Hamiltonicity of Amalgams

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Abstract

An amalgam is obtained from two Halin graphs having skirting cycles of the same length. We are interested in Hamiltonicity of amalgams constructed from two identical Halin graphs without any shift along the skirting cycle. We establish Hamiltonicity of amalagams constructed from cubic Halin graphs. We give a sufficient condition for Hamiltonicity of non-cubic amalgams and characterize infinite classes of non-Hamiltonian amalgams. We also characterize Hamiltonicity of amalgams constructed by shifting the component Halin graphs by one and of general amalgams of higher degree.

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1 Definitions and preliminary results

We consider simple graphs without self-loops. The number of neighbor vertices of a given vertex is called its *degree*. A *tree* is a connected acyclic graph, with vertices of degree greater than 1 called *internal vertices* and degree 1 vertices called *leaves*. An internal vertex with at most one internal neighbor is called a *remote vertex*. A tree with all internal vertices of the same degree d is called d-regular. A 3-regular tree is called *cubic*. In a rooted tree, the *down-degree* of an internal vertex v is the number of neighbors in the direction away from the root. Subtrees rooted in those neighbors are called *principal subtrees* of v.

A *plane* graph is a graph that has been embedded onto the plane with no crossing edges. Given a plane tree T with no degree 2 vertices, the *Halin graph* of T, $\mathcal{H}(T)$, is the graph that results from the union of T and the *skirting cycle* C on its leaf vertices that preserves their order on the periphery of T, see Figure 1(a). A *face* of a plane graph is a connected plane region bounded by edges of the graph. Two faces are *independent* if their boundaries do not share a vertex. All Halin graphs are Hamiltonian, *i.e.*, have a simple cycle that includes all vertices. Every Hamiltonian cycle in a Halin graph corresponds uniquely to a cover of all internal vertices of the graph by a collection of independent faces [6], see Figure 1(b). We call such a collection a *facial cover* of the Halin graph.

The *weak dual* of a plane graph G is the graph G^* with faces of G as its vertices, two vertices being adjacent if the corresponding faces of G share a boundary edge. An *outerplanar graph* has a planar embedding in which all vertices lie on the outer (infinite) face. A graph is *maximal outerplanar (mop)* if the addition of any edge would violate its outerplanarity.

Lemma 1. [1, 3] The weak dual graph of a Halin graph is outerplanar.

Given two plane trees with the same number of leaves and no degree 2 vertices, T_1 and T_2, and a bijection φ between their leaf sets which preserves their order in the plane, the *amalgam* $A = \mathcal{A}(T_1, T_2, \varphi)$ is the union of the corresponding Halin graphs $\mathcal{H}(T_1)$ and $\mathcal{H}(T_2)$ in which the leaf vertices v and $\varphi(v)$ are identified. Let ι be the identity mapping. Figure 2 shows the amalgam $A = \mathcal{A}(T, T, \iota)$ for the tree of the Halin graph in Figure 1(a).

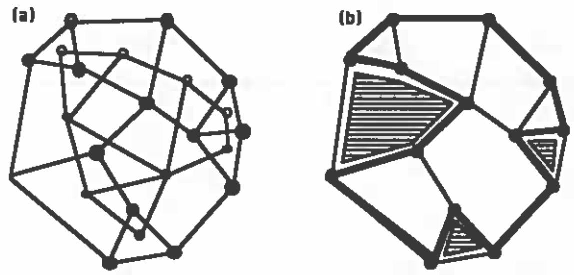

Figure 1: A Halin graph (solid vertices) and its outerplanar week dual (a), a Hamiltonian cycle and the corresponding facial cover (b).

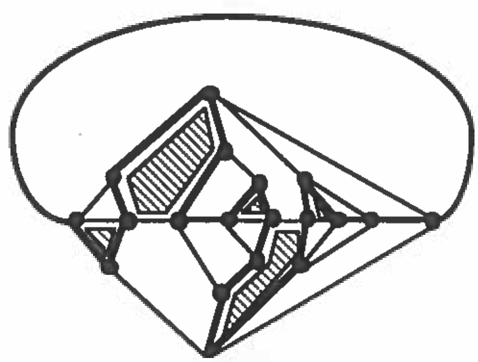

Figure 2: An amalgam and its Hamiltonian cycle resulting from the merge of two Hamiltonian cycles in the component Halin graphs.

Since all Halin graphs are Hamiltonian, it is reasonable to ask about the Hamiltonicity of amalgams. In this paper, we focus on hamiltonicity of amalgams of the form $A = \mathcal{A}(T, T, \gamma_k)$, where γ_k is a cyclic k-shift of the leaves of T, (thus, $\iota = \gamma_0$). One approach to the construction of a Hamiltonian cycle of an amalgam is by merging Hamiltonian cycles of the two component Halin graphs (their facial covers). Given Hamiltonian cycles C_1 and C_2 in the component Halin graphs of the amalgam A, we define the *merge* of the two cycles as the graph $\mathcal{M}(C_1, C_2)$ with the same vertex set as A and the edge set being the union of edges of C_1 and C_2 without the edges of the skirting cycle that belong to only one of the cycles C_1 or C_2. In general, $\mathcal{M}(C_1, C_2)$ is not necessarily a cycle. The following theorem answers the hamiltonicity question for $k = 1$ for a general tree T.

Theorem 2.[4] For any plane tree T, the amalgam $A = \mathcal{A}(T, T, \gamma_1)$ has a Hamiltonian cycle that is the merge of copies of the same Hamiltonian cycle from both component Halin graphs isomorphic to $\mathcal{H}(T)$.

Proof: No two consecutive edges of the skirting cycle are on the boundaries of faces of any given facial cover of a Halin graph. Thus, it is possible to merge instances of the same Hamiltonian cycle (corresponding to a facial cover) in the two copies of $\mathcal{H}(T)$ into a Hamiltonian cycle of A.
■

We extend this result in the next section by investigating hamiltonicity of amalgams constructed from cubic trees. In the subsequent section, we consider amalgams constructed from general trees. There, we expand on statements in [4] regarding hamiltonicity of general amalgams of high vertex degree.

Lemma 3. [4] Any amalgam with minimum vertex degree at least 4 is Hamiltonian.

For amalgams that are Eulerian, Hamiltonian cycles can be constructed from such cycles of the component Halin graphs.

Lemma 4. [4] Any Eulerian amalgam has a Hamiltonian cycle that is the merge of two Hamiltonian cycles corresponding to facial covers of the component Halin graphs.

2 Cubic trees

Our interest in amalgams constructed from cubic trees stems from the fact that any amalgam with minimum degree at least 4 is Hamiltonian [4]. We first characterize hamiltonicity of amalgams constructed from Halin graphs of identical cubic trees with small (by 0 or 1) circular shifts of vertices. Given a cubic tree T and its isomorphic copy, we will denote by $\hat{w}$ the vertex in the copy that corresponds to a vertex w in T.

Theorem 5. Given a cubic tree T, the amalgam $A = \mathcal{A}(T, T, \gamma_0)$ has a Hamiltonian cycle that results from merging two Hamiltonian cycles, one from each copy of $\mathcal{H}(T)$.

Proof: The theorem follows by induction. The smallest such amalgam $\mathcal{A}(K_{1,3}, K_{1,3}, \gamma_0)$ has a Hamiltonian cycle with no two consecutive edges on the skirting cycle corresponding to different facial covers of the component Halin graphs. Every cubic tree T of more than four vertices can be reduced to $K_{1,3}$ by a series of operations that collapse a remote vertex and its two adjacent leaves to a single leaf vertex. For a remote vertex a of T, let T' denote the cubic tree resulting from the collapse of a and its adjacent leaves p and q into the vertex $\tilde{a}$. We will show that if $A' = \mathcal{A}(T', T', \gamma_0)$ has a Hamiltonian cycle corresponding to facial covers of the component Halin graphs with no two consecutive edges on the skirting cycle, so does $A = \mathcal{A}(T, T, \gamma_0)$ (see Figure 3). Let e be the non-leaf neighbor of a in T and u and v be the neighbors of vertex $\tilde{a}$ on the skirting cycle of A'. Consider two cases of the Hamiltonian cycle of A' passing through $\tilde{a}$:

(i) $(e, \tilde{a}), (\tilde{a}, \hat{e})$. Then in $\mathcal{A}(T, T, \gamma_0)$, the corresponding Hamiltonian cycle can be obtained by replacing the two edges by the path $(e, a, p, q, \hat{a}, \hat{e})$ or the path $(e, a, q, p, \hat{a}, \hat{e})$, depending on whether the edge $(u, \tilde{a})$ or the edge $(v, \tilde{a})$ is in the Hamiltonian cycle of $\mathcal{H}(T')$ (the graph in Figure 3 illustrates the latter case);

(ii) $(u, \tilde{a}), (\tilde{a}, e)$. The corresponding Hamiltonian cycle in $\mathcal{A}(T, T, \gamma_0)$ will have the path $(u, p, \hat{a}, q, a, e)$.

■

The amalgam $A = \mathcal{A}(T, T, \gamma_0)$ can be viewed as a special case of shifting one isomorphic copy of T along the skirting cycle. We will consider the hamiltonicity question for $\mathcal{A}(T, T, \gamma_k)$, where T is cubic and γ_k is the circular shift by k. Theorems 2 and 5 have resolved the issue for $k = 0$ and 1.

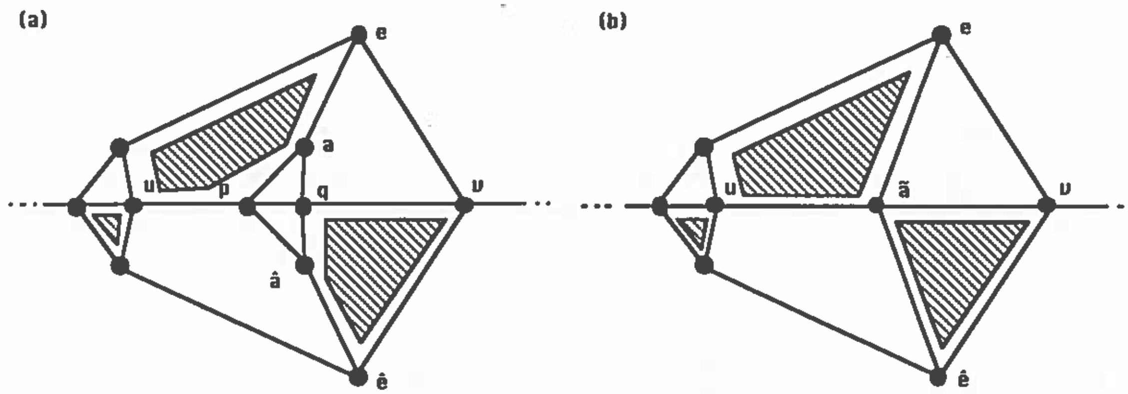

Figure 3: The amalgams A (a) and A' (b) with the corresponding facial covers.

We can view Hamiltonian cycles of cubic Halin graphs (equivalently, facial covers of its vertices) in terms of vertex colorings of its weak dual, which is maximal outerplanar.

Lemma 6. There is a bijection between the set of cubic trees, the corresponding Halin graphs, and the set of maximal outerplanar graphs.

Proof: Each mop has the associated cubic tree as its weak dual (see [1, 3]). Conversely, the weak dual graph H^* of a Halin graph $H = \mathcal{H}(T)$ is a maximal outerplanar graph; each face of H shares an edge with the outer face, and each internal face of H^* is triangular, since T is cubic.

■

Theorem 7. Every cubic Halin graph has exactly three independent facial covers.

Proof: The weak dual H^* of a cubic Halin graph H is a maximal outerplanar graph. As such, H^* has a unique 3-coloring of vertices [2]. Each monochromatic vertex set corresponds to a facial cover of H.

■

Given a cubic Halin graph H of an n-leaf tree, consider labeling the edges of the skirting cycle by the color (in the above sense) of the adjacent faces. This defines an n-character circular string $\sigma \in \{1,2,3\}^n$, such that each of the three alphabet symbols is present, and no two adjacent string characters are the same. The string σ represents the labeling of cycle edges of H. Two strings are equivalent if one can be obtained from the other by a permutation of symbols (*i.e.,* by renaming the corresponding facial covers).

For *complete* cubic trees, in which all leaves are at the same distance from the central vertex, we find an easy resolution of the hamiltonicity question for the associated amalgam.

Theorem 8. For a complete cubic tree T and any integer k, the amalgam $\mathcal{A}(T, T, \gamma_k)$ has a Hamiltonian cycle that is the result of merging two Hamiltonian cycles in the two copies of $\mathcal{H}(T)$.

Proof: It is easy to see by induction on the height of T that the string σ representing independent facial covers of $\mathcal{H}(T)$ is of the form $\langle(123)^{n/3}\rangle$. Given the cyclic periodicity of σ, we need only consider two cases, $k = 0$ and $k = 1$. These cases have been resolved in Theorems 2 and 5 for a general cubic tree T. For a complete cubic tree, any circular shift of σ allows choosing symbols i and j, $i, j \in \{1, 2, 3\}$ so that they do not appear as the same character of σ and $\gamma_k(\sigma)$, respectively. For $k = 0$, any two Hamiltonian cycles corresponding to independent facial covers i and j, $i \neq j$ will work, while for $k = 1$, we need $i \neq j - 1 \bmod 3$.

■

We observe a correspondence between restricted ternary strings (representing facial covers of a graph) and cubic Halin graphs.

Lemma 9. For every ternary string σ of length n with all three symbols present and no two adjacent characters identical, there is a Halin graph H of an n-leaf cubic tree such that σ corresponds to the labeling of cycle edges of H by the three facial coverings of H.

Proof: We will construct an n-vertex maximal outerplanar graph M with the 3-coloring of vertices defined by σ. We start with an n-cycle C colored according to σ. If $n = 3$, the only M is the triangle, K_3. For $n \geq 4$, there are at least two vertices of C of the same color $i \in \{1, 2, 3\}$. Take two non-adjacent vertices of the remaining colors and subdivide C by a chord connecting these vertices into two cycles, each of fewer than n vertices. The substrings σ_1 and σ_2 of σ defined by these two cycles have the required property and thus, by inductive assumption, correspond to mops M_1 and M_2 with the corresponding 3-colorings. The union of M_1 and M_2 identifying the end vertices of the original chord results in a mop that has a 3-coloring defined by σ. By Lemma 6, there is a unique Halin graph corresponding to M with facial covers represented by σ.

■

One might expect that for every cubic tree T (not necessarily complete) and integer k, there exists a Hamiltonian cycle in the amalgam $A = \mathcal{A}(T, T, \gamma_k)$ that is a merge of Hamiltonian cycles in the component Halin graphs. Unfortunately, it is possible to construct an n-string σ so that there exists k, $1 < k < n-1$, such that in σ and $\gamma_k(\sigma)$, for each pair $i, j \in \{1, 2, 3\}^2$ there is a position $l, 1 \leq l \leq n$ such that $(\gamma_k(\sigma)[l], \sigma[l]) = (i, j)$. This means that for any choice of a Hamiltonian cycle in each of the two component Halin graphs, there is an edge of the common skirting cycle in A that belongs to none of them. Hence, there is no Hamiltonian cycle of A that is the result of merging two Hamiltonian cycles in the component Halin graphs. A small example with $n = 18$ and $k = 2$ is given below (here, x stands for any suitable symbol, different from its neighbors):

$$\sigma = 1x1x2x2x3x3x2x1x3x$$

This example also contradicts the expectation that hamiltonicity of amalgams is similar to Halin graphs, *i.e.,* in that every Hamiltonian cycle corresponds to a facial cover of vertices. We have established, by inspection, that the amalgam corresponding to one realization of the above string σ is Hamiltonian. Thus, existence of a Hamiltonian cycle in an amalgam $\mathcal{A}(T, T, \gamma_k)$ does not depend on the ability to merge two Hamiltonian cycles in the component Halin graphs.

3 General trees

In this section, we discuss hamiltonicity of amalgams constructed from Halin graphs of trees with the minimum vertex degree of at least 4. We first generalize Theorem 5 for the case of regular trees and then investigate substructures that deny hamiltonicity of amalgams.

Theorem 10. Given a regular tree T with internal vertex degrees at least 4, the amalgam $A = \mathcal{A}(T, T, \gamma_0)$ has a Hamiltonian cycle that is the result of merging two facial covers of the component Halin graphs.

Proof: The proof is similar to that in Theorem 5. A tree T is either a *star* $K_{1,k}$ (for some $k > 3$) or can be reduced to a star by a series of operations that collapse a remote vertex and its adjacent leaves to a single leaf vertex. For a remote vertex a of T, let T' denote the tree resulting from the collapse of a and its adjacent leaves into the vertex $\tilde{a}$. We will show that if A' has a Hamiltonian cycle corresponding to facial covers of the component Halin graphs, so does A. (Note that we do not require that no two consecutive edges of the Hamiltonian cycle are on the skirting cycle C, as is necessary in the proof of Theorem 5.) We maintain the notation introduced in Figure 3 with some additional leaf vertices between p and q on C. Let e be the non-leaf neighbor of a in T, and let u and v be the neighbors on C of vertex $\tilde{a}$ on the skirting cycle of A'. Consider three cases of the Hamiltonian cycle of A' passing through $\tilde{a}$:

(i) $(e, \tilde{a}), (\tilde{a}, \hat{e})$. The corresponding Hamiltonian cycle in A can be obtained by replacing the two edges with the path $(e, a, p, ..., q, \hat{a}, \hat{e})$ or the path $(e, a, q, ..., p, \hat{a}, \hat{e})$, depending on whether the edge $(u, \tilde{a})$ or the edge $(v, \tilde{a})$ is in the Hamiltonian cycle of $\mathcal{H}(T')$;

(ii) $(u, \tilde{a}), (\tilde{a}, e)$. The corresponding Hamiltonian cycle in A will have a subpath of the form $(u, p, ..., r_i, \hat{a}, r_{i+1}, ..., q, a, e)$, where r_i and r_{i+1} are any two consecutive leaves between p and q (inclusive);

(iii) $(u, \tilde{a}), (\tilde{a}, v)$ (the additional case, not present in the proof of Theorem 5). The corresponding Hamiltonian cycle in A will have a subpath of the form $(u, p, ..., r_i, a, r_{i+1}, ..., r_j, \hat{a}, r_{j+1}, ..., q, v)$, where r_i and r_j are any two (possibly consecutive) leaves between p and q (inclusive).

The theorem follows by induction from the fact that the smallest such amalgam $\mathcal{A}(K_{1,k}, K_{1,k}, \gamma_0)$ has a Hamiltonian cycle corresponding to facial covers of the component Halin graphs.

■

We now describe construction of a Hamiltonian cycle of the amalgam $A = \mathcal{A}(T, T, \gamma_0)$ for an arbitrary tree T. We do it by arbitrarily rooting T and traversing all vertices of the corresponding subtrees in the two copies of T (symmetric about the skirting cycle). We refer to this rooted tree as $\bar{T}$. We observe that vertices of each (non-trivial) subtree S of $\bar{T}$ and its isomorphic copy are separated from the remainder of A by four vertices (called, w.l.o.g., *upper, left, right* and *lower* vertices). These vertices determine the *diamond* $d(S)$. There are only five ways that a Hamiltonian cycle of A can traverse

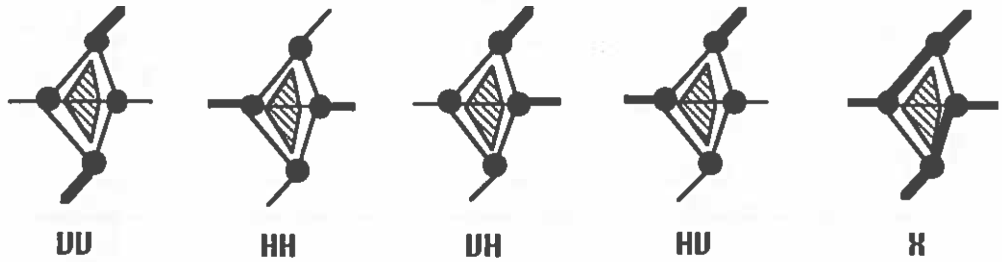

Figure 4: The five cases of a Hamiltonian cycle traversing a diamond in an amalgam.

a diamond (below, H and V signify horizontal and vertical access to the separator vertices, left or right and upper or lower, respectively), as follows:

VV: from the upper to the lower vertex,

HH: from the left to the right vertex,

VH: from the upper (or lower) vertex to the right vertex,

HV: from the left to the upper (or lower) vertex,

X: in two segments: from the left to upper and from the right to lower vertices (or *vice versa*).

We observe that there is no directionality of the Hamiltonian cycle; thus, the "from" and "to" above can be reversed. Because of the symmetry about the skirting cycle, the specification of upper and lower vertices is arbitrary.

We can now define the five traversals of diamonds for the general subtree $S = (S_1 \ldots S_d)$. The strings below use as components *local* traversals of principal subdiamonds, with "(HH)*" denoting zero or more repetitions of local HH traversal.

Lemma 11. The five traversals of the diamond for a general subtree $S = (S_1 \ldots S_d)$, $d > 1$, can only be realized by the following sequences of traversals of the principal subdiamonds $d(S_i)$:

HH: (HH)*(HV)(VV)(VH)(HH)* or (HH)*(X)(HH)*(X)(HH)* or
(HH)*(HV)(VH)(HH)*(X)(HH)* or (HH)*(X)(HH)*(HV)(VH)(HH)*
or (HH)*(HV)(VH)(HH)*(HV)(VH)(HH)*,

VH: (VV)(VH)(HH)* or (VH)(HH)*(HV)(VH)(HH)* or (VH)(HH)*(X)(HH)*,

HV: (HH)*(HV)(VV) or (HH)*(X)(HH)*(HV) or (HH)*(HV)(VH)(HH)*(HV),

X: (HH)*(X)(HH)* or (HH)*(HV)(VH)(HH)*,

VV: (VH) (HH)* (HV).

Proof: The traversals requiring entry (exit) through an upper or lower vertex are the most restrictive, as they can provide connection to only one principal subdiamond. In the case of VV, the path from the upper vertex has to pass through the first or last principal subdiamond, and symmetrically for the lower vertex. The same restriction of connecting only one principal subdiamond each with the upper and the lower vertices applies in the case of X traversal, where "vertical swings" of the path can be performed by the (X) or by the (HV)(VH) traversals of adjacent principal subdiamond, resulting in the two possible sequences.

In the case of HV (a symmetrical argument applies to VH), one of the lower and upper vertices allows two principal subdiamonds to be connected by the traversal. The left of those two can be traversed by an (X) traversal, in which case the rightmost principal subdiamond is necessarily traversed by (HV), or by an (HV) traversal, which requires in addition that the adjacent principal subdiamond be traversed by a (VH) traversal, unless it is also the rightmost principal subdiamond, in which case it has to be traversed by a (VV) traversal.

The case of HH allows the most varying subdiamond traversals. All of them share the possibility of (HH) traversals of some number of principal subdiamonds. Traversal of the upper and lower vertices of the diamond can be accomplished in "vertical swings" of the path from the skirting cycle through the sequence (HV)(VV)(VH) (both vertices traversed), an (X) traversal (accessing both vertices), or the sequence (HV)(VH) (traversing one vertex or connecting each vertex with a path segment). Combinations of the latter two lead to the four listed sequences.

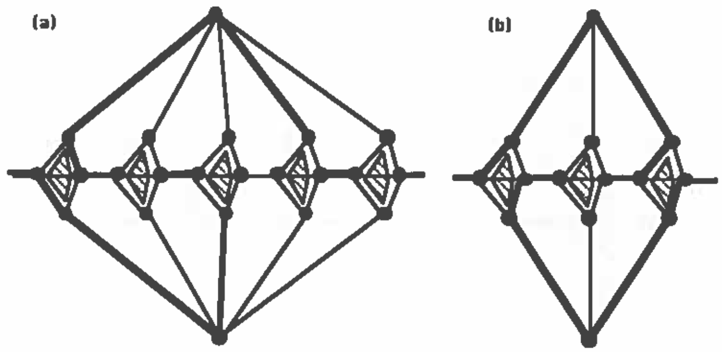

Figure 5: The sequences (X)(HH)(HV)(VH)(HH) (a) and (X)(HH)(X) (b) realizing an HH traversal of a diamond in an amalgam.

The necessity of these traversals follows by an inductive proof on the degree d of the root of subtree S. Since (HH)* traversals can be inserted between compatible traversals of two adjacent subdiamonds, we need only consider (HH)-less traversals, limiting the degree d in the traversals: to 2 in (X) and (VV), to 3 in (VH) and (HV), and to 4 in (HH). These cases can be resolved by inspection.

■

Figure 5 illustrates some of the traversals listed in the above lemma.

Lemma 12. For down-degree $d > 1$ of a remote vertex in rooted tree $\bar{T}$, all the above traversals can be realized with the exception of HH for $d = 2$. For the trivial subtree of $\bar{T}$ ($d = 1$), the traversal X cannot be realized.

Proof: The traversal X is not possible in the "trivial diamond" ($d = 1$), as it requires two disjoint paths through the diamond. In the four-vertex diamond ($d = 2$), entering at the left (right) vertex precludes traversal of the upper or lower vertex, unless the traversing path leaves the diamond through one of these vertices. This allows X, VH, HV, and VV, but not HH. In diamonds with the remote vertex having higher degree, all five traversals can be accomplished, as indicated by Lemma 11.

■

To simplify our discussion of the existence of traversals defining a Hamiltonian cycle in an amalgam, we observe the following:

Lemma 13. In every Hamiltonian amalgam $A = \mathcal{A}(T, T, \gamma_0)$, a Hamiltonian cycle must include an edge of the skirting cycle between two adjacent subtrees of T.

Proof: Assume that no edge of the skirting cycle between subdiamonds of A is traversed by any Hamiltonian cycle of A. We can therefore remove those edges; Hamiltonicity of the remaining graph would imply Hamiltonicity of $K_{2,d}$, $d > 2$, contradicting the well-known fact that such graphs are non-Hamiltonian. Thus, any Hamiltonian cycle of A must include an edge of the skirting cycle.

■

By the above lemma, we need only consider the case HH of traversing the diamond of a tree T, independent of its rooting. We must consider each edge between adjacent subtrees as the possible edge of the Hamiltonian cycle.

We can further characterize Hamiltonian amalgams with minimum vertex degree 3 by considering the location of degree 3 vertices.

Theorem 14. Any amalgam $A = \mathcal{A}(T, T, \gamma_0)$ for a tree T with no vertex of degree 3 adjacent to a leaf vertex is Hamiltonian.

Proof: We prove the theorem by showing that any diamond corresponding to a subtree T' of T can be traversed in each of the five ways introduced above. The proof will be by induction on the length h of the longest path from the root of the diamond to a vertex of the skirting cycle (*i.e.*, the *height* of the tree T'). The base case ($h = 1$, the root is a remote vertex of down-degree at least 3) can be traversed in all five ways by Lemma 11.

We now assume that for $h > 1$, the diamond corresponding to any tree of height less than h with no degree 3 vertex adjacent to a leaf can be traversed in any of the ways as above. We consider the case of a tree with height h. We consider two cases of the down-degree d of the root, as follows:

(i) $d = 2$. Each of the two principal subtrees of T' has height at least 1. By the inductive assumption, each subtree can be traversed in all ways. As such, T' is traversable as follows: (VH)(HV) realizing VV traversal of T, (X)(X) realizing HH, (VV)(VH) realizing VH, (HV)(VV) realizing HV, and (HV)(VH) realizing X.

(ii) $d \geq 3$. It suffices to consider three principal subtrees of T', any additional subtree to be traversed by HH. By the inductive assumption, the only possible hindrance to a traversal can come from the attempt to traverse the trivial subtree by X. Since one of the above three subtrees must be non-trivial (as $h > 1$) and for each of the five traversals we can find a realization that does not involve two X's, T' can be traversed in all five ways.

The Hamiltonicity of A follows from the above and Lemma 13.

■

By the above theorem, we restrict amalgams that may not be Hamiltonian to those that have degree 3 vertices adjacent to the skirting cycle. We further characterize non-Hamiltonian amalgams in the following theorem.

Theorem 15. All non-Hamiltonian amalgams of the form $\mathcal{A}(T, T, \gamma_0)$ can be defined by rooted trees with a degree 3 root, one trivial (one vertex) principal subtree and one principal subtree that does not allow any of the traversals VV, VH, HV, or X.

Proof: Consider the tree T that defines a non-Hamiltonian amalgam $A = \mathcal{A}(T, T, \gamma_0)$. Since T must have a degree 3 vertex adjacent to a leaf, we can select such a vertex of minimum height as the root of $\bar{T}$, leading to two possible configurations, depending on whether the vertex is adjacent to one or two leaves. (Not all three neighbors are leaves, since then we would have $T = K_{1,3}$ for which A is Hamiltonian.) There are only six possible HH traversals of a diamond with degree 3 root: (X)(X)(HH), (HH)(X)(X), (X)(HH)(X),(HV)(VH)(X), (HV)(VV)(VH) or (X)(HV)(VH). To deny Hamiltonicity of A in the case of two adjacent leaves, the nontrivial subtree of $\bar{T}$ must not be traversable by either VH, HV, VV or X; the two leaves do not allow an X traversal, precluding the first three HH traversals above. In the case of one adjacent leaf, the vertices of one of the subtrees of $\bar{T}$ can be traversed in any manner; the other non-trivial subtree again must not allow any of the four traversals.

■

4 Non-Hamiltonian amalgams

Lastly, we consider several small examples of non-Hamiltonian amagams $\mathcal{A}(T, T, \gamma_0)$. We focus on the structure of tree T, rooted at some arbitrary vertex, remembering there is an isomorphic copy of T intersecting at a skirting cycle in the associated amalgam, as described above. We will consider

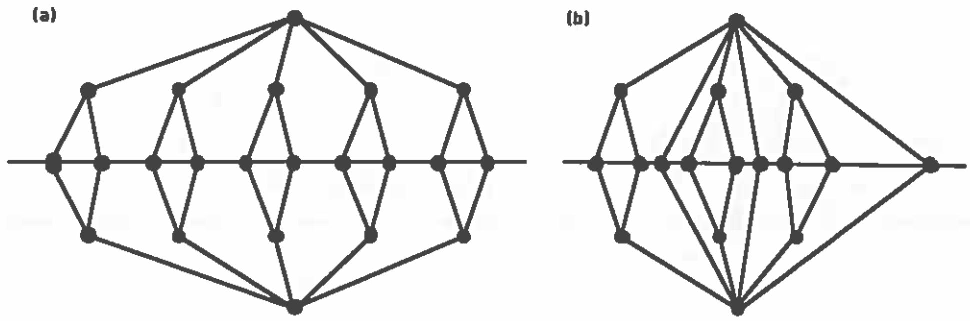

Figure 6: Two small examples of non-Hamiltonian amalgams $\mathcal{A}(T, T, \gamma_0)$.

trees T of limited height. By definition of an amalgam, the root of tree T must be of degree 3 or higher. Recall that the amalgam associated with a tree T is Hamiltonian if and only if the diamond corresponding to a tree isomorphic to T as the plane rooted tree is HH traversable.

First, we consider trees T of height 1. By our previous discussion, we see that all relevant amalgams are Hamiltonian, as we can avoid any impossible, local X traversal of a trivial subtree to realize an HH traversal.

Next, we consider trees T of height 2. We classify the subtrees of the root as trivial (a single vertex on the skirting cycle, denying a local X traversal), of down-degree 2 (denying a local HH traversal), or of higher down-degree (allowing any traversal). By definition, a tree T of height 2 has at least one non-trivial subtree. For a root of degree at most 4, all associated amalgams are Hamiltonian, as there exist HH traversals not requiring a local X or HH traversal of any subtree. When we consider a degree 5 vertex, any HH traversal of tree T must involve a local HH traversal of at least one subtree. Thus, if all subtrees are of down-degree 2, the associated amalgam is not Hamiltonian. Figure 6(a) presents this non-Hamiltonian amalgam. If any of the subtrees is trivial or has a root of higher down-degree, that subtree can be HH traversed first, reducing the problem to one of down-degree 4, yielding an amalgam that is Hamiltonian.

Next, consider T of height 2 with root of degree 6 or higher. We see that any HH traversal will require two or more HH traversals. If the root has at least three subtress with roots of down-degree 2 that are separated by at least one arbitrary subtree regardless of rotation, then the associated amalgam is not Hamiltonian. There exists no HH traversal of T, as it would require an

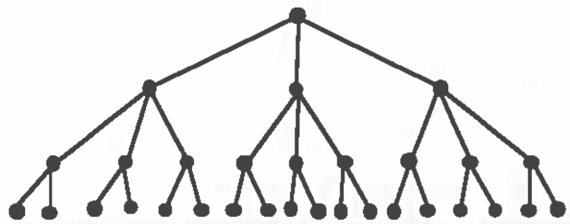

Figure 7: A tree T of height 3 resulting in a non-Hamiltonian amalgam $\mathcal{A}(T, T, \gamma_0)$.

HH traversal of a subtree with down-degree 2, which is impossible. Figure 6(b) shows the smallest non-Hamiltonian amalgam $\mathcal{A}(T, T, \gamma_0)$ based on one of these trees T.

Finally, we consider trees of greater height. One set of non-Hamiltonian amalagams is generated according to the last pattern noted above. As long as such amalgams have three separated subtrees with roots of down degree 2 adjacent to the skirting cycle, they are not Hamilitonian. We find another tree T of height 3 having a root of degree 3, as shown in Figure 7. Each subtree of T precludes traversal by X or VV. Thus, there are no HH traversals of T, and the associated amalgam is not Hamiltonian. One can add an arbitrary principal subtree to this structure and maintain the non-Hamiltonicity.

5 Conclusions

In this paper, we have characterized several classes of Hamiltonian and non-Hamiltonian amalgams. We establish Hamiltonicity of all amalgams in which all vertices are of degree 4 or more. All amalagams of the form $\mathcal{A}(T, T, \gamma_1)$ are Hamiltonian. For the case of $\mathcal{A}(T, T, \gamma_0)$, we establish Hamiltonicity of amalgams based on cubic trees T. We also characterize amalgams $\mathcal{A}(T, T, \gamma_0)$ that are not Hamiltonian and provide several small examples. A challenging research goal remains to find a complete set of smallest examples of non-Hamiltonian amalgams that are substructures of all such amalgams.

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