = \{x_1, x_2, ..., x_k\}$ and $Out_2(v) = \{y_1, y_2, ..., y_l\}$. The covering sequence of $Out_1(v)$ must be $R(x_1), R(x_2), ..., R(x_k)$. Similarly, the covering sequence of $Out_2(v)$ must be $L(y_1), L(y_2), ..., L(y_l)$.

Assume now that c_v is not in $E(v)$. If $In_1(v)||Out_1(v) = \emptyset$ and $In_2(v)||Out_2(v) = \emptyset$, then $E(v)$ does not exist. Otherwise, we need to distinguish between the following cases:

- The exterior face is covered by a covering sequence of $Out_1(v)$, and the face corresponding to v is covered by a covering sequence of $In_2(v)$. We can assume that $Out_2(v) = In_1(v) = \emptyset$. Covering sequences of $Out_1(v)$ and $In_2(v)$ must be the same as the covering sequence of $In_2(v)$ for $I(v)$ (Fig. 11).

- The exterior face is covered by a covering sequence of $Out_2(v)$, and the face corresponding to v is covered by a covering sequence of $In_1(v)$. This case is analogous to the previous case.

- The exterior face is covered by a covering sequence of $Out_1(v)$, and the face corresponding to v is covered by a covering sequence of $In_1(v)$. Then we can assume that $In_2(v) = \emptyset$. Let $Out_2(v) = \{y_1, y_2, ..., y_l\}$. The covering sequence of $Out_2(v)$ must be the same as the covering sequence of $Out_1(v)$ for $I(v)$. Let $In_1(v) = \{x_1, x_2, ..., x_p\}$ and $Out_1(v) = \{x_{p+1}, x_{p+2}, ..., x_k\}$, $1 \leq p < k$. If $p = 1$ then $G(x_1)$ is covered by $E(x_1)$. Suppose that $p > 1$. $G(x_1)$ must be covered by either $F(x_1)$ or by $E(x_1)$; otherwise the face corresponding to v in the embedding of $G(v)$ would be uncovered. Suppose that $G(x_1)$ is covered by $E(x_1)$ (Fig. 15a). Then $G(x_2)$ must be covered by either $I(x_2)$ or $B(x_2)$. Let $Out_1'(v) = Out_1(v)||\{x_2, x_3, ..., x_p\}$ and $In_1'(v) = \{x_1\}$. $E(v)$ is still a *FIVC* for all faces (Fig. 15b).

  Suppose that $G(x_1)$ is covered by $F(x_1)$ (Fig. 16a). Then $G(x_2)$ must be covered by either $F(x_2)$ or $E(x_2)$. Let $Out_1'(v) = \{x_1\}||Out_1(v)$ and $In_1'(v) = \{x_2, x_3, ..., x_p\}$. $E(v)$ is still a *FIVC* for all faces (Fig. 16b). Hence, the cardinality of $In_1(v)$ has been reduced by one. Either $|In_1(v)| = 1$ or one of the above two cases is applicable.

  For each choice of a node x in $In_1(v)||Out_1(v)$ as x_1 (the only node in $In_1(v)$), we find the covering sequence of nodes in $Out_1(v)$. It can be determined in the same manner as the covering sequence of $In_2(v)$ for $I(v)$ (*cf.* Fig. 11). We select the covering sequence whose cardinality together with $|E(x)|$ is the smallest.

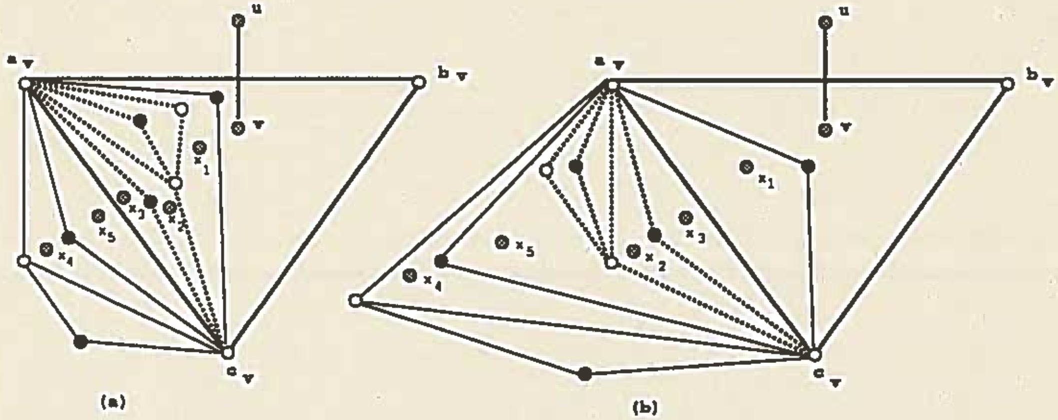


Figure 15: Equivalent covering sequences of $In_1(v)$ and $Out_1(v)$ in $E(v)$

- The exterior face is covered by a covering sequence of $Out_2(v)$, and the face corresponding to v is covered by a covering sequence of $In_2(v)$. This case is analogous to the previous case.

The smallest among the above five covers is the desired $E(v)$.

4.1.7 Final FIVC Determination

Suppose that $I(r)$, $B(r)$, $F(r)$, $L(r)$, $R(r)$, $E(r)$ have been determined for the root node r. Then the minimum cardinality perfect *FIVC* for G is the smallest of the covers $L(r)$, $R(r)$, $E(r)$.

4.2 Covering faces of partial 2-trees

The *FIVC* problem for a plane embedding of partial 2-trees is solvable in an almost identical manner as that of 2-trees. Given a partial 2-tree H, one has to produce an imbedding in a 2-tree G and then process G as for full 2-trees, with small modifications. To avoid repetition, we only will give the basic case of covering the subgraph $G(v)$, for some node v of the in-graph of G, by $I(v)$, *i.e.*, a set of vertices covering all faces of $G(v)$ except for the exterior one. We will refer to the analysis of the section 4.1.1.

Let us assume that triangles corresponding to the children $x_1, \ldots, x_k$ of v in $T(G)$ (constituting, w.l.o.g., $Out_1(v)$) have as the common base an *added* edge, (a_v, c_v), which is in G but not in H and that triangles corresponding to children $y_1, \ldots, y_l$ of v (constituting $In_2(v) || Out_2(v)$) have as the common base the other edge of the triangle v of G, (b_v, c_v), which is also in H. We will consider two cases of constructing an $I(v)$ *FIVC*, depending on the manner in which the interior face corresponding to v is covered.

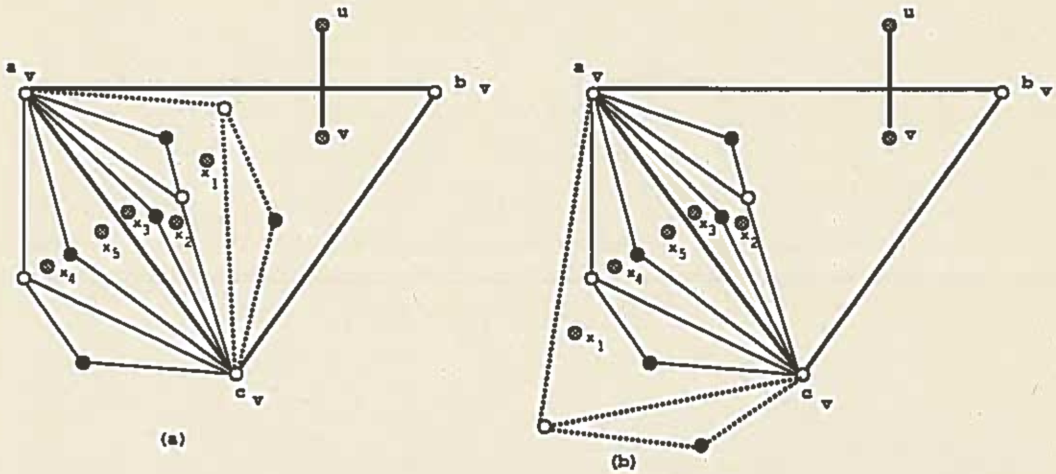


Figure 16: Equivalent covering sequences of $In_1(v)$ and $Out_1(v)$ in $E(v)$

Suppose first that the interior face corresponding to v is covered by a vertex from $G(x_k)$. Then the analysis of the covering sequence of $Out_1(v)$ is the same as in the (full) 2-tree case for $I(v)$. By considering the result of setting $Out_2'(v) = Out_2(v)||In_2(v)$ (*cf.* Figure 6), we can assume that $In_2'(v) = \emptyset$. Thus, the analysis of the covering sequence of $Out_2'(v)$ is the same as for $Out_1(v)$ in the 2-tree case.

Suppose next that the interior face corresponding to v is covered by a vertex from $G(y_1)$. We can then assume that $Out_2'(v) = \emptyset$, setting $In_2'(v)$ to $In_2(v)||Out_2(v)$ (*cf.* Figure 7). The covering sequence of $In_2'(v)$ must be the same as the covering sequence of $In_2(v)$ in the (full) 2-tree case. The covering sequence of $Out_1(v)$ is almost the same as in the $I(v)$ covering of 2-trees. The only difference is that it must end in either $B(x_k)$ or $F(x_k)$. Since every $F(x_i)$ can be replaced by $I(x_i)$ and placed in front of the sequence, we can assume that the sequence ends in $B(x_k)$. In the remainder of this section, we discuss the analysis of the covering sequence of $Out_1(v)$ in this case.

If k is even, then the feasible sequences are of the form

$$I(x_1), \ldots, I(x_i), B(x_{i+1}), E(x_{i+2}), B(x_{i+3}), E(x_{i+4}), \ldots E(x_{k-1}), B(x_k)$$

where i is odd.

To find the minimum cardinality *FIVC* we define a graph K as for the 2-tree case, with similarly weighted edges. For each choice of two vertices x_m and x_n of K, we find M_{mn}, the minimum weight perfect matching for the subgraph of K induced by the remaining vertices of K. We then select the solution that minimizes the sum of the cost of M_{mn} and $|I(x_m)| + |B(x_n)|$, and construct the minimum cardinality *FIVC* as in the 2-tree case (with x_m and x_n as the vertices x_i, and x_{i+1}, respectively).

If k is odd, the feasible sequences are similar as above but the number of initial I covers, i, is even. To find the minimum cardinality *FIVC*, one has to remove a vertex x_m of K, find the solution M_m for the matching problem for the subgraph of K induced by the remaining vertices and then join with the removed vertex as x_k, minimizing the sum of the cost of M_m and $|B(x_m)|$.

The other five types of covers for partial 2-trees can be constructed in a similar manner, based on the analyses of the full 2-tree cases.

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