Axioms for Domain Theory and for Modeling Denotational Semantics

—Part I—

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The Axiom List

Axiom I: Approximating

Axiom II: Lowering

Axiom III: Pairing

Axiom IV: Mapping

These axioms refer to the structure of the ***semilattice*** underlying our ***universal domain.***

Approximating

$$0 < 1$$

$$0 \leqq p \leqq 1$$

$$p \leqq p$$

$$p \leqq q \;\&\; q \leqq r \Rightarrow p \leqq r$$

$$p \leqq q \;\&\; q \leqq p \Rightarrow p = q$$

$$p < q \Leftrightarrow p \leqq q \;\&\; \neg\, q \leqq p$$

$$p \vee q \leqq r \Leftrightarrow p \leqq r \;\&\; q \leqq r$$

0 means ***no*** information.
1 means ***too much*** information.
$p \leqq q$ means p's information ***approximates*** q's.
$p \vee q$ means ***joining*** information.

Ideal & Domain Completion

$$\mathbf{Id}(S) = \{X \subseteq S \mid 0 \in X \;\&\; \forall p,q \in S\,[(p \vee q) \in X \Leftrightarrow p \in X \;\&\; q \in X\,]\}$$

$$\mathbf{Dom}(S) = \{X \in \mathbf{Id}(S) \mid 1 \notin X\}$$

$$\mathbf{Id}(S) = \mathbf{Dom}(S) \cup \{S\}$$

Theorem: Every ***directed*** subset of $\mathbf{Dom}(S)$ has a least upper bound in $\mathbf{Dom}(S)$.

Definition: $\downarrow p = \{q \mid q \leqq p\}$ and $\downarrow X = \bigcup\{\downarrow p \mid p \in X\}$

Theorem: For $X, Y \in \mathbf{Id}(S)$, we have $X = \bigvee\{\downarrow p \mid p \in X\}$, and $X \vee Y = \downarrow\{p \vee q \mid p \in X \;\&\; q \in Y\}$.

Lowering & Pairing

$$p^* < 1 \;\&\; 0^* = 0$$

$$(p \vee q)^* = p^* \vee q^*$$

$$p^* = q^* \Rightarrow p = q$$

$$\langle\!\langle 0,0 \rangle\!\rangle = 0$$

$$\langle\!\langle p,q \rangle\!\rangle = 1 \Leftrightarrow p = 1 \text{ or } q = 1$$

$$\langle\!\langle p,q \rangle\!\rangle \vee \langle\!\langle p',q' \rangle\!\rangle = \langle\!\langle p \vee p', q \vee q' \rangle\!\rangle$$

$$\langle\!\langle p',q' \rangle\!\rangle < 1 \Rightarrow$$

$$\langle\!\langle p,q \rangle\!\rangle \leqq \langle\!\langle p',q' \rangle\!\rangle \Leftrightarrow p \leqq p' \,\&\, q \leqq q'$$

$$p^* > 0 \;\&\; \langle\!\langle q,r \rangle\!\rangle > 0 \Rightarrow p^* \vee \langle\!\langle q,r \rangle\!\rangle = 1$$

Mapping

$$(p \to q) \text{ defined} \Leftrightarrow p < 1$$

$$\bigvee_{i<k}(p_i \to q_i) = 1 \Leftrightarrow \exists\, r < 1.\ \bigvee\{q_i \mid p_i \leqq r\} = 1$$

$$\bigvee_{i<k}(p_i \to q_i) < 1 \Rightarrow$$

$$(r \to s) \leqq \bigvee_{i<k}(p_i \to q_i) \Leftrightarrow s \leqq \bigvee\{q_i \mid p_i \leqq r\}$$

$$p^* > 0 \ \&\ (q \to r) > 0 \Rightarrow p^* \vee (q \to r) = 1$$

$$\langle\!\langle p,q \rangle\!\rangle > 0 \ \&\ (r \to s) > 0 \Rightarrow \langle\!\langle p,q \rangle\!\rangle \vee (r \to s) = 1$$

$(p \to q)$ symbolizes a ***one-step function*** that takes the value q as soon as an input is ***approximated*** by p.

Joins of functions will evaluate ***argumentwise.***

A Universal Semilattice

Definition: We take U as the ***minimal solution*** of:

$$U = U^* \cup (U \times U) \cup (U \to U).$$

There are many ways of defining ***subsemigroups***:

$$S^* = \{1\} \cup \{p^* \mid p \in S\}$$

$$S \times T = \{\langle\!\langle p,q \rangle\!\rangle \mid p \in S \,\&\, q \in T\}$$

$$S \to T = \{\bigvee_{i<k}(p_i \to q_i) \mid k \in N \,\&\, \forall i < k.\ p_i \in S \setminus \{1\} \,\&\, q_i \in T\}$$

Note: The three parts of U are each ***subsemigroups*** and intersect each other only in $\{0,1\}$.

Domain Isomorphism

Definition: A mapping F: **Dom**$(S) \longrightarrow$ **Dom**(T) of one domain ***onto*** another is an ***isomorphism*** iff

$$\forall P_0, P_1 \in \textbf{Dom}(S)\ [\ P_0 \subseteq P_1 \Leftrightarrow F(P_0) \subseteq F(P_1)\ ].$$

Write: $S \cong T$ to mean **Dom**$(S) \cong$ **Dom**(T).

Question: Does this also mean *semilattice* isomorphism?

Theorem: Some very basic properties:

(i) $((R \times S) \times T) \cong (R \times (S \times T))$; and

(ii) $((R \times S) \to T) \cong (R \to (S \to T))$; and

(iii) $(R \to (S \times T)) \cong (R \to S) \times (R \to T)$.

Question: What does it take to *prove* these?

A Universal Domain

$$\mathbf{Sub}(U) = \{S \subseteq U \mid 0 \in S \,\&\, 1 \in S \,\&\, \forall p,q \in S.\ (p \vee q) \in S\}$$

Theorem: $\mathbf{Sub}(U)$ can be considered as the ***domain completion*** of $\mathbf{Fin}(\mathbf{Sub}(U)) \cup \{U\}$.

$$\underset{\text{New}}{\mathbf{Id}(S)} = \{\downarrow X \mid X \in \underset{\text{Old}}{\mathbf{Id}_0(S)}\} \subseteq \mathbf{Id}(U) \text{ and } \underset{\text{New}}{\mathbf{Dom}(S)} = \mathbf{Id}(S) \setminus \{U\}.$$

Theorem: For $S,T \in \mathbf{Sub}(U)$ we have:

$$\mathbf{Dom}(S^*) \cong \mathbf{Id}(S) \cong \mathbf{Id}_0(S) \text{ and}$$

$$\mathbf{Dom}(S \times T) \cong \mathbf{Dom}(S) \times \mathbf{Dom}(T).$$

For $P \in \mathbf{Id}(S)$ we have $P^* = \{p^* \mid p \in P\} \in \mathbf{Dom}(S^*)$.

For $P \in \mathbf{Dom}(S)$ and $Q \in \mathbf{Dom}(T)$ we have
$\langle\!\langle P,Q \rangle\!\rangle = \{\langle\!\langle p,q \rangle\!\rangle \mid p \in P \,\&\, q \in Q\} \in \mathbf{Dom}(S \times T)$.

Continuous Functions

Definition: Functions $F: \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$ are ***continuous***

iff $\forall P \in \mathbf{Dom}(S) \forall q \in T[\, q \in F(P) \Leftrightarrow \exists p \in P.\, q \in F(\downarrow p)\,]$

iff $\forall P \in \mathbf{Dom}(S).\ F(P) = \bigvee\{F(\downarrow p) \mid p \in P\}.$

A finite amount of information about an output is already exactly determined by a finite amount of information about the input.

Theorem: Isomorphisms between domains ***are*** continuous.

Definition: For $p \in S\backslash\{1\}$, $q \in T\backslash\{1\}$ and $P \in \mathbf{Dom}(S)$,

$${}_S[p \to q]_T(P) = \bigvee\{\downarrow q \mid p \in P\} \in \mathbf{Dom}(T).$$

Theorem: ${}_S[p \to q]_T: \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$ ***is*** continuous; and for every continuous F, as an argumentwise l.u.b.,

$$F = \bigvee\{{}_S[p \to q]_T \mid q \in F(\downarrow p)\}.$$

Function Spaces

Definition: For continuous F,G: **Dom**(S) ⟶ **Dom**(T),
write F ≦ G ⇔ ∀ P ∈ **Dom**(S). F(P) ≦ G(P).

Theorem: In view of the ***structure*** of the universal semilattice U, we have for S,T ∈ **Sub**(U),
the ***space*** of continuous F: **Dom**(S) ⟶ **Dom**(T)
is ***isomorphic*** to **Dom**(S→T).

As a corollary we see that the universal domain **Dom**(U) ***contains copies of the product spaces and function spaces for all of the subsemilattices in*** **Sub**(U) ***including*** U ***itself.***

End of Part I

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Representing Integers

Definition: For $n \in \mathbb{N}$ define recursively:

$$[0] = 1^* \quad \text{and} \quad [n+1] = \langle\!\langle [n], [0] \rangle\!\rangle.$$

Theorem: For all $n, m \in \mathbb{N}$:

$$0 < [n] < 1, \text{ and } n < m \Rightarrow [n] \vee [m] = 1.$$

Definition: $[\mathbb{N}] = \{0, 1\} \cup \{[n] \mid n \in \mathbb{N}\}$.

Theorem: **Dom**$([\mathbb{N}]) = \{\downarrow 0\} \cup \{\downarrow [n] \mid n \in \mathbb{N}\}$.

Question: How to interpret **Dom**$([\mathbb{N}] \to [\mathbb{N}])$?

Computable Domains and Mappings

Definition. The ***computable*** elements of **Sub**(U) are those that are ***recursively enumerable*** subsets of U.

Definition. The ***computable*** elements of **Dom**(S) are those that are ***recursively enumerable*** subsets of U.

Definition. The ***computable*** mappings F: **Dom**(S) ⟶ **Dom**(T) are those that are continuous and where the set $\{(p \to q) \mid q \in F(\downarrow p)\}$ is a ***recursively enumerable*** subset of U.

Theorem: ***Compositions*** of computable mappings are computable, and the ***application*** of a computable mapping to a computable argument is computable.

A Review of Semilattice Constructs

$$H = \{0\} \cup \{\langle\!\langle p,q \rangle\!\rangle \mid p > 0 \,\&\, q > 0\}$$

$$S^* = \{1\} \cup \{p^* \mid p \in S\}$$

$$S_\perp = \{0\} \cup \{\langle\!\langle p,[0] \rangle\!\rangle \mid p \in S\}$$

$$S \times T = \{\langle\!\langle p,q \rangle\!\rangle \mid p \in S \,\&\, q \in T\}$$

$$S \times_S T = \{\langle\!\langle p,q \rangle\!\rangle \mid p \in S \,\&\, q \in T\} \cap H$$

$$S + T = \{0\} \cup \{\langle\!\langle p,[0] \rangle\!\rangle \mid p \in S\} \cup \{\langle\!\langle q,[1] \rangle\!\rangle \mid q \in T\}$$

$$S +_C T = (\{0\} \cup \{\langle\!\langle p,[0] \rangle\!\rangle \mid p \in S\} \cup \{\langle\!\langle q,[1] \rangle\!\rangle \mid q \in T\}) \cap H$$

$$S \to T = \{\bigvee_{i<k}(p_i \to q_i) \mid k \in N \,\&\, \forall i<k.\, p_i \in S \setminus \{1\} \,\&\, q_i \in T\}$$

$$S \to_S T = \{0,1\} \cup \{\bigvee_{i<k}(p_i \to q_i) \mid k \in N \,\&\, \forall i<k.\, p_i \in S \setminus \{0,1\} \,\&\, q_i \in T \setminus \{0,1\}\}$$

Theorem: All these operations on **Sub**(U) are ***continuous*** and ***computable.***

Strict Functions

Definition: A continuous $F: \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$ is said to be ***strict*** if $F(\downarrow 0) = \downarrow 0$.

Theorem: The space of strict $F: \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$ is isomorphic to $\mathbf{Dom}(S \to_s T)$.

The computable elements of $\mathbf{Dom}([N] \to_s [N])$ correspond exactly to the classical ***partial recursive functions***.

Least Fixed Points

Theorem: Every continuous F: **Dom**$(S) \longrightarrow$ **Dom**(S) has a ***least fixed point***; and if F is ***computable***, then so is this fixed point.

Proof: Note that continuous functions are such that whenever $P_0 \leqq P_1 \leqq P_2 \leqq P_3 \leqq \ldots \leqq P_n \leqq P_{(n+1)} \leqq \ldots$ in **Dom**(S), then $F(\bigvee\{P_n \mid n \in N\}) = \bigvee\{F(P_n) \mid n \in N\}$. Hence,

$$\textbf{lfp}(F) = \bigvee\{F^n(\downarrow 0) \mid n \in N\}.$$

And, if F is computable, then so is **lfp**(F).

Note: **lfp** is also a ***computable operator***.

Applied to **Dom**$([N]\to_s[N])$ ***least fixed points give us the recursive definitions needed for partial recursive functions.***

Applied to **Sub**(U) ***we get domain equations and recursive definitions of computable domains.***

Some Domain Equations

There is a ***computable*** $V = V^* +_C ((V \times V) +_C (V \to V))$,
and we have $U \cong V$.

Another computable example:

Let $V = V \times (U \times V)$. Let $C = (U \times V)$.

Then $C \cong (C \times C)$.

Let $D = (D \to C)$.

Then $D \cong (D \times D)$.

Finally $D \cong (D \to D)$.

D must be ***computable***.

Note: **Dom**(U) is isomorphic to a ***subdomain*** of **Dom**(D).

Modeling λ-Calculus

Definition: The ***ideal generated by*** $V \subseteq U$ is defined by:

$$\mathbf{id}(V) = \downarrow\{\bigvee\{p_i \mid i<k\} \mid k \in N \;\&\; \forall i<k.\; p_i \in V\}.$$

Note: We always have $\mathbf{id}(V) \in \mathbf{Id}(U)$, but it is ***not*** always true that $\mathbf{id}(V) \in \mathbf{Dom}(U)$, even though $1 \notin V$.

Definition: ***Functional application*** for $F,X \in \mathbf{Dom}(U)$, is defined by: $F(\!(X)\!) = \mathbf{id}(\{q \mid \exists\, p \in X.\; (p \to q) \in F\})$.

Definition: λ***-Abstraction*** for continuous $\Phi: \mathbf{Dom}(U) \longrightarrow \mathbf{Dom}(U)$ is defined by: $\lambda X.\Phi(X) = \mathbf{id}(\{(p \to q) \mid q \in \Phi(\downarrow p) \;\&\; p<1\})$.

Theorem: The ***values*** of the applications and abstractions belong to $\mathbf{Dom}(U)$, and the ***operations*** respect continuity and computability.

Moreover, β***-conversion*** holds:

$$(\lambda X.\Phi(X))(\!(P)\!) = \Phi(P)$$

A Simple Recursion

$$\mathbf{zero} = \lambda X.\lambda P.\lambda Q.\ \mathbf{id}(\{p \in P \mid [0] \in X\} \cup \{q \in Q \mid \langle\!\langle 0,[0] \rangle\!\rangle \in X\})$$

$$\mathbf{pred} = \lambda X.\ \mathbf{id}(\{p \mid \langle\!\langle p,[0] \rangle\!\rangle \in X\})$$

$$\mathbf{succ} = \lambda X.\ \mathbf{id}(\{\langle\!\langle q,[0] \rangle\!\rangle \mid q \in X\})$$

$$\mathbf{plus} = \lambda X.\lambda Y.\ \mathbf{zero}(X)(Y)(\mathbf{succ}(\mathbf{plus}(\mathbf{pred}(X))(Y)))$$

$$\mathbf{tims} = \lambda X.\lambda Y.\ \mathbf{zero}(X)(\downarrow[0])(\mathbf{plus}(\mathbf{tims}(\mathbf{pred}(X))(Y))(Y))$$

OK, now ***you*** define **fact**,
(and other p.r. functions).

Partial Equivalence Relations

Definition: For $P,Q \in \mathbf{Dom}(U)$, let

$$\langle\!\langle P,Q \rangle\!\rangle = \downarrow\{ \langle\!\langle p,q \rangle\!\rangle \mid p \in P \;\&\; q \in Q \} \in \mathbf{Dom}(U \times U).$$

Definition: In **PER**(U), the collection of ***partial equivalence relations*** over **Dom**(U), are the subsets $\mathcal{A} \subseteq \mathbf{Dom}(U \times U)$, where, for all elements $P,Q,R \in \mathbf{Dom}(U)$,

$$\langle\!\langle P,Q \rangle\!\rangle \in \mathcal{A} \Rightarrow \langle\!\langle Q,P \rangle\!\rangle \in \mathcal{A} \text{ and}$$

$$\langle\!\langle P,Q \rangle\!\rangle \in \mathcal{A} \;\&\; \langle\!\langle Q,R \rangle\!\rangle \in \mathcal{A} \Rightarrow \langle\!\langle P,R \rangle\!\rangle \in \mathcal{A}.$$

As a shorthand we write: $P \mathcal{A} Q \Leftrightarrow \langle\!\langle P,Q \rangle\!\rangle \in \mathcal{A}$.

The ***range*** of $\mathcal{A}$ is denoted by: $\mathbf{Rng}(\mathcal{A}) = \{ P \mid P \mathcal{A} P \}$.

***The idea of a* PER *is to give a new kind of element somehow constructed from* U.**

PERs as Types

Definition: The ***product*** of two PERs is defined by:

$$P_0\ (\mathcal{A} \times \mathcal{B})\ P_1 \Leftrightarrow \exists\ X_0, Y_0, X_1, Y_1\ [\ P_0 = \langle\!\langle X_0, Y_0 \rangle\!\rangle\ \&\ P_1 = \langle\!\langle X_1, Y_1 \rangle\!\rangle\ \&\ X_0\ \mathcal{A}\ X_1\ \&\ Y_0\ \mathcal{B}\ Y_1\ ]$$

Definition: The ***sum*** of two PERs is defined by:

$$P_0\ (\mathcal{A} + \mathcal{B})\ P_1 \Leftrightarrow \exists\ X_0, X_1 [\ P_0 = \langle\!\langle \downarrow [0], X_0 \rangle\!\rangle\ \&\ P_1 = \langle\!\langle \downarrow [0], X_1 \rangle\!\rangle\ \&\ X_0\ \mathcal{A}\ X_1\ ]$$
$$\text{or } \exists\ Y_0, Y_1 [\ P_0 = \langle\!\langle \downarrow [1], Y_0 \rangle\!\rangle\ \&\ P_1 = \langle\!\langle \downarrow [1], Y_1 \rangle\!\rangle\ \&\ Y_0\ \mathcal{B}\ Y_1\ ]$$

Definition: The ***function space*** of two PERs is defined by:

$$F_0\ (\mathcal{A} \to \mathcal{B})\ F_1 \Leftrightarrow \forall\ X_0, X_1\ [\ X_0\ \mathcal{A}\ X_1 \Rightarrow F_0(\!|X_0|\!)\ \mathcal{B}\ F_1(\!|X_1|\!)\ ].$$

Definition: A ***mapping*** $F: \mathcal{A} \longrightarrow \mathcal{B}$ between two PERs is defined as an element $F \in$ **Dom**(U) where $F\ (\mathcal{A} \to \mathcal{B})\ F$ (i.e., $F \in$ **Rng**$(\mathcal{A} \to \mathcal{B})$).

Isomorphisms of PERs

Definition: Two PERS $\mathcal{A}$ and $\mathcal{B}$ are said to be ***isomorphic*** (in symbols, $\mathcal{A} \cong \mathcal{B}$) provided there are mappings

$$F: \mathcal{A} \longrightarrow \mathcal{B} \text{ and } G: \mathcal{B} \longrightarrow \mathcal{A} \text{ where}$$

$$\forall X \in \mathbf{Rng}(\mathcal{A}).\, X\, \mathcal{A}\, G(\!(F(\!(X)\!)\!) \;\&\; \forall Y \in \mathbf{Rng}(\mathcal{B}).\, Y\, \mathcal{B}\, F(\!(G(\!(Y)\!)\!).$$

Theorem: For all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{PER}(U)$, we have:

$$\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A} \text{ and } \mathcal{A} + \mathcal{B} \cong \mathcal{B} + \mathcal{A}\text{; and}$$

$$((\mathcal{A} \times \mathcal{B}) \to \mathcal{C}) \cong (\mathcal{A} \to (\mathcal{B} \to \mathcal{C}))\text{; and}$$

$$(\mathcal{A} \to (\mathcal{B} \times \mathcal{C})) \cong (\mathcal{A} \to \mathcal{B}) \times (\mathcal{A} \to \mathcal{C})\text{; and}$$

$$((\mathcal{A} + \mathcal{B}) \to \mathcal{C}) \cong (\mathcal{A} \to \mathcal{C}) \times (\mathcal{B} \to \mathcal{C})\text{;}$$

Question: Is it possible there are also ***associative*** and ***distributive*** laws for + and × on PERs?

Dependent Types

Definition: For $\mathcal{A} \in \mathbf{PER}(U)$, an ***$\mathcal{A}$-indexed family*** of PERs is a function $\mathcal{B}: \mathbf{Dom}(U) \longrightarrow \mathbf{PER}(U)$ where we have

$$\forall\, X_0, X_1\, [\, X_0\, \mathcal{A}\, X_1 \Rightarrow \mathcal{B}(X_0) = \mathcal{B}(X_1)\,].$$

Definition: The ***dependent product*** of an ***$\mathcal{A}$-indexed family*** of PERs is defined as a relation on $\mathbf{Dom}(U)$ by:

$$F_0\, (\textstyle\prod X{:}\mathcal{A}.\, \mathcal{B}(X))\, F_1 \Leftrightarrow \forall\, X_0, X_1\, [\, X_0\, \mathcal{A}\, X_1 \Rightarrow F_0\langle\!\langle X_0 \rangle\!\rangle\, \mathcal{B}(X_0)\, F_1\langle\!\langle X_1 \rangle\!\rangle\,].$$

Definition: The ***dependent sum*** of an ***$\mathcal{A}$-indexed family*** of PERs is defined as a relation on $\mathbf{Dom}(U)$ by:

$$P_0\, (\textstyle\sum X{:}\mathcal{A}.\, \mathcal{B}(X))\, P_1 \Leftrightarrow$$
$$\exists\, X_0, Y_0, X_1, Y_1\, [\, P_0 = \langle\!\langle X_0, Y_0 \rangle\!\rangle\, \&\, P_1 = \langle\!\langle X_1, Y_1 \rangle\!\rangle\, \&\, X_0\, \mathcal{A}\, X_1\, \&\, Y_0\, \mathcal{B}(X_0)\, Y_1\,]$$

Systems of Dependent Types

Example: We will say that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ form a ***system*** of dependent types if:

$$\forall\, X_0, X_1\ [\ X_0\ \mathcal{A}\ X_1 \Rightarrow \mathcal{B}(X_0) = \mathcal{B}(X_1)\ ] \text{ and}$$

$$\forall\, X_0, X_1, Y_0, Y_1\ [\ X_0\ \mathcal{A}\ X_1\ \&\ Y_0\ \mathcal{B}(X_0)\ Y_1 \Rightarrow \mathcal{C}(X_0, Y_0) = \mathcal{C}(X_1, Y_1)\ ] \text{ and}$$

$$\forall\, X_0, X_1, Y_0, Y_1, Z_0, Z_1\ [\ X_0\ \mathcal{A}\ X_1\ \&\ Y_0\ \mathcal{B}(X_0)\ Y_1\ \&\ Z_0\ \mathcal{C}(X_0, Y_0)\ Z_1 \Rightarrow \mathcal{D}(X_0, Y_0, Z_0) = \mathcal{D}(X_1, Y_1, Z_1)\ ],$$

where $\mathcal{A} \in$ **PER**(U), and $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are mappings on **Dom**(U) to **PER**(U) with the indicated number of arguments.

Theorem: Under the above assumptions on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, the following compound is indeed in **PER**(U):

$$\prod X{:}\mathcal{A}.\ \sum Y{:}\mathcal{B}(X).\ \prod Z{:}\mathcal{C}(X,Y).\ \mathcal{D}(X,Y,Z).$$

Identity Types

Definition: The ***identity type*** over $\mathcal{A} \in$ **PER**(U) is defined as:

$$Z\ [\ X \equiv_{\mathcal{A}} Y\ ]\ W \Leftrightarrow Z\ \mathcal{A}\ X\ \mathcal{A}\ Y\ \mathcal{A}\ W.$$

Example: Given $F\ (\mathcal{A} \to (\mathcal{A} \to \mathcal{A}))\ F$, we can think of F as a ***binary operation*** on **Rng**($\mathcal{A}$). Then this PER is ***non-empty***:

$$\prod X{:}\mathcal{A}.\prod Y{:}\mathcal{A}.\prod Z{:}\mathcal{A}.[\ F(\!(X)\!)(\!(F(\!(Y)\!)(\!(Z)\!))\!) \equiv_{\mathcal{A}} F(\!(F(\!(X)\!)(\!(Y)\!))\!)(\!(Z)\!)\ ],$$

we then have "evidence" that – up to the equivalence $\mathcal{A}$ – the operation F is ***associative***.

End of Part II