NOTES ON AXIOMS FOR DOMAIN THEORY AND FOR MODELING DENOTATIONAL SEMANTICS

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BACKGROUND: A countably based ***Scott-Ershov domain*** can be characterized as the "completion" by ***proper*** ideals of a countable semilattice. (This will all be explained below.) Different semilattices give differently structured domains, and there are several kinds of "universal" domains which enable us to realize a variety of structures among their subdomains.

The purpose of this report is to axiomatize the construction of ***one kind*** of universal domain in order to show how its subdomains can be found via choosing subsemilattices of a master semilattice by very elementary definitions and via easy solutions to "domain equations".

CHAPTER I. SEMILATTICES

DISCUSSION: A ***semilattice*** is a system $\langle S,0,1,\leqq,\vee\rangle$ (with a set of elements and with special elements, relations, and operations) which can be defined in words as a ***bounded partial ordering where any pair of elements has a least upper bound.***

Intuitively, a semilattice should be interpreted as giving "finite" amounts of "information" encoded in each of its elements. The element 0 stands for "no information"; the element 1 stands for "too much information" (i.e., "broken" or "inconsistent" information).

A relationship $p \leqq q$ stands for the information in p being "contained in" the information in q, or, as we shall sometimes say, p "approximates" q. Finally $p \vee q$ stands for the "join" or "union" of the information of p with that of q.

The extra, defined relationship $p < q$ stands for p having "strictly less" information than q. By convention we suppose $0 < 1$ in order to avoid trivial structures.

Symbolically, the axioms can be presented in various equivalent ways. The first is our preferred version.

DEFINITION: Structures satisfying the following axiom set are called ***semilattices***. (The conditions are assumed to hold for ***all*** elements of the semilattice. The symbol ¬ means "not".)

AXIOM I: Approximating

$0 < 1$

$0 \leqq p \leqq 1$

$p \leqq p$

$p \leqq q \ \&\ q \leqq r \ \Rightarrow\ p \leqq r$

$p \leqq q \ \&\ q \leqq p \ \Rightarrow\ p = q$

$p < q \ \Leftrightarrow\ p \leqq q \ \&\ \neg\, q \leqq p$

$p \vee q \leqq r \ \Leftrightarrow\ p \leqq r \ \&\ q \leqq r$

EXERCISE: Prove that the axiom set below is ***equivalent*** to the one above:

$0 < 1$

$0 \vee p = p$

$1 \vee p = 1$

$p \vee p = p$

$p \vee q = q \vee p$

$p \vee (q \vee r) = (p \vee q) \vee r$

$$p \leqq q \iff p \vee q = q$$

$$p < q \iff p \leqq q \ \& \ \neg\, q \leqq p$$

EXERCISE: (i) Show that semilattices can also be axiomatized in ***first-order logic*** using $\leqq$ as the *only* non-logical symbol. (ii) What is different in this approach from the first two axiom sets above?

EXERCISE: (i) Give an example of an infinite structure that is a semilattice with respect to $\leqq$ ***and*** the converse relation $\geqq$ at the same time. (Such structures are called ***lattices***.) (ii) Give an example that is not. (iii) Show that a finite semilattice, however, is ***always*** a lattice.

DISCUSSION: Sometimes "broken" elements need to be looked at and computed with. Now any semilattice can have an "extra" 1 joined "at the top" of the partial ordering. In our universal semilattice which we will construct below, we make the adjunction by "lowering" a copy of the ***whole semilattice*** with the aid of a new unary operation.

AXIOM II: Lowering

$$p^* < 1$$

$$0^* = 0$$

$$(p \vee q)^* = p^* \vee q^*$$

$$p^* = q^* \Rightarrow p = q$$

EXERCISE: What is the simplest example of a semilattice that has a lowering operation?

EXERCISE: Let S be the set of all ***infinite sequences*** p of 0's and 1's (that is, $p = \langle p_i \mid i \in N \rangle$, where each $p_i = 0$ or 1, and N is the set of non-negative integers). Define

$$p \vee q = \langle p_i \vee q_i \mid i \in N \rangle.$$

(i) Show S is a semilattice. (ii) Is S a lattice?
(iii) Would S always be a lattice even if the p_i are chosen from a given semilattice A instead of just from {0,1}?
(iv) Does S have a lowering operations?

DEFINITION: If P and Q are two semilattices, their ***cartesian product*** is defined as the set $(P \times Q)$ of all ordered pairs (p,q) with $p \in P$ and $q \in Q$. We then define on $(P \times Q)$:

$$0 = (0,0)$$

$$1 = (1,1)$$

$$(p,q) \leqq (p',q') \Leftrightarrow p \leqq p' \ \& \ q \leqq q'$$

$$(p,q) \vee (p',q') = (p \vee p', q \vee q')$$

EXERCISE: (i) Show that $(P \times Q)$ is indeed a semilattice if P and Q are. (ii) Does a similar result hold for lattices? (iii) Recall the definition of a ***Boolean algebra***. Does a similar result hold for such algebras?

DISCUSSION: The above notion of product is unsatisfactory from the "informational" point of view, because the two elements (1,0) and (0,1) are different from the 1 of the product, but conceptually they are "partly broken". In order to remedy this we have to define a product where ***all*** partly broken elements are ***identified*** with the single top element 1 of the product.

It is easy to make this repair, but we shall go further and assume that the ***universal semilattice*** we want to construct is ***closed under*** the appropriate pairing. This means that it contains among its ***subsemilattices*** many such products. Thus, in the next axiom set we allow the pairing to be ***iterated*** to form elements such as $\langle\!\langle\langle\!\langle p_0,p_1\rangle\!\rangle,\langle\!\langle\langle\!\langle p_2,p_3\rangle\!\rangle,\langle\!\langle p_4,p_5\rangle\!\rangle\rangle\!\rangle\rangle\!\rangle$, etc. We also decree that non-zero lowered elements are ***inconsistent*** with non-zero pairs.

AXIOM III: Pairing

$\langle\langle 0,0 \rangle\rangle = 0$

$\langle\langle p,q \rangle\rangle = 1 \Leftrightarrow p = 1$ or $q = 1$

$\langle\langle p,q \rangle\rangle \vee \langle\langle p',q' \rangle\rangle = \langle\langle p \vee p', q \vee q' \rangle\rangle$

$$p' < 1 \ \& \ q' < 1 \ \Rightarrow$$
$$[\,\langle\langle p,q \rangle\rangle \leqq \langle\langle p',q' \rangle\rangle \Leftrightarrow p \leqq p' \ \& \ q \leqq q'\,]$$

$p^* > 0 \ \& \ \langle\langle q,r \rangle\rangle > 0 \Rightarrow p^* \vee \langle\langle q,r \rangle\rangle = 1$

EXERCISE: Let S/E be the ***quotient*** of the S defined above by the ***equivalence relation*** E that identifies two infinite sequences if they ***both*** contain ***infinitely many*** 1's. (i) Show S/E is a semilattice. Is it a lattice? (ii) Show that S/E has a paring operation which satisfies all but the last of the conditions above.

EXERCISE: (i) Show that the semilattice S/E ***does not*** have a pair of elements p and q satisfying $p < 1 \ \& \ q < 1 \ \& \ p \vee q = 1$. (ii) Show that any Boolean algebra with more than two elements ***does*** have such a pair.

EXERCISE: Show, on the basis of our axioms, that by ***iterating pairing*** we can obtain a copy of the set N of non-negative integers using this definition:

$$[0] = 1^* \quad \text{and} \quad [n+1] = \langle\langle [n], 0 \rangle\rangle$$

and satisfying these conditions for all $n,m \in N$:

$0 < [n] < 1$, and

$[n] \vee [m] < 1 \Rightarrow n = m$.

EXERCISE: (i) Find a ***primitive recursive*** operation (n,m) of two variables that puts the Cartesian square $N \times N$ of the set N of integers into a one-one correspondence with the whole set N itself. In this way we can consider $N \times N = N$.
(ii) We may then define

$$\langle\!\langle P,Q \rangle\!\rangle = \{(n,m) \mid n \in P \text{ or } m \in Q\}$$

for sets $P,Q \subseteq N$. (Note the use of "or" here.) Prove that the countable Boolean algebra **Rec** of all ***recursive sets*** of integers is closed under ***this*** pairing operation and satisfies all but the last of the conditions of Axiom III. (iii) Give a ***modification*** of this definition so that, with a suitable lowering operation, **Rec** satisfies all of Axioms II and III.

CHAPTER II. COMPLETION

DEFINITION: Finitely ***repeated*** joining is defined by a recursive definition, where for $k \in N$ we set:

$$\bigvee\{t_i \mid i < 0\} = 0$$

$$\bigvee\{t_i \mid i < k+1\} = \bigvee\{t_i \mid i < k\} \vee t_k$$

EXERCISE: Prove that in any semilattice and for any integer $k \in N$ we have: $\bigvee\{t_i \mid i < k\} \leqq u \Leftrightarrow \forall i < k.\ t_i \leqq u$.

DEFINITION: A semilattice is ***complete*** if, and only if for ***all families*** $\{t_i \mid i \in I\}$ of elements of any size I, there is an element $\bigvee\{t_i \mid i \in I\}$ where we have:

$$\bigvee\{t_i \mid i \in I\} \leqq u \Leftrightarrow \forall i \in I.\ t_i \leqq u$$

EXERCISE: Prove that the element $\bigvee\{t_i \mid i \in I\}$ above is ***uniquely determined***.

EXERCISE: (i) Prove that any finite semilattice is complete. (ii) Is a complete semilattice always a lattice?

EXERCISE: (i) Show that the ***powerset*** **Pow**(N) of any non-empty set is a complete semilattice. (ii) But show that the countable semilattice of ***finite*** and ***cofinite*** subsets of N is ***not*** complete. (iii) Is the ***unit interval*** [0,1] of real numbers with the usual meaning of ≦ a complete semilattice? A lattice?

DISCUSSION: In order to ***create*** a complete semilattice from a given semilattice S, we want to adjoin to S "limits" of systems of elements from S. In the case of a countable S, a limit can be thought of the ***join*** of an infinite chain:

$$p_0 \leqq p_1 \leqq p_2 \leqq p_3 \leqq \ldots \leqq p_n \leqq p_{(n+1)} \leqq \ldots.$$

In other words, we can think of collecting "more and more" information ***ad infinitum***. Now, if we have another infinite chain:

$$q_0 \leqq q_1 \leqq q_2 \leqq q_3 \leqq \ldots \leqq q_n \leqq q_{(n+1)} \leqq \ldots,$$

when are their limits ***equal***? Thinking in terms of approximations, having ***equal limits*** can be defined as:

$$\forall i \exists j.\ p_i \leqq q_j \ \&\ \forall j \exists i.\ q_j \leqq p_i.$$

This means that any approximation in one tower is "bettered" by an approximation in the other.

The difficulty with this approach is that the "limit" has to be identified with "equivalence classes" of towers. To avoid having ***multiple representations*** of limits, we note that the two towers above are equivalent in that sense if, and only if, these two ***sets*** are ***identical***:

$$\{r \in S \mid \exists i.\ r \leqq p_i\} = \{r \in S \mid \exists j.\ r \leqq q_j\}.$$

In the case of a countable semilattice S, these sets are arbitrary "ideals" of S according to the next definition. We use the word "ideal" in the sense of ***imagining*** a limit corresponding to the join of all the bits of information in all the elements of the ideal.

DEFINITION: Given a semilattice S, the ***ideal completion*** of S (or ***completion by ideals***), in symbols **Id**(S), consists of all subsets $X \subseteq S$ where we have:

$$0 \in X \ \&\ \forall p,q \in S\ [(p \vee q) \in X \Leftrightarrow p \in X \ \&\ q \in X].$$

The ***domain*** determined by S is defined by:

$$\mathbf{Dom}(S) = \{X \in \mathbf{Id}(S) \mid 1 \notin X\}.$$

COMMENT: Not containing 1 is what we mean by a "proper" ideal. Note that there is only one ***improper*** ideal and so $\mathbf{Id}(S) = \mathbf{Dom}(S) \cup \{S\}$.

DEFINITION: The ***downset*** of an element p of a semilattice S is defined as: $\downarrow p = \{q \mid q \leqq p\}$. For $X \subseteq S$ we also define:

$$\downarrow X = \{q \mid \exists\, p \in X.\ q \leqq p\}.$$

EXERCISE: (i) Prove that $\mathbf{Id}(S)$ is a complete semilattice under the relation of set inclusion. (ii) Show that for $X, Y \in \mathbf{Id}(S)$ we have $X \vee Y = \downarrow\{p \vee q \mid p \in X \;\&\; q \in Y\}$. (iii) Show for all $X \in \mathbf{Id}(S)$ we have $X = \bigcup\{\downarrow p \mid p \in X\} = \bigvee\{\downarrow p \mid p \in X\}$. (iv) Show that the mapping $p \longrightarrow \downarrow p$ embeds S ***isomorphically*** into $\mathbf{Id}(S)$ as a semilattice embedding.

EXERCISE: $\mathbf{Id}(S)$ is, as we know, a complete lattice. Show that the ***greatest lower bound*** in $\mathbf{Id}(S)$ has a very simple set-theoretical meaning.

EXERCISE: (i) Show that every ***bounded*** subset of $\mathbf{Dom}(S)$ has a least upper bound in $\mathbf{Dom}(S)$. (ii) Show that every subset of $\mathbf{Dom}(S)$ that is ***directed*** (in the sense of having an upper bound in the set for every pair of elements of the set) has a least upper bound in $\mathbf{Dom}(S)$.

DEFINITION: A mapping $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$ of one domain ***onto*** another is said to be an ***isomorphism*** if, and only if, for all $P_0, P_1 \in \mathbf{Dom}(S)$, we have: $P_0 \subseteq P_1 \Leftrightarrow F(P_0) \subseteq F(P_1)$.

EXERCISE: If a mapping $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$ is an isomorphism, then show so is the inverse function $F^{-1}\colon \mathbf{Dom}(T) \longrightarrow \mathbf{Dom}(S)$.

EXERCISE: Let $S = F \cup \{N\}$, where F is the set of all ***finite subsets*** of the set of integers N. S is a semilattice under set inclusion. Show $\mathbf{Dom}(S)$ is isomorphic to the powerset

Pow(N) of N.

EXERCISE: A ***partial*** function from N into N can be identified with a set f of ordered pairs where whenever $(n,m),(n,k) \in f$, then $m = k$ (i.e., ***function values are unique***). Let **Part** be the set of all partial functions and let $S = \mathbf{Part} \cup \{N \times N\}$.
(i) Prove S is a complete semilattice under set inclusion.
(ii) Let F be the set of all finite elements of **Part.** Show that S can be regarded as the completion of $F \cup \{N \times N\}$.
(iii) Show that **Part** is isomorphic to $\mathbf{Dom}(F \cup \{N \times N\})$.

DEFINITION: Let S be any ***complete*** semilattice. Define the ***(abstractly) finite elements*** of S as being those elements p such that whenever $\{q_i \mid i \in I\}$ is a family of elements such that $p \leqq \bigvee\{q_i \mid i \in I\}$, then $p \leqq \bigvee\{q_i \mid i \in J\}$ for some ***finite*** $J \subseteq I$. Let **Fin**(S) be the set of all ***finite elements*** of S.

EXERCISE: Prove that $\mathbf{Fin}(S) \cup \{1\}$ is always a subsemilattice of a given complete semilattice S.

EXERCISE: Determine the finite elements of (i) **Pow**(N), (ii) the complete semilattice $\mathbf{Part} \cup \{N \times N\}$ made from partial functions, and (iii) the complete lattice **Id**(S) for a given semilattice S.

EXERCISE: Let S be a given semilattice, and let **Sub**(S) be the collection of all ***sub***semilattices of S. (i) Explain what this means. (ii) Show **Sub**(S) is a complete semilattice under set inclusion. (iii) Assuming S is countable, find a countable semilattice of which **Sub**(S) can be regarded as the ***domain completion***.

EXERCISE: In general, under what conditions is a given complete lattice the ideal completion of ***some*** semilattice?

CHAPTER III. CONTINUOUS FUNCTIONS

DISCUSSION: The powerset **Pow**(N) of the set of integers has infinite elements (of course, as N is an infinite set), but among the mappings (or operations) $F\colon \mathbf{Pow}(N) \longrightarrow \mathbf{Pow}(N)$ there is a rich collection which work by "finite approximation". This means that taking any $P \in \mathbf{Pow}(N)$, then a finite subset $R \subseteq F(P)$ is already a subset of $F(Q)$, where Q is some finite subset of P, and conversely. In symbols we can say for all $P \in \mathbf{Pow}(N)$:

$$\forall R \in \mathbf{Fin}(\mathbf{Pow}(N))[\, R \subseteq F(P) \Leftrightarrow \exists Q \in \mathbf{Fin}(\mathbf{Pow}(P)).\, R \subseteq F(Q)\,]$$

Note that $X \in \mathbf{Fin}(\mathbf{Pow}(Y))$ means that X is a ***finite subset*** of the set Y.

As a "slogan" we can say concerning such operations:

> ***A finite amount of information about an output is already exactly determined by a finite amount of information about the input.***

We call these operations ***continuous***, because as you approximate the input, you have some means of approximating the output "little by little".

EXERCISE: Show that continuous $F\colon \mathbf{Pow}(N) \longrightarrow \mathbf{Pow}(N)$ are always ***monotone*** in the sense that whenever $P_0 \subseteq P_1 \in \mathbf{Pow}(N)$, then $F(P_0) \subseteq F(P_1)$.

EXERCISE: Generalize the idea of continuity to operations of ***two*** variables (or inputs):

$$G\colon \mathbf{Pow}(N) \times \mathbf{Pow}(N) \longrightarrow \mathbf{Pow}(N).$$

EXERCISE: (i) Suppose $f\colon N \longrightarrow N$ is a function from integers to integers. Prove that both these operations (called ***image*** and ***inverse image***) on $P \in \mathbf{Pow}(N)$ are continuous:

$$f(P) = \{f(n) \mid n \in P\} \text{ and } f^{-1}(P) = \{n \mid f(n) \in P\}.$$

(ii) Suppose R as a subset of $N \times N$ is considered as a binary relation. Prove that both these operations on $P \in \mathbf{Pow}(N)$ are continuous:

$\{m \mid \exists\, n \in P.\ (n,m) \in R\}$ and

$\{k \mid \exists\, n,m \in P.\ n < m\ \&\ (n,k) \in R\ \&\ (m,k) \in R\}$.

EXERCISE: (i) Prove that if $F,G\colon \mathbf{Pow}(N) \longrightarrow \mathbf{Pow}(N)$ are both continuous operations, then so is H defined by ***composition***:

$H(P) = F(G(P))$.

(ii) Does a Principle of Composition generalize to operations of two variables? (iii) Prove that if the operation of two variables $K\colon \mathbf{Pow}(N) \times \mathbf{Pow}(N) \longrightarrow \mathbf{Pow}(N)$ is continuous, then $D(P) = K(P,P)$ is also continuous.

DISCUSSION: In the case of ***domains*** we can phrase the idea of continuity in terms of the underlying semilattice – and with the added condition that the output of an operation is never broken as long as the input is proper.

DEFINITION: Given a semilattice S, the ***continuous operations*** $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(S)$ are the functions where for $P \in \mathbf{Dom}(S)$ we have: $\forall\, r \in S[\, r \in F(P) \Leftrightarrow \exists\, q \in P.\ r \in F(\downarrow q)\,]$.

EXERCISE: Show that continuous $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(S)$ are always ***monotone*** in the sense that whenever $P_0 \subseteq P_1 \in \mathbf{Dom}(S)$, then $F(P_0) \subseteq F(P_1)$.

EXERCISE: (i) Prove if $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(S)$ is continuous, then for all $P \in \mathbf{Dom}(S)$, $F(P) = \bigvee\{F(\downarrow q) \mid q \in P\}$. (ii) Show that this condition is ***equivalent*** to being continuous.

EXERCISE: (i) Expand these definitions to $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$ between different domains. (ii) Show that an isomorphism is always continuous.

THE FUNCTION-SPACE THEOREM: Given a countable semilattice S, the ***set*** of all continuous functions $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(S)$ under the partial ordering $F \leqq G \Leftrightarrow \forall P \in \mathbf{Dom}(S).\ F(P) \leqq G(P)$ forms a ***domain*** over some countable semilattice.

PROOF: The idea will be to discover a countable semilattice derived from the function space of ***all*** functions

$$\varphi\colon S \longrightarrow \mathbf{Id}(S).$$

This family of functions forms a complete lattice under these definitions:

$$0(p) = \{0\}$$

$$1(p) = \downarrow 1 = S$$

$$\varphi \leqq \psi \;\Leftrightarrow\; \forall p \in S.\ \varphi(p) \leqq \psi(p)$$

But this space is ***much too large*** for what we need to represent continuous functions over $\mathbf{Dom}(S)$. So, let us ***cut down*** to the subspace **FUN** to where:

$$\varphi(1) = 1$$

$$\exists r < 1.\ \varphi(r) = 1 \;\Rightarrow\; \forall p \in S.\ \varphi(p) = 1$$

$$p \leqq q \;\Rightarrow\; \varphi(p) \leqq \varphi(q)$$

The intuition here is that if $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(S)$ is continuous, then the there is a function $\varphi \in \mathbf{FUN}$ such that for all $p < 1$, we have $\varphi(q) = F(\downarrow q)$. Then for $P \in \mathbf{Dom}(S)$, we will also have: $F(P) = \bigvee\{\varphi(p) \mid p \in P\}$. And the only function in **FUN** that does not arise in this way is the "broken" function $1(p) = 1$.

So, we will now hope to have a isomorphism between the space of continuous $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(S)$ and the elements $\varphi \in \mathbf{FUN}$ with $\varphi < 1$. But we will have to prove that this works out by a series of steps.

LEMMA 1: FUN is a semilattice.

PROOF: Let $\varphi, \psi \in \mathbf{FUN}$. If there is some $r < 1$ with

$\varphi(r) \vee \psi(r) = 1$ in **Id**(S), then the $1 \in$ **FUN** is the only element $\eta \in$ **FUN** where $\varphi \leqq \eta$ and, at the same time, $\psi \leqq \eta$. Hence, $\varphi \vee \psi = 1$ in **FUN**.

If, on the other hand, $\varphi(r) \vee \psi(r) < 1$ for all $r < 1$, then the function η defined by $\eta(p) = \varphi(p) \vee \psi(p)$ is a function in **FUN**, and moreover it follows that $\varphi \vee \psi = \eta$ in **FUN**. **QED**

LEMMA 2: FUN is a complete semilattice.

PROOF: We can argue in a way that is similar to the last proof. Suppose $\{\varphi_i \mid i \in I\}$ is a family of functions in **FUN**. There are two cases.

First, there may be no upper bound to the the functions in this family other than the broken function 1. Hence, we have

$$\bigvee\{\varphi_i \mid i \in I\} = 1$$

in the sense of least upper bounds in **FUN**.

Second, there may be a function $\psi < 1$ in **FUN** such that $\forall\, i \in I.\ \varphi_i \leqq \psi$. Now by the definition of **FUN**, it must be the case that $\psi(p) \in$ **Dom**(S) for all $p < 1$. This means that $\{\varphi_i(p) \mid i \in I\}$ is a subset of **Dom**(S). We can then define a function for all $p \in S$ by: $\eta(p) = \bigvee\{\varphi_i(p) \mid i \in I\}$. It is easy to show $\eta \in$ **FUN** and so $\eta = \bigvee\{\varphi_i \mid i \in I\}$ in **FUN**. **QED**

Now that we know **FUN** is a complete semilattice, we need to set about finding its ***finite elements***. It turns out we can give ***explicit formulas*** for them.

Define first for $p < 1$ and any $q \in S$:

$[p \to q](r) = 1$ if $q = 1$; otherwise if $q < 1$,

$$[p \to q](r) = \begin{cases} 0 & \text{if } \neg\, p \leqq r \\ \downarrow q & \text{if } p \leqq r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Such a function $[p \to q]$ is called a ***step function*** – actually, a ***one-step*** function. The general step functions are the ***finite joins*** of these. Note that it is easy to show that $[p \to q] \in \mathbf{FUN}$, and, hence, the finite joins are members, too.

LEMMA 3: The broken function 1 is a finite element of **FUN**.

PROOF: Suppose that $\{ \varphi_i \mid i \in I \}$ is a family of elements of **FUN** such that $\bigvee \{ \varphi_i \mid i \in I \} = 1$. This implies that $\bigvee \{ \varphi_i \mid i \in I \} \notin \mathbf{Dom}(S)$.

By way of contradiction, suppose that for ***all*** $J \in \mathbf{Fin}(\mathbf{Pow}(I))$ we have $\bigvee \{ \varphi_i \mid i \in J \} < 1$. Hence, the family

$$\{ \bigvee \{ \varphi_i \mid i \in J \} \mid J \in \mathbf{Fin}(\mathbf{Pow}(I)) \}$$

is a directed subset of $\mathbf{Dom}(S)$. But then we know that the least upper bound of this family is in fact in $\mathbf{Dom}(S)$. But the join of the family is just $\bigvee \{ \varphi_i \mid i \in I \}$, which is not in $\mathbf{Dom}(S)$ by assumption. Therefore, there must be at least one $J \in \mathbf{Fin}(\mathbf{Pow}(I))$ where $\bigvee \{ \varphi_i \mid i \in J \} = 1$.

That proves the element 1 is ***finite*** in **FUN**. **QED**

LEMMA 4: For $p < 1$ and $q \in S$, $[p \to q]$ is finite in **FUN**.

PROOF: Let p and q be as above. Suppose that $\{ \varphi_i \mid i \in I \}$ is a family of elements of **FUN** such that $[p \to q] \leqq \bigvee \{ \varphi_i \mid i \in I \}$.

Case 1. $\bigvee \{ \varphi_i \mid i \in I \} = 1$.

By Lemma 3, $[p \to q] \leqq \bigvee \{ \varphi_i \mid i \in J \}$, for some $J \in \mathbf{Fin}(\mathbf{Pow}(I))$.

Case 2. $\bigvee \{ \varphi_i \mid i \in I \} < 1$.

This means that this least upper bound is in $\mathbf{Dom}(S)$, and so is $[p \to q]$. By definition, then $[p \to q](p) = {\downarrow} q$ and $q < 1$. We have ${\downarrow} q \leqq \bigvee \{ \varphi_i(p) \mid i \in I \}$. Since ${\downarrow} q$ is a finite element of $\mathbf{Id}(S)$, we know ${\downarrow} q \leqq \bigvee \{ \varphi_i(p) \mid i \in J \}$ for some $J \in \mathbf{Fin}(\mathbf{Pow}(I))$. But this implies $[p \to q] \leqq \bigvee \{ \varphi_i \mid i \in J \}$.

And so by the two cases, $[p \to q]$ is finite in **FUN.** **QED**

COROLLARY: If $k \in N$ and each $p_i < 1$, then $\bigvee\{[p_i \to q_i] \mid i < k\}$ is a finite element of **FUN.**

LEMMA 5: For each $\varphi \in$ **FUN**, $\varphi = \bigvee\{[p \to q] \mid p < 1 \;\&\; q \in \varphi(p)\}$.

PROOF: If $p < 1$ and $q \in \varphi(p)$, then $[p \to q] \leqq \varphi$. So, φ is an upper bound for all these $[p \to q]$. If φ is not the least upper bound, then there is some ψ in **FUN** where $\psi < \varphi$ and ψ is an upper bound for all the relevant $[p \to q]$.

Now, if $\psi < \varphi$, then for some $p < 1$, we have ***not*** $\varphi(p) \leqq \psi(p)$. This means that there must be a $q \in \varphi(p)$ where $q \notin \psi(p)$. Inasmuch as $[p \to q] \leqq \varphi$, we would have also $[p \to q] \leqq \psi$. But this can only be so if $q \in \psi(p)$.

Having reached a contradiction, the lemma is proved. **QED**

COROLLARY: The elements $\bigvee\{[p_i \to q_i] \mid i < k\}$, where $k \in N$ and each $p_i < 1$, are ***all*** and the ***only*** finite elements of **FUN.**

COROLLARY: **FUN** is the completion of the countable semilattice **Step** generated by the $[p \to q]$ with $p < 1$.

COROLLARY: The space of continuous F: **Dom**(S) $\longrightarrow$ **Dom**(S) is isomorphic to **Dom**(**Step**).

And this completes the analysis of the function space of a given domain.

NOTE: We could have been more general and treated continuous functions F: **Dom**(S) $\longrightarrow$ **Dom**(T) between ***different*** domains. The work of the next two chapters will show that the extra effort is not especially necessary, however, owing to the properties of the universal domain.

CHAPTER IV. A UNIVERSAL SEMILATTICE

DISCUSSION: Axioms I-III have been ***closure conditions*** in the sense that the operations introduced can be iterated, one on top of another. The discussion of the function space, on the other hand, went from one domain just ***one level up*** to the function-space domain. To have a truly "universal" domain, we also need to be able to ***iterate*** function-space formations.

Looking back to the proofs in Chapter III, we see the key was to introduce step-functions $[p \to q]$ to get at the finite functions. In our next-to-last axiom set, we will take the ***properties*** we discovered above ***axiomatically*** and ***symbolically*** to allow for the desired iteration.

AXIOM IV: Mapping

$$(p \to q) \text{ defined} \Leftrightarrow p < 1$$

$$\bigvee\{(p_i \to q_i) \mid i < k\} = 1 \Leftrightarrow \exists\, r < 1.\ \bigvee\{q_i \mid p_i \leqq r\} = 1$$

$$\bigvee\{(p_i \to q_i) \mid i < k\} < 1 \Rightarrow$$
$$(r \to s) \leqq \bigvee\{(p_i \to q_i) \mid i < k\} \Leftrightarrow s \leqq \bigvee\{q_i \mid p_i \leqq r\}$$

$$p^* > 0 \ \&\ (p \to q) > 0 \Rightarrow p^* \vee (p \to q) = 1$$

$$\langle\!\langle p,q \rangle\!\rangle > 0 \ \&\ (p \to q) > 0 \Rightarrow \langle\!\langle p,q \rangle\!\rangle \vee (p \to q) = 1$$

DISCUSSION: In summary, we now have these axiom sets:

AXIOM I: Approximating
AXIOM II: Lowering
AXIOM III: Pairing
AXIOM IV: Mapping

Any semilattice (with extra operations) satisfying these axiom sets would be interesting, and there are many possibilities. But we would like to focus on the ***minimal*** model.

DEFINITION: The ***universal semilattice*** U is formed by ***equivalence classes*** of symbolic expressions generated by the ***constants*** 0, and 1, with the aid of one ***unary operation*** p* and three ***binary operations*** $(p \vee q)$, $\langle\!\langle p,q \rangle\!\rangle$, and $(p \to q)$. On the basis of Axioms I-IV such expressions ξ and ζ can be proved to be ***equal*** ($\xi = \zeta$) and/or ***approximating*** ($\xi \leqq \zeta$) and/or ***strictly approximating*** ($\xi < \zeta$).

NOTE: ***No variables*** are allowed in these symbolic expressions. Also keep in mind that ***before*** using a combination ($\xi \to \zeta$), we need to have proved $\xi < 1$.

THEOREM: U forms a semilattice where all the operations and relations are ***recursive*** and where – save when a combination is proved to be 0 or 1 – none of the combinations

$$\xi^*, \ \langle\!\langle \xi, \zeta \rangle\!\rangle, \text{ and } (\xi \to \zeta)$$

are proved equal or approximating if of different kinds.

PROOF SKETCH: Observe that the Axioms of Lowering, Pairing, and Mapping show explicitly how to reduce any equality (or approximation) back to compound expressions that are ***simpler*** (i.e., less compound) than the given ones. Note, too, that we included in the axioms conditions that make the different kinds of values unequal and non-approximating. Eventually, then, any assessment is pushed back to the constants where all the possible relationships are explicitly provable.

This kind of reductive argument shows that the axioms taken together are ***consistent***. That means that U so constructed is an non-trivial semilattice. **QED**

DISCUSSION: Though the construction of U is done with equivalence classes of symbolic expressions, we will just use the letters p, q, r, ... for elements of U as we have done with all the semilattices before.

EXERCISE: (i) Prove that $[N] = \{0,1\} \cup \{[n] \mid n \in N\} \in \mathbf{Sub}(U)$. (ii) Prove that $U^* = \{1\} \cup \{p^* \mid p \in U\} \in \mathbf{Sub}(U)$. (iii) Prove that $U \times U = \{\langle\!\langle p,q \rangle\!\rangle \mid p \in U \ \& \ q \in U\} \in \mathbf{Sub}(U)$. (iv) Prove that $U \to U = \{\bigvee\{(p_i \to q_i) \mid i < k\} \mid k \in N \ \& \ \forall i < k.\ p_i \in U \backslash \{1\} \ \& \ q_i \in U\} \in \mathbf{Sub}(U)$. (v) Prove that U is the set-theoretical union of three of its subsemilattices: $U = U^* \cup (U \times U) \cup (U \to U)$. (vi) What elements do these three subsemilattices have in common (two-by-two).

EXERCISE: **Fin**(**Sub**(U)) is the collection of all ***finite*** subsemilattices of S. Recall that $V = \mathbf{Fin}(\mathbf{Sub}(U)) \cup \{U\}$ is a ***semilattice***. Show that V is ***isomorphic*** to

$$\{ \bigvee\{(p_i \to p_i) \mid i < k\} \mid k \in N \;\&\; \forall i < k.\ p_i \in U \backslash \{1\} \} \in \mathbf{Sub}(U).$$

(That is the subsemilattice ***generated*** by the elements (p→p).)

DISCUSSION: Though we see that U has very many interesting subsemilattices, we will argue in the next chapter that U gives us a domain **Dom**(U) where the subdomains form a very rich category. This is not the only category suitable for Domain Theory, but it is a good one to test out ideas.

CHAPTER V. A UNIVERSAL DOMAIN

From now on, ***all*** semilattices considered will be taken as ***subsemilattices*** of U. The ideal completion of U is, as we know, $\mathbf{Id}(U) = \mathbf{Dom}(U) \cup \{U\}$, and it is a complete lattice, but we will concentrate on **Dom**(U) and its (many) subdomains.

If $S \in \mathbf{Sub}(U)$, then ***strictly speaking*** **Dom**(S) is not really a subcollection of **Dom**(U). We will slightly change our definitions so it henceforth becomes a ***subdomain***.

REVISED DEFINITION: For $S \in \mathbf{Sub}(U)$, we now write

$$\mathbf{Dom}(S) = \{\downarrow(X \cap S) \mid X \in \mathbf{Dom}(U)\} \subseteq \mathbf{Dom}(U).$$

The idea here is that every ideal $X \in \mathbf{Dom}(U)$ gives an ideal $(X \cap S) \in \mathbf{Dom}(S)$. But then $\downarrow(X \cap S) \in \mathbf{Dom}(U)$ and we can check that $\downarrow(X \cap S) \cap S = (X \cap S)$. These comments can be used to give the solutions to the next exercise.

EXERCISE: (i) Show that the new meaning of **Dom**(S) is ***isomorphic*** to the old meaning of domain completion.
(ii) Find a continuous function $F\colon \mathbf{Dom}(U) \longrightarrow \mathbf{Dom}(U)$ where the image $F(\mathbf{Dom}(U)) = \mathbf{Dom}(S)$ (using revised definition).

EXERCISE: (i) For $P \in \mathbf{Id}(U)$, let $P^* = \{p^* \mid p \in P\}$. Show that $P^* \in \mathbf{Dom}(U)$. (ii) Prove that $\mathbf{Dom}(\{1\} \cup U^*) = \{P^* \mid P \in \mathbf{Id}(U)\}$. (iii) Prove that $\mathbf{Dom}(\{1\} \cup U^*)$ is isomorphic to $\mathbf{Id}(U)$.

EXERCISE: Prove that $\mathbf{Dom}([N]) = \{0,1\} \cup \{\downarrow[n] \mid n \in N\}$.

EXERCISE: (i) Show $H = \{0\} \cup \{\langle\!\langle p,q \rangle\!\rangle \mid p > 0 \;\&\; q > 0\} \in \mathbf{Sub}(U)$.
(ii) For $S,T \in \mathbf{Sub}(U)$, show

$$S \times T = \{\langle\!\langle p,q \rangle\!\rangle \mid p \in S \;\&\; q \in T\} \in \mathbf{Sub}(U).$$

(iii) Prove that $\mathbf{Dom}(S \times T)$ is isomorphic to the ***product*** of $\mathbf{Dom}(S)$ and $\mathbf{Dom}(T)$ as ***partially ordered sets***. **HINT:** Show that these relationships:

$$P = \{p \in S \mid \langle\!\langle p,0 \rangle\!\rangle \in R\} \text{ and } Q = \{q \in T \mid \langle\!\langle 0,q \rangle\!\rangle \in R\} \text{ and}$$
$$R = \downarrow\{\langle\!\langle p,q \rangle\!\rangle \mid p \in P \;\&\; q \in Q\}$$

establish a one-one correspondence between pairs of elements of $\mathbf{Dom}(S)$ and $\mathbf{Dom}(T)$ and elements of $\mathbf{Dom}(S \times T)$.
(iv) Discuss the meaning of $\mathbf{Dom}((S \times T) \cap H)$.

EXERCISE: For $S,T \in \mathbf{Sub}(U)$, let $S \to T$ the subsemilattice ***generated by*** $\{(p \to q) \mid p \in S \backslash \{1\} \;\&\; q \in T\}$. Prove that $\mathbf{Dom}(S \to T)$ is isomorphic to the domain of continuous functions $F\colon \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$.

EXERCISE: For $S,T \in \mathbf{Sub}(U)$, let

$$S + T = \{0\} \cup \{\langle\!\langle p,[0] \rangle\!\rangle \mid p \in S\} \cup \{\langle\!\langle q,[1] \rangle\!\rangle \mid q \in T\}.$$

Show that $S + T \in \mathbf{Sub}(U)$ and discuss the meanings of $\mathbf{Dom}(S+T)$ and $\mathbf{Dom}((S+T) \cap H)$.

EXERCISE: For $S \in \mathbf{Sub}(U)$, let $S_\perp = \{0\} \cup \{\langle\!\langle p,[0] \rangle\!\rangle \mid p \in S\}$. Show that $S_\perp \in \mathbf{Sub}(U)$ and discuss the meaning of $\mathbf{Dom}(S_\perp)$.

EXERCISE: Show that the following operations on $\mathbf{Sub}(U)$ are ***continuous*** and preserve ***finiteness***:

S^*, $S_\perp$, $S \times T$, $S + T$, $S \to T$, and $S \cap T$.

CHAPTER VI. SOLVING DOMAIN EQUATIONS

The last chapter has shown us that that the domains **Dom**(S) associated with semilattices $S \in$ **Sub**(U) can have a large variety of structure – especially if we iterate the many operations that form semilattices from given semilattices. The objective now is to show that this iteration can also be carried on ***ad infinitum*** to create ***recursively defined*** domains.

EXERCISE: Show that the continuous functions

$$F: \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(T)$$

are exactly the same as the functions such that whenever

$$P_0 \leqq P_1 \leqq P_2 \leqq P_3 \leqq \ldots \leqq P_n \leqq P_{(n+1)} \leqq \ldots \text{ in } \mathbf{Dom}(S),$$

then $F(\bigvee\{P_n \mid n \in N\}) = \bigvee\{F(P_n) \mid n \in N\}$.

We can apply the last exercise in giving the proof of the next very basic result.

THE FIXED-POINT THEOREM. Every continuous function

$$F: \mathbf{Dom}(S) \longrightarrow \mathbf{Dom}(S)$$

has a ***least fixed point*** $P \in \mathbf{Dom}(S)$ which can be found as

$$P = \bigvee\{F^n(0) \mid n \in N\}.$$

PROOF: Here F^n is the n^{th}-iterate of the function F. Now $0 \leqq F(0)$, and so it follows that $F^n(0) \leqq F^{(n+1)}(0)$. Applying the exercise we find

$$F(P) = \bigvee\{F^{(n+1)}(0) \mid n \in N\} = \bigvee\{F^n(0) \mid n \in N\} = P.$$

This shows that P is a fixed point. If Q were another fixed point, then, because $0 \leqq Q$, we would have for all $n \in N$, $F^n(0) \leqq Q$. This proves that $P \leqq Q$. **QED**

Inasmuch as **Sub**(U) is isomorphic to **Dom**(**FSub**(U) ∪ {U}), we can apply the Fixed-Point Theorem to **Sub**(U) and to the many continuous functions we now know. Keep in mind here that the zero element of **Sub**(U) is actually the subsemilattice {0,1}.

EXERCISE: (i) By the Fixed-Point Theorem we know there is a B ∈ **Sub**(U), where B = B + B. Is such a B ***unique***?
(ii) Discuss the nature of **Dom**(B).

EXERCISE: (i) By the Fixed-Point Theorem we know there is an M ∈ **Sub**(U), where M = ({0,1}$_\perp$ + M) ∩ H. Is such an M ***unique***?
(ii) Are **Dom**([N]) and **Dom**(M) ***isomorphic***?.

EXERCISE: (i) By the Fixed-Point Theorem we know there is a Z ∈ **Sub**(U), where Z = ([N] × Z) ∩ H. Is such a Z ***unique***?
(ii) Discuss the nature of **Dom**(Z). (iii) Is there an element E ∈ **Dom**(Z) such that E = ↓{《↓[0],e》| e ∈ E }?

EXERCISE: Is there a ***sequence*** of elements E_n ∈ **Dom**(Z) such that for n ∈ N we have E_n = ↓{《↓[n],e》| e ∈ $E_{(n+1)}$ }?

EXERCISE: Given two continuous functions F and G mapping ***pairs*** of elements of **Sub**(U) to **Sub**(U), must there exist C,D ∈ **Sub**(U) such that C = F(C,D) and D = G(C,D)? (This is a ***Double Fixed-Point Theorem.***)

EXERCISE: Discuss why solving equations X = F(X) in **Sub**(U) give us "equations" involving the domain **Dom**(X) and others.

NOTE: ***Equations between domains*** often have to be interpreted as ***isomorphisms.***

CHAPTER VII. MODELING LAMBDA CALCULUS

By the very construction of the universal semilattice U we know that U → U is a subsemilattice of U. It follows that

Dom$(U \to U)$ is a subdomain of **Dom**(U). But, as we have checked, **Dom**$(U \to U)$ is isomorphic to the continuous function space from **Dom**(U) to **Dom**(U). In other words:

> ***The domain* Dom**(U) ***contains a copy of its own function space.***

Generally, standard mathematical spaces, such as, say, the unit interval [0,1] do not have this property — even when we restrict attention to continuous functions.

The possibility of this self-containment brings up the question: ***Can we make elements of* Dom**(U) ***play the roles of continuous functions?*** The answer is ***Yes***, and a ***notation*** for doing so is one way of giving a model for the so-called $\boldsymbol{\lambda}$-*calculus*.

COMMENT: The historical reason for using the Greek letter $\boldsymbol{\lambda}$ will be explained below.

DEFINITION: For $V \subseteq U$, the ***ideal generated by*** V is defined as: $\mathbf{id}(V) = {\downarrow}\{\bigvee\{p_i \mid i<k\} \mid k \in N \;\&\; \forall i < k.\ p_i \in V\}$.

COMMENT: We always have $\mathbf{id}(V) \in \mathbf{Id}(U)$, but it is not always true that $\mathbf{id}(V) \in \mathbf{Dom}(U)$, even though $1 \notin V$. So we have to watch out for the cases where $\mathbf{id}(V) = U$ might be possible.

DEFINITION: For $F, X \in \mathbf{Dom}(U)$, ***functional application*** is defined by: $F⟪X⟫ = \mathbf{id}(\{q \mid \exists\, p \in X. (p \to q) \in F\})$.

DISCUSSION: The idea here is that F works as a "look-up table" for mapping finite amounts of information. When an approximation p is found in X (and remember that we know $X = \bigvee\{{\downarrow}p \mid p \in X\}$ in **Dom**(U)), then we look to see if F allows us to map p to a q. The totality of these mappings gives us $F⟪X⟫$. (Note the value could also have been written as $\bigvee\{{\downarrow}q \mid \exists\, p \in X. (p \to q) \in F\}$.) For this all to make sense, however, it must be checked that we ***stay in*** **Dom**(U).

EXERCISE: (i) Prove that for $F, X \in \mathbf{Dom}(U)$, we always have

$F⟪X⟫ \in \mathbf{Dom}(U)$. (ii) Show that for any fixed F, the mapping from X to $F⟪X⟫$ over $\mathbf{Dom}(U)$ is always continuous. (iii) Show that for any fixed X, the mapping from F to $F⟪X⟫$ over $\mathbf{Dom}(U)$ is always continuous.

DEFINITION: For a given continuous $\Phi\colon \mathbf{Dom}(U) \longrightarrow \mathbf{Dom}(U)$, let ***λ-abstraction*** be defined by:

$$\lambda X.\, \Phi(X) = \mathbf{id}(\{(p \to q) \mid q \in \Phi(\downarrow p) \;\&\; p < 1\}).$$

EXERCISE: (i) Prove for continuous $\Phi\colon \mathbf{Dom}(U) \longrightarrow \mathbf{Dom}(U)$, we always have $\lambda X.\, \Phi(X) \in \mathbf{Dom}(U)$. (ii) For all $Y \in \mathbf{Dom}(U)$, we also have $(\lambda X.\, \Phi(X))⟪Y⟫ = \Phi(Y)$.

DISCUSSION: These results demonstrate two points about the universal domain: (a) ***every*** element of $\mathbf{Dom}(U)$ can be used to make a continuous function; and (b) ***every*** continuous function from $\mathbf{Dom}(U)$ to itself comes up in this way. In fact, we could show the isomorphism with the function space with the next fact.

EXERCISE: (i) Show for continuous $\Phi, \Psi\colon \mathbf{Dom}(U) \longrightarrow \mathbf{Dom}(U)$, we have $\Phi \leqq \Psi \Leftrightarrow \lambda X.\, \Phi(X) \leqq \lambda X.\, \Psi(X)$. (ii) Prove that $\{\lambda X.\, F⟪X⟫ \mid F \in \mathbf{Dom}(U)\} = \mathbf{Dom}(U \to U)$. (iii) Prove that we have $\mathbf{Dom}(U \to U) = \{F \in \mathbf{Dom}(U) \mid F = \lambda X.\, F⟪X⟫\}$. (iv) Show that we have $\lambda X.\, F⟪X⟫ \vee \lambda X.\, G⟪X⟫ = \lambda X.(F⟪X⟫ \vee G⟪X⟫)$.

BACKGROUND: The λ in the λ-notation is not a "quantity" but is rather a ***variable-binding operator***. Alonzo Church chose the Greek letter "at random" in analogy with Bertrand Russell's iota-operator for ***definite descriptions*** ($\iota x.\, \Phi(x)$ means "the unique x satisfying $\Phi(x)$) and David Hilbert's epsilon-operator for ***arbitrary choice*** ($\varepsilon x.\, \Phi(x)$ means "an x satisfying $\Phi(x)$, so that $\Phi(\varepsilon x.\, \Phi(x)) \Leftrightarrow \exists x.\, \Phi(x)$ and $\Phi(\varepsilon x.\, \neg\Phi(x)) \Leftrightarrow \forall x.\, \Phi(x)$.) After working with a type-free system for some time, Church turned to higher-order logic and a calculus of typed λ-terms. Another approach to types will be discussed in Chapter IX.

EXERCISE: (i) Explain why ***iterated application***, $F⟪X⟫⟪Y⟫$, gives us all continuous functions of ***two variables***. Connect this idea to ***iterated abstraction***, as in $\lambda X \lambda Y. \Phi(X,Y)$. (ii) Show that $\lambda Y. \Phi(X,Y)$ is always a continuous function of ***one*** variable X when $\Phi(X,Y)$ is a continuous function of ***two***.

EXERCISE: Consider the binary operation on **Dom**(U) defined by $F \circ G = \lambda X. F⟪G⟪X⟫⟫$. (i) Is the operation ***continuous***? (ii) Is it ***associative***? (iii) Is it ***commutative***? (iv) Restricted to **Dom**$(U \rightarrow U)$, does it have a ***two-sided unit***?

EXERCISE: Look up information on Haskell Curry and on ***Combinatory Logic***. (Wikipedia does not seem all that helpful at the moment.) Does **Dom**(U) give a a model for the algebra of combinators? Give some examples of combinators in **Dom**$(U \rightarrow U)$.

CHAPTER VIII. ASSESSING COMPUTABILITY

Recall that the ***universal semilattice*** U is formed by using ***equivalence classes*** of symbolic expressions formed by the ***constants*** 0 and 1 with the aid of one ***unary operation*** p* and three ***binary operations*** $(p \vee q)$, $⟪p,q⟫$, and $(p \rightarrow q)$. On the basis of Axioms I-IV such expressions ξ and ζ can be proved to be ***equal*** ($\xi = \zeta$) and/or ***approximating*** ($\xi \leqq \zeta$) and/or ***strictly approximating*** ($\xi < \zeta$). By sorting through such proofs, we can always effectively choose the ***simplest*** (or a ***canonical***) representative in each equivalence class.

And we can thus regard the operations of ***lowering***, ***joining***, ***pairing***, and ***mapping*** as being ***effectively computable*** operations on U. In this way, the semilattice U becomes an example of a ***recursive algebra***. In any such algebra, there is a notion of being a ***recursively enumerable*** subset of the the set of elements of the algebra. Such subsets can be regarded as effective "limits" of effectively defined infinite sequences of finite subsets.

NOTE: If we had more time, we could make this discussion more precise by the use of ***Gödel numberings***.

DEFINITION: The ***computable elements*** of $\mathbf{Dom}(U)$ are the subsets in $\mathbf{Dom}(U)$ that are recursively enumerable.

DEFINITION: The ***computable operations*** $\Phi: \mathbf{Dom}(U) \longrightarrow \mathbf{Dom}(U)$ are the continuous mappings where $\lambda X.\, \Phi(X)$ is a computable element of $\mathbf{Dom}(U)$.

EXERCISE: (i) Prove that if $F \in \mathbf{Dom}(U)$ is computable, then the mapping from X to $F⟪X⟫$ is a computable operation. (ii) And if $X \in \mathbf{Dom}(U)$ is computable, then so is $F⟪X⟫$.

EXERCISE: Prove that a continuous $\Phi: \mathbf{Dom}(U) \longrightarrow \mathbf{Dom}(U)$ is a computable operation on $\mathbf{Dom}(U)$ if, and only if, the set $\{(p \to q) \mid q \in \Phi(\downarrow p)\ \&\ p < 1\}$ is recursively enumerable.

EXERCISE: If $F \in \mathbf{Dom}(U)$ is computable, will the ***least fixed point*** of F also be computable?

EXERCISE: Why are there ***non-computable*** elements of $\mathbf{Dom}(U)$?

EXERCISE: (i) Discuss the computable $\Phi: \mathbf{Dom}([N]) \longrightarrow \mathbf{Dom}([N])$.
(ii) Which elements of $\mathbf{Dom}(U)$ represent the same mappings?

CHAPTER IX. MOVING TO HIGHER TYPES

DISCUSSION: Every $S \in \mathbf{Sub}(U)$ gives a ***subdomain*** $\mathbf{Dom}(S) \subseteq \mathbf{Dom}(U)$. Such subdomains can be – as we have seen – quite differently structured. Each represents a certain "type" of element in $\mathbf{Dom}(U)$. Moreover, the construction $\mathbf{Dom}(S \to T)$ allows us to move to "higher types", in the sense of passing from individual inputs and outputs to the space of (continuous) operations. But these subtypes have special properties; for example, each has a ***least element*** and all those operations have ***least fixed points***.

Now there are many ***subsets*** of $\mathbf{Dom}(U)$ that are not subdomains. They give us many kinds of new types. But an even more flexible way of creating special types is to take ***quotients*** of subsets of $\mathbf{Dom}(U)$ by ***equivalence relations***. The reason for doing so is that there could be many elements of $\mathbf{Dom}(U)$ which represent a quantity we want to focus on. Picking a ***canonical*** example from each equivalence class might not be easy to do, so it is in general better to keep the whole equivalence class around. (Previously – because it was possible – we avoided equivalence classes and chose suitable representatives.)

DEFINITION: With a slight abuse of notation we let

$$\langle\!\langle P,Q \rangle\!\rangle = \downarrow\{ \langle\!\langle p,q \rangle\!\rangle \mid p \in P \;\&\; q \in Q \} \in \mathbf{Dom}(U \times U)$$

denote ordered pairs of elements $P,Q \in \mathbf{Dom}(U)$.

EXERCISE: Show that $\langle\!\langle \downarrow p, \downarrow q \rangle\!\rangle = \downarrow \langle\!\langle p,q \rangle\!\rangle$ for $p,q \in U \backslash \{1\}$ in the two senses of pairing.

DEFINITION: We denote by $\mathbf{PER}(U)$ the collection of ***partial equivalence relations*** over $\mathbf{Dom}(U)$. These are the subsets $\mathcal{A} \subseteq \mathbf{Dom}(U \times U)$ where we have

$$\langle\!\langle P,Q \rangle\!\rangle \in \mathcal{A} \Rightarrow \langle\!\langle Q,P \rangle\!\rangle \in \mathcal{A} \text{ and}$$

$$\langle\!\langle P,Q \rangle\!\rangle \in \mathcal{A} \;\&\; \langle\!\langle Q,R \rangle\!\rangle \in \mathcal{A} \Rightarrow \langle\!\langle P,R \rangle\!\rangle \in \mathcal{A}$$

for all elements $P,Q,R \in \mathbf{Dom}(U)$. As a shorthand we write

$P \mathcal{A} Q \Leftrightarrow \langle\langle P,Q \rangle\rangle \in \mathcal{A}$.

Additionally, let the ***range*** of $\mathcal{A}$ be denoted by:

$\mathbf{Rng}(\mathcal{A}) = \{ P \mid P \mathcal{A} P \}$.

COMMENT: These relations $\mathcal{A}$ are called "partial" because they are not equivalences over the ***whole*** of $\mathbf{Dom}(U)$, but only over a ***subset***, namely, $\mathbf{Rng}(\mathcal{A})$. If $\mathbf{Rng}(\mathcal{A}) = \mathbf{Dom}(U)$, then we would call $\mathcal{A}$ a "total" equivalence relation. We will call partial equivalence relations **PERs** for short.

If we wanted we could take the ***quotient set*** $\mathbf{Rng}(\mathcal{A})/\mathcal{A}$ of the ***equivalence classes*** with respect to $\mathcal{A}$. But it not really necessary to do so. The equivalence relation $\mathcal{A}$ itself gives us all the information we need. And so we let $\mathcal{A}$ be the new, expanded kind of ***type***. The next task is to introduce the typical operations on types to define additional types from given types — in much the same way as we introduced the subdomains.

EXERCISE: (i) Prove that if $\mathcal{A},\mathcal{B} \in \mathbf{PER}(U)$, then $\mathcal{A} \cap \mathcal{B} \in \mathbf{PER}(U)$. (ii) What is the $\mathbf{Rng}(\mathcal{A} \cap \mathcal{B})$? (iii) Prove that $\varnothing \in \mathbf{PER}(U)$, and that $\mathbf{Dom}(U \times U) \in \mathbf{PER}(U)$. (iv) For any $A \subseteq \mathbf{Dom}(U)$, show that $\{ \langle\langle X,X \rangle\rangle \mid X \in A \} \in \mathbf{PER}(U)$.

DEFINITION: The ***product*** of two PERs is defined by:

$$P_0 \, (\mathcal{A} \times \mathcal{B}) \, P_1 \Leftrightarrow \exists\, X_0,Y_0,X_1,Y_1 \, [\, P_0 = \langle\langle X_0,Y_0 \rangle\rangle \;\&\; P_1 = \langle\langle X_1,Y_1 \rangle\rangle \;\&\; X_0 \mathcal{A} X_1 \;\&\; Y_0 \mathcal{B} Y_1 \,]$$

EXERCISE: (i) Prove that if $\mathcal{A},\mathcal{B} \in \mathbf{PER}(U)$, then $\mathcal{A} \times \mathcal{B} \in \mathbf{PER}(U)$. (ii) What is the $\mathbf{Rng}(\mathcal{A} \times \mathcal{B})$?

DEFINITION: The ***sum*** of two PERs is defined by:

$$P_0 \, (\mathcal{A} + \mathcal{B}) \, P_1 \Leftrightarrow \exists\, X_0,X_1 [\, P_0 = \langle\langle \downarrow[0],X_0 \rangle\rangle \;\&\; P_1 = \langle\langle \downarrow[0],X_1 \rangle\rangle \;\&\; X_0 \mathcal{A} X_1 \,]$$
$$\text{or } \exists\, Y_0,Y_1 [\, P_0 = \langle\langle \downarrow[1],Y_0 \rangle\rangle \;\&\; P_1 = \langle\langle \downarrow[1],Y_1 \rangle\rangle \;\&\; Y_0 \mathcal{B} Y_1 \,]$$

EXERCISE: (i) Prove that if $\mathcal{A},\mathcal{B} \in \textbf{PER}(U)$, then $\mathcal{A} + \mathcal{B} \in \textbf{PER}(U)$.
(ii) What is the $\textbf{Rng}(\mathcal{A} + \mathcal{B})$?

DEFINITION: The ***function space*** of two PERs is defined by:

$$F_0 \, (\mathcal{A} \to \mathcal{B}) \, F_1 \Leftrightarrow \forall \, X_0, X_1 \, [\, X_0 \, \mathcal{A} \, X_1 \Rightarrow F_0 (\!(X_0)\!) \, \mathcal{B} \, F_1 (\!(X_1)\!) \,].$$

EXERCISE: (i) Prove that if $\mathcal{A},\mathcal{B} \in \textbf{PER}(U)$, then $\mathcal{A} \to \mathcal{B} \in \textbf{PER}(U)$.
(ii) What is the $\textbf{Rng}(\mathcal{A} \to \mathcal{B})$?

DEFINITION: A ***mapping*** $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ between two PERs is defined as an element $F \in \textbf{Dom}(U)$ where $F \, (\mathcal{A} \to \mathcal{B}) \, F$ (i.e., $F \in \textbf{Rng}(\mathcal{A} \to \mathcal{B})$).

NOTE: ***Different*** F's can represent the ***same*** mapping.
Equivalent mappings are those where $F \, (\mathcal{A} \to \mathcal{B}) \, G$ holds.

DEFINITION: Two PERS $\mathcal{A}$ and $\mathcal{B}$ are said to be ***isomorphic*** (in symbols, $\mathcal{A} \cong \mathcal{B}$) provided there are mappings

$$F\colon \mathcal{A} \longrightarrow \mathcal{B} \text{ and } G\colon \mathcal{B} \longrightarrow \mathcal{A} \text{ where}$$

$$\forall \, X \in \textbf{Rng}(\mathcal{A}).\ X \, \mathcal{A} \, G(\!(F(\!(X)\!))\!) \ \& \ \forall \, Y \in \textbf{Rng}(\mathcal{B}).\ Y \, \mathcal{B} \, F(\!(G(\!(Y)\!))\!).$$

EXERCISE: Prove that for $\mathcal{A},\mathcal{B},\mathcal{C} \in \textbf{PER}(U)$, we have:

(i) $\mathcal{A} \cong \mathcal{A}$;

(ii) $\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{B} \cong \mathcal{A}$; and

(iii) $\mathcal{A} \cong \mathcal{B} \ \& \ \mathcal{B} \cong \mathcal{C} \Rightarrow \mathcal{A} \cong \mathcal{C}$.

EXERCISE: (i) Prove that for $\mathcal{A},\mathcal{B} \in \textbf{PER}(U)$, we have:

$$\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A} \text{ and } \mathcal{A} + \mathcal{B} \cong \mathcal{B} + \mathcal{A} .$$

(ii) Is it possible there are also associative and distributive laws for + and × on PERs?

PROJECT: Give examples of several ***non-isomorphic*** $\mathcal{D} \in \mathbf{PER}(U)$ where $\mathcal{D} \cong \mathcal{D} + \mathcal{D}$.

EXERCISE: Prove that for $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathbf{PER}(U)$, we have:

(i) $((\mathcal{A} \times \mathcal{B}) \to \mathcal{C}) \cong (\mathcal{A} \to (\mathcal{B} \to \mathcal{C}))$; and

(ii) $(\mathcal{A} \to (\mathcal{B} \times \mathcal{C})) \cong (\mathcal{A} \to \mathcal{B}) \times (\mathcal{A} \to \mathcal{C})$; and

(iii) $((\mathcal{A} + \mathcal{B}) \to \mathcal{C}) \cong (\mathcal{A} \to \mathcal{C}) \times (\mathcal{B} \to \mathcal{C})$;

DEFINITION: Let $\mathcal{A} \in \mathbf{PER}(U)$. An ***$\mathcal{A}$-indexed family*** of PERs is a function $\mathcal{B}\colon \mathbf{PER}(U) \longrightarrow \mathbf{PER}(U)$ where we have

$$\forall\, X_0, X_1\, [\, X_0\, \mathcal{A}\, X_1 \Rightarrow \mathcal{B}(X_0) = \mathcal{B}(X_1)\,].$$

Note that the values of $\mathcal{B}$ outside $\mathbf{Rng}(\mathcal{A})$ are ***irrelevant.***

DEFINITION: The ***dependent product*** of an ***$\mathcal{A}$-indexed family*** of PERs is defined as a relation on $\mathbf{Dom}(U)$ by:

$$F_0\, (\prod X{:}\mathcal{A}.\ \mathcal{B}(X))\, F_1 \Leftrightarrow \forall\, X_0, X_1\, [\, X_0\, \mathcal{A}\, X_1 \Rightarrow F_0(\!(X_0)\!)\, \mathcal{B}(X_0)\, F_1(\!(X_1)\!)\,].$$

EXERCISE: **(i)** Prove that if $\mathcal{B}$ is an ***$\mathcal{A}$-indexed family*** of PERs, then $\prod X{:}\mathcal{A}.\ \mathcal{B}(X) \in \mathbf{PER}(U)$. (ii) If $\mathcal{B}$ is a ***constant*** family, how does this relate to the notion of function space defined earlier?

DEFINITION: The ***dependent sum*** of an ***$\mathcal{A}$-indexed family*** of PERs is defined as a relation on $\mathbf{Dom}(U)$ by:

$$P_0\, (\sum X{:}\mathcal{A}.\ \mathcal{B}(X))\, P_1 \Leftrightarrow \exists\, X_0, Y_0, X_1, Y_1\, [\, P_0 = \langle\!\langle X_0, Y_0 \rangle\!\rangle \,\&\, P_1 = \langle\!\langle X_1, Y_1 \rangle\!\rangle \,\&\, X_0\, \mathcal{A}\, X_1 \,\&\, Y_0\, \mathcal{B}(X_0)\, Y_1\,]$$

EXERCISE: **(i)** Prove that if $\mathcal{B}$ is an ***$\mathcal{A}$-indexed family*** of PERs, then $\sum X{:}\mathcal{A}.\ \mathcal{B}(X) \in \mathbf{PER}(U)$. (ii) If $\mathcal{B}$ is a ***constant*** family, how does this relate to the notion of ***binary sum*** and ***binary product*** (+ and ×) defined earlier?

DISCUSSION: We have already seen that the constructs $\mathcal{A}\times\mathcal{B}$, $\mathcal{A}+\mathcal{B}$, and $\mathcal{A}\rightarrow\mathcal{B}$ have many properties, say under isomorphism. It is not surprising, then, that the more general $\sum$ and $\prod$ constructs have even more properties. A full investigation of such properties would bring us to some fairly advanced notions of ***Category Theory***. A different direction of thinking brings us to ***Per Martin-Löf's Theory of Types***. More precisely, PERs ***model*** Martin-Löf's Theory.

To understand the connections, we need to look at ***iterations*** of $\sum$ and $\prod$. Let us take an example. We could say that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, form a ***system*** of dependent types if:

$$\forall\, X_0,X_1\; [\; X_0\,\mathcal{A}\,X_1 \Rightarrow \mathcal{B}(X_0)=\mathcal{B}(X_1)\;] \text{ and}$$
$$\forall\, X_0,X_1,Y_0,Y_1\; [\; X_0\,\mathcal{A}\,X_1 \;\&\; Y_0\,\mathcal{B}(X_0)\,Y_1 \Rightarrow \mathcal{C}(X_0,Y_0)=\mathcal{C}(X_1,Y_1)\;] \text{ and}$$
$$\forall\, X_0,X_1,Y_0,Y_1,Z_0,Z_1\; [\; X_0\,\mathcal{A}\,X_1 \;\&\; Y_0\,\mathcal{B}(X_0)\,Y_1 \;\&\; Z_0\,\mathcal{C}(X_0,Y_0)\,Z_1 \Rightarrow \mathcal{D}(X_0,Y_0,Z_0)=\mathcal{D}(X_1,Y_1,Z_1)\;],$$

$\mathcal{A}\in$ **PER**(U), and $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are mappings on **PER**(U) to **PER**(U) with the indicated number of arguments.

EXERCISE: Under the above assumptions on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, prove that the following compound is indeed in **PER**(U):

$$\prod X{:}\mathcal{A}.\sum Y{:}\mathcal{B}(X).\prod Z{:}\mathcal{C}(X,Y).\,\mathcal{D}(X,Y,Z).$$

DEFINITION: The ***identity type*** over $\mathcal{A}\in$ **PER**(U) is defined as:

$$Z\,\lbrace X \equiv_{\mathcal{A}} Y\rbrace\,W \Leftrightarrow Z\,\mathcal{A}\,X\,\mathcal{A}\,Y\,\mathcal{A}\,W$$

EXERCISE: Show that if $\mathcal{A}\in$ **PER**(U) and $X,Y\in$ **Dom**(U), then $\lbrace X\equiv_{\mathcal{A}} Y\rbrace\in$ **PER**(U).

DISCUSSION: One application of these PER types is to represent ***constructive reasoning***. The idea goes back to the so-called ***Curry-Howard Correspondence***. Showing that a complex PER is ***non-empty*** is interpretable as a ***logical statement***. For example, if we know that $F\,(\mathcal{A}\rightarrow(\mathcal{A}\rightarrow\mathcal{A}))\,F$, we can think of F as a ***binary operation*** on **Rng**($\mathcal{A}$). Then if we can show that this PER is ***non-empty***:

$$\prod X{:}\mathcal{A}.\prod Y{:}\mathcal{A}.\prod Z{:}\mathcal{A}.\lceil F\langle\!\langle X\rangle\!\rangle\langle\!\langle F\langle\!\langle Y\rangle\!\rangle\langle\!\langle Z\rangle\!\rangle\rangle\!\rangle \equiv_{\mathcal{A}} F\langle\!\langle F\langle\!\langle X\rangle\!\rangle\langle\!\langle Y\rangle\!\rangle\rangle\!\rangle\langle\!\langle Z\rangle\!\rangle\rceil,$$

we then have "evidence" that – up to the equivalence $\mathcal{A}$ – the operation F is ***associative***. Of course, if F just ***is*** associative, then the above PER is non-empty, but finding an element to show this might be difficult. Compounds which have ***recursive*** evidence are especially representative of constructively verifiable facts. And there are many such.

EXERCISE: Prove that if $\mathcal{A} \in$ **PER**(U) and $\mathcal{B}$ is an $\mathcal{A}$-indexed family of PERs of two arguments, then these PERS contain recursive elements:

(i) $(\prod X{:}\mathcal{A}.\prod Y{:}\mathcal{A}.\ \mathcal{B}(X,Y) \to \prod Y{:}\mathcal{A}.\prod X{:}\mathcal{A}.\ \mathcal{B}(X,Y))$

(ii) $(\prod F{:}(\mathcal{A} \to \mathcal{A}).\prod X{:}\mathcal{A}.\prod Y{:}\mathcal{A}.(\lceil X \equiv_{\mathcal{A}} Y\rceil \to \lceil F\langle\!\langle X\rangle\!\rangle \equiv_{\mathcal{A}} F\langle\!\langle Y\rangle\!\rangle\rceil))$

(iii) $(\prod F{:}(\mathcal{A} \to \mathcal{A}).\prod X{:}\mathcal{A}.\sum Y{:}\mathcal{A}.\lceil F\langle\!\langle X\rangle\!\rangle \equiv_{\mathcal{A}} Y\rceil)$

NOTE: In giving an informal reading to these compounds, read $\prod$ as a ***universal quantifier*** and $\sum$ as an ***existential quantifier***. Read the $\to$ as ***if ... then***. How should + and × be read?

PROJECT: Look up details on the (extensional) ***Martin-Löf Theory*** and check out further properties of this PER-based modeling.

CHAPTER X. ALTERNATIVE LAMBDA MODELS

As we have shown every $S \in$ **Sub**(U) gives a ***subdomain*** **Dom**(S) $\subseteq$ **Dom**(U), which can also be regarded as a PER.

EXERCISE: Prove that if $S \in$ **Sub**(U) then we have:

$$\{\langle\!\langle X,X\rangle\!\rangle \mid X \in \mathbf{Dom}(S)\} \cong \{\langle\!\langle X,Y\rangle\!\rangle \mid X,Y \in \mathbf{Dom}(U)\ \&\ X \cap S = Y \cap S\}$$

NOTE: One of these is a PER and the other is total. But we have to regard isomorphic PERs as giving is the ***same type***.

One key difference between subdomains and PERs is that the construct ($\mathbf{Dom}(S) \to \mathbf{Dom}(T)$) comes from a ***continuous*** operation ($S \to T$) on subsemilattices. For PERs, ($\mathcal{A} \to \mathcal{B}$) fails to have any simple ***monotone*** property in the variable $\mathcal{A}$. For the subsemilattices, we can take advantage of the Fixed-Point Theorem. So we now do this to give some steps to obtain a more "tidy" $\boldsymbol{\lambda}$-calculus model. In the following we will write $S \cong T$ to mean $\mathbf{Dom}(S) \cong \mathbf{Dom}(T)$.

EXERCISE: Let $V \in \mathbf{Sub}(U)$ satisfy $V = V \times (U \times V)$. Let $C = (U \times V)$. Prove that $C \cong (C \times C)$.

EXERCISE: Let $D = (D \to C)$. Prove that $D \cong (D \times D)$.

EXERCISE: Prove that $D \cong (D \to D)$.

EXERCISE: Prove that $\mathbf{Dom}(U)$ is isomorphic to a subdomain of $\mathbf{Dom}(D)$.

As a $\boldsymbol{\lambda}$-calculus model, the application and abstraction constructs on $\mathbf{Dom}(D)$ can be redefined so that we have:

$$F = \boldsymbol{\lambda} X.\, F⟪X⟫.$$

Whether that is important enough to go to the trouble (of tracing through several isomorphism) is a question. It would make $\mathbf{Dom}(D)$ a semigroup with identity under the operation $F \circ G = \boldsymbol{\lambda} X.\, F⟪G⟪X⟫⟫$ we studied earlier. And that is nice.

The other domain equation would mean we can redefine ordered pairs so that all elements of $\mathbf{Dom}(D)$ are uniquely pairs in such a way all possible pairs are so obtained. With more work we could assure that $\langle\!\langle F,G\rangle\!\rangle = \lambda X. \langle\!\langle F(\!(X)\!),G(\!(X)\!)\rangle\!\rangle$. That's a very "cute" identification, but in the longer run it probably does not save us essential work.

EXERCISE: As a ***temporary notation*** write: $S \oplus T = (S + T) \cap H$. Prove that for $S,T,R \in \mathbf{Dom}(U)$:

(i) $S \oplus \{0,1\} \cong S$; and

(ii) $S \oplus T \cong T \oplus S$; and

(iii) $(R \oplus S) \oplus T \cong R \oplus (S \oplus T)$; and

(iv) $\mathbf{Sub}(U)$ has many non-isomorphic $B \cong B \oplus B$; and

(v) $U \cong U^* \oplus ((U \times U) \oplus (U \to U))$.

A different alternative to the "universal" domain $\mathbf{Dom}(U)$ is to find a super-semilattice L which has ***all*** semilattices as subsemilattices (up to isomorphism, of course). Our especially constructed U has only certain kinds of subsemilattices – though the number and complexity is high owing to the use of the Fixed-Point Theorem to gives us recursive domain equations.

It turns out we do not have to search far for L, as it is a very well known semilattice. The L we need is just the ***Boolean algebra of propositional formulae*** in a denumerable number of variables. It is also called the ***free Boolean algebra*** on a denumerable number of generators. For the purposes of semilattices we use only a portion of the structure: 0 = False, 1 = True, and $\vee$ = Or.

There are many connections between semilattices and Boolean algebras. First, we see that any semilattice is a subsemilattice of a Boolean algebra.

EXERCISE: Let S be a semilattice and let $P = \mathbf{Pow}(S \backslash \{1\})$, regarded as a Boolean algebra in the normal way. Consider the function $\rho : S \longrightarrow P$ defined by:

$\rho(p) = \{ q \in S \mid p \not\leq q \}$.

(i) Show that ρ is a one-one mapping of S into P; and

(ii) $\rho(1) = S \backslash \{1\}$; and

(iii) $\rho(0) = \varnothing$; and

(iv) $\rho(p \vee q) = \rho(p) \cup \rho(q)$.

If S is a denumerable semilattice, then the Boolean algebra P, being a powerset, is ***uncountable***. That is much too large. But we only need a subalgebra of P.

EXERCISE: If S is a recursive semilattice, let Q be the subalgebra of the Boolean algebra $P = \mathbf{Pow}(S \backslash \{1\})$ generated by the ***range*** $\rho(S)$ of the function $\rho: S \longrightarrow P$ defined in the last exercise. Show that Q — as a Boolean algebra — can be given a ***recursive structure***. (And, of course, S embedds isomorphically into Q as a semilattice and $\rho: S \longrightarrow Q$ becomes a recursive function.)

THE EMBEDDING THEOREM: Every countable Boolean algebra can be ***isomorphically embedded*** as a Boolean subalgebra into L; and, moreover, if the Boolean algebra has a ***recursive structure***, then the embedding can also be made ***recursive***.

PROOF SKETCH: First, recall that every finitely generated Boolean algebra is ***finite***. This follow from, for example, the use of ***disjunctive normal form***, inasmuch as with finitely many generators there are only finitely many forms to write out. And, if the given Boolean algebra has a recursive structure, then the listing of all the possible elements is recursive.

If A_0 is a finite subalgebra of a given Boolean algebra A, then we must keep in mind that A_0 is ***atomistic***. An ***atom*** is a ***minimal, non-zero element***, and an easy argument shows that all the elements of A_0 are least upper bounds of the atoms they individually contain.

Next, if $b \in A$ is a new element we want to ***adjoin*** to A_0 forming a larger finite subalgebra A_1, then we remark that b meets each of the atoms $a \in A_0$ so that

$$a \wedge b = 0 \text{ or } a \wedge b = a \text{ or } 0 < a \wedge b < a.$$

In the second case, $a \wedge b$ is the atom a. In the third case, b creates a possible pair of atoms for A_1: $a \wedge b$ and $a \wedge \neg b$, a splitting of a in A_0. If we add together the expanded set of new atoms, then the subalgebra A_1 generated by them contains A as well as the new element b. (And the procedure is recursive, if that is required.)

Remembering that the given algebra A is ***countable***, we may then form a ***tower*** of finite subalgebras:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq A_{(n+1)} \subseteq \ldots,$$

so that $A = \bigcup \{ A_n \mid n \in N \}$. Again, if A has a recursive structure, the construction of the tower is also recursive.

Turning attention now to L, we note first that L is ***atomless***. For, suppose $0 < p \in L$. Let $v \in L$ be a propositional variable ***not*** in p. (A formula for an element of L can contain only a finite number of symbols.) We see, therefore, that we have $0 < p \wedge v < p$. This means p is ***not*** and atom of L.

Now, suppose A_n has been (recursively) embedded into L. The passage from A_n to $A_{(n+1)}$ can be thought of as a process of ***dividing*** the atoms of A_n into several pieces. The isomorphic image of A_n in L, being finite, gives us the opportunity of choosing new elements of L to subdivide the images of the atoms into the ***same*** pattern of pieces. The resulting subalgebra of L generated by this subdivision is then isomorphic to $A_{(n+1)}$. In other words, the (recursive) embedding of A_n into L can be ***extended*** to a (recursive) embedding of $A_{(n+1)}$ into L. Continuing these expansions of the embeddings will give us a (recursive) embedding of the whole of A into L. **QED**

COROLLARY: Every countable semilattice can be ***isomorphically embedded*** as a subsemilattice into L; and, moreover, if the semilattice has a ***recursive structure***, then the embedding can also be made ***recursive***.

PROOF: Take a countable semilattice S. Embed it into a countable Boolean algebra A (as checked earlier). Keep things recursive as needed. Now embed A into L. **QED**

DISCUSSION: This means that the domain **Dom**(L) is likely to be much richer than **Dom**(U), because U can be taken as a subsemilattice of L. (**PROBLEM:** *It is just possible that L is isomorphic to a subsemilattice of U, but the author has not been able to settle this question.*)

Now, by analogy to what we proved for U (by construction), the domain (**Dom**(L) $\to$ **Dom**(L)) can be shown to be a domain of the form **Dom**(M) for a countable semilattice. Embedding this into L gives us a subsemilattice $(L \to L) \subseteq L$, much as we had $(U \to U) \subseteq U$. We can then make **Dom**(L) into a λ-calculus model in the same way we did for U.

By comparison, however, the construction of U is much more direct, since for L we have to invoke the Embedding Theorem which says ***there is some embedding*** without really showing it explicitly.

But passing to higher types, we can show, for example that ***every countably based topological space*** is represented by a PER over **Dom**(L). Moreover, products and function spaces correspond to topological products of domains and spaces of topologically continuous functions with a fairly easy to work with topology on the function space. (Further higher types with iterated function spaces are harder to analyze, however.)

SOME REFERENCES

The literature on Domain Theory has become vast, but pointers to historical background, textbooks, and much research can be found in these references.

(1) There are many other constructions of λ-calculus models. For an extensive review containing many references see: G.D. Plotkin, *Set-theoretical and other elementary models of the λ-calculus*, ***Theoretical Computer Science***, vol. 121 (1993), pp. 351-409.

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http://www.di.ens.fr/users/longo/download.html)

(3) A long and detailed survey with many references can be found in: S. Abramsky and A. Jung, *Domain Theory*. In: ***Handbook of Logic in Computer Science***, volume 3, edited by S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, Oxford University Press, 1994, pp. 1–168. (Revised version for download at:
http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf)

(4) Very readable reports on the life and work of Strachey can be found in: O. Danvy and C. Talcott (eds.) *A Special Issue Dedicated to Christopher Strachey*. ***Higer-Order and Symbolic Computation***, vol. 13, Nos. 1/2 (2000), 152 pp.

(5) An advanced textbook covering a wide range of topics and applications is: R.M. Amadio and P.-L. Curien, ***Domains and Lambda-Calculi*** (Cambridge Tracts in Theoretical Computer Science, No. 46), Cambridge University Press, Paperback edition, 2008, xiv + 484 pp. (First edition 1998.)

(6) A large-scale text on the lattice-theoretical and topological development of Domain Theory, together with a long bibliography, can be found in: G. Gierz, K.H. Hofmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott, ***Continuous Lattices and Domains***. Cambridge University Press, 2003, xxxvi + 591 pp.