Datum / Date: 6/19/15

Harper

Identification Type:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{refl}_A(M) : M =_A M}\,(I)$$

$$\frac{\Gamma, x:A, y:A, z: x =_A y \vdash C_{x,y,z} : U \qquad \begin{array}{l}\Gamma, x:A \vdash c : [x,x,\mathrm{refl}(x)/x,y,z]C \\ \Gamma \vdash \alpha : M =_A N\end{array}}{\Gamma \vdash J[x.y.z.C](x.c)(\alpha) : [M,N,\alpha/x,y,z]C}\ \text{(path induction)}$$

Special case of path induction:

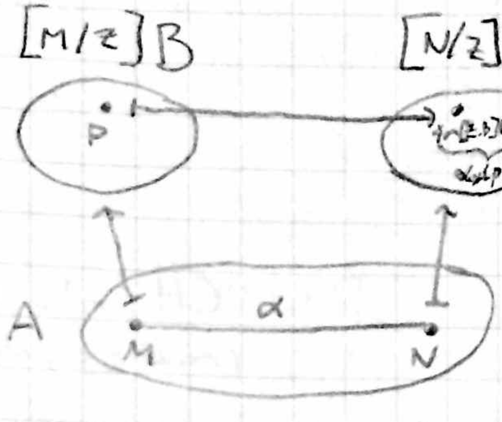
$$\frac{\begin{array}{l}\Gamma, x:A, y:A \vdash C_{x,y} : U \\ \Gamma, x:A \vdash c : [x,x/x,y]C \\ \Gamma \vdash \alpha : M =_A N\end{array}}{\Gamma \vdash J'[x,y.C](x.c) : [M,N/x,y]C}$$

Computation Rules:

$$J[x,y,z,C](x.c)(\mathrm{refl}_A(M)) \equiv [M/x]c$$
$$J'[x,y.C](x.c)(\mathrm{refl}_A(M)) \equiv [M/x]c$$

Transport:

$$\frac{\begin{array}{l}\Gamma, z:A \vdash B : U \\ \Gamma \vdash \alpha : M =_A N\end{array}}{\Gamma \vdash tr[z.B](\alpha) : [M/z]B \to [N/z]B} \qquad tr[z.B](\mathrm{refl}_A(M)) \equiv id_{[M/z]B}$$



Path Over (or Correlation): suppose $z:A \vdash B:U$ $\alpha : M =_A N$

$$P =^{z.B}_{\alpha} Q \triangleq \alpha_*(P) =_{[N/z]B} Q$$

Identification is an Equivalence Relation: prove using path induction

Functionality (and Functoriality):

$$f : A \to B, x:A, y:A, p : x =_A y \vdash ap_f(p) : f(x) =_B f(y)$$
$$ap_f(p) \triangleq J'[u,v.\ f(u) =_B f(v)](u.\mathrm{refl}_B(f(u)))(p)$$
$$f : \Pi(x:A).B, x:A, y:A, p : x =_A y \vdash apd_f(p) : f(x) =^{z.B}_{p} f(y)$$

Exercise: define $apd_f(p)$

Harper (cont.)

Identification as an ∞-groupoid:

Groupoid Laws (can be proven) // Kan Condition (in math)

$\mathrm{unit}_L : \mathrm{refl}_A(M) \cdot \alpha =_{M =_A N} \alpha$

$\mathrm{unit}_R : \alpha =_{M =_A N} \alpha \cdot \mathrm{refl}_A(N)$

$\mathrm{inv}_L : \alpha^{-1} \cdot \alpha =_{N =_A N} id$

$\mathrm{inv}_R : \alpha \cdot \alpha^{-1} =_{M =_A M} id$

$\mathrm{assoc} : \alpha \cdot (\beta \cdot \gamma) =_{M =_A P} (\alpha \cdot \beta) \cdot \gamma$ Exercize: prove the groupoid laws

Structure

	(globular)	(cubical)	(simplicial)	
0	points	point	points	
1	lines	lines	lines	equality
2	disk	squares	triangles	equality of equality
3	sphere	cubes	tetrahedra	equality of equality of...
⋮				

$\mathrm{Id}_{A \times B}(M, N) \simeq \mathrm{Id}_A(\mathrm{fst}\, M, \mathrm{fst}\, N) \times \mathrm{Id}_B(\mathrm{snd}\, M, \mathrm{snd}\, N)$

etc... (derive equivalences as exercize)

$\mathrm{Id}_{A \to B}(f, g) \underset{\text{funext}}{\overset{\text{happly}}{\rightleftarrows}} \prod_{x:A} f(x) =_B g(x) \simeq \prod_{x,y:A} x =_A y \to f(x) =_B g(y)$

funext DOES NOT EXIST!

Univalence Axiom: $\mathrm{Id}_U(A, B) \underset{\text{axiom}}{\overset{\text{provable}}{\rightleftarrows}} A \simeq B$, or $(A =_U B) \simeq (A \simeq B)$

Higher Inductive Types:

"The interval is a proof relevant quotient"

The Interval $\quad 0 \xrightarrow{seg} 1$, $\mathbb{I}$

- can map into types, identifying equalities

$\mathbb{I} : U$ where

$0 : \mathbb{I}$

$1 : \mathbb{I}$

$seg : 0 =_{\mathbb{I}} 1$

$$\frac{\Gamma, z:\mathbb{I} \vdash C_z : U \qquad \Gamma \vdash c_0 : [0/z]C \quad \Gamma \vdash c_1 : [1/z]C \qquad \Gamma \vdash s : c_0 =^{z.C}_{seg} c_1}{\Gamma, z:\mathbb{I} \vdash \mathrm{rec}_{\mathbb{I}}[z.C](c_0, c_1, s)(z) : C}$$

DTT + Id + $\mathbb{I}$ yields FunExt!

- $\mathbb{I} \to A \simeq (\Sigma x,y{:}A.\ x =_A y)$
- $\mathbb{I} \to (A \to B) \simeq (\mathbb{I} \times A) \to B$
  $\simeq (A \times \mathbb{I}) \to B$
  $\simeq A \to (\mathbb{I} \to B)$

Problem: Computational Interpretation of Higher Type Theory?

$J[\quad](x.c)(\mathrm{refl}_A(M)) \equiv [M/x]c$

$J[\quad](x.c)(seq) \equiv ?$

$J[\quad](x.c)(\mathrm{ua}(f)) \equiv ?$