Proof Theory - Lecture 4
Cut Elim; Linear Logic Part 1

Questions from the homework:

Q1.)

\[
\begin{align*}
\Gamma, A & \vdash \Delta \\
\Gamma, B & \vdash \Delta \\
\vdash & \vdash \\
A \lor B & \vdash \Delta, \text{x, y, \ldots}
\end{align*}
\]

\[\text{CD}\]

\[\Gamma, A \lor B \Rightarrow \Delta \quad \Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta \]

Means... A sequent for this,
so, there is no rule in sequent calculus
(this rule is bad), so we'd have to
 generalize the sequent calculus.

Q2.) Notation, \(\Rightarrow\), why not include in sequent:

\[\Gamma \vdash A\] in Natural Deduction;
\[\Gamma \Rightarrow A\] in sequent calculus.

\(\ldots\) be "convention" and
\(\ldots\) to separate it from ND.

Q3.)

\[\begin{align*}
\text{A \circ t+1} & \Rightarrow \text{O}\circ t \\
\Rightarrow & \text{O}\circ t
\end{align*}\]

Similarly,

\[\begin{align*}
\text{A \circ t+1} & \Rightarrow \text{O}\circ t \\
\Rightarrow & \text{O}\circ t
\end{align*}\]

Useful for:

\(\checkmark\) limiting (and counting)
the number of proof's
of any given theorem,
proof searches. (Elaboration)

Chain - Prove - Arrow.

(\(\uparrow\downarrow\uparrow\downarrow\))

Q) the arrows point in the
direction of the
"smaller terms"

\(\uparrow\): want to show, "to prove..."
\(\downarrow\): can assume,
are given, can use,
"if we know this..."
\(\uparrow\downarrow\): if working through a proof
else (i.e., if done proving sth)
\(\uparrow\downarrow\): represents our belief that
we have a proof.
\[ A_1 \ldots A_n \]

\[ \vdash \]

\[ C \]

\[ \Gamma \vdash A \quad (\text{in } ND) \]

\[ \Gamma \Rightarrow A \quad (\text{in } SEQ) \]

**Theorem (version 1)**

(i) if \( \Gamma \vdash A \uparrow \),

\[ \text{then } \Gamma \Rightarrow A. \]

**Case**

\[ \Delta = \Gamma \vdash A \uparrow, \Gamma \vdash A \downarrow \]

\[ \frac{\Gamma \vdash A, \Gamma \vdash A \downarrow \downarrow \Rightarrow }{\Gamma \vdash A, A \downarrow \downarrow} \]

\[ \Gamma \Rightarrow A \quad \text{by Ind. Hypothesis } \Delta_1 \]

\[ \Gamma \Rightarrow A_1 \quad \text{by Ind. Hypothesis } \Delta_2 \]

\[ \Gamma \Rightarrow A, A \downarrow \downarrow \quad \text{by } \land \text{R} \]

Can

\[ \Delta' = \Gamma \vdash A \uparrow \]

\[ \frac{\Gamma \vdash A \uparrow \downarrow \downarrow }{\Gamma \vdash A \downarrow \downarrow} \]

**Theorem (revised)**

(i) if \( \Gamma \vdash A \uparrow \), then \( \Gamma \Rightarrow A \).

(ii) if \( \Gamma \vdash A \downarrow \downarrow \) and \( \Gamma, A \Rightarrow C \)

\[ \text{then } \Gamma \Rightarrow C. \]
\[ \Gamma, A \vdash C \quad \text{by Assumption} \]

\[ \Gamma, A \vdash B, A \vdash C \quad \text{by Weakening (Lemma)} \]

\[ \Gamma, A \vdash B, C \quad \text{by}\ \mathcal{N}^1 \]

\[ \Gamma \vdash C \quad \text{by Ind, Hyp, and } \mathcal{D} \]

Case

\[ \mathcal{D} = \underbrace{\Gamma, A \downarrow + A \downarrow}_{\text{in } \mathcal{N}^P} \]

\[ \Gamma, A, A \vdash C \quad \text{by Assumption} \]

\[ \text{Proof!} \]

\[ \text{Proof: by Induction} \]

\[ \Gamma, A \vdash C \quad \text{by contraction (Lemma)} \]

Now, back to the program... 

\[ A \quad \langle A \uparrow \rangle \]

\[ A \vdash B \uparrow \quad A \uparrow \quad \times \]

hypothesized, consider the "rule":

\[ A \uparrow \quad \frac{\text{which we won't actually do}}{A \uparrow \downarrow \quad \times, \text{it would render our arrows meaningless}} \]
Case 1:
\[ D = \frac{\Gamma A \vdash A}{\Gamma A} \]

\[ \Gamma, A \vdash C \quad \text{assumption} \]

\[ \Gamma, A \vdash A, A \vdash C \quad \text{Weakening (Lemma)} \]

\[ \Gamma, A \vdash C \quad \text{by } \Lambda \]

\[ \Gamma \vdash C \quad \text{by Ind. Hyp. on } D \]

Case 2:
\[ D = \frac{\Gamma \vdash A}{\Gamma} \]

\[ \Gamma, A \vdash C \quad \text{assumption, trim (ii) (justified)} \]

\[ \Gamma \vdash A \quad \text{by I.H. on } D \]

To show: \( \Gamma \vdash C \)

"Hauptsatz"\

\[ \Gamma \vdash A \quad \Gamma A \vdash C \quad \text{cut} \]

\[ \Gamma \vdash C \]
Thm (Admissibility of cut):

If \( \Gamma \vdash A \)

and \( \Gamma, A \vdash E \)

then \( \Gamma \vdash C \)

**Case**

\[
\Gamma' = \begin{array}{c}
\Gamma \\
\Gamma' \vdash A_1, A_2
\end{array}
\]

\( \Gamma' \vdash A_1 \land A_2 \)

\( E_1 \)

\[
\Gamma', A_1, \land A_2 \vdash C \land \Gamma_1 = E
\]

\( \Gamma, A_1 \land A_2 \vdash C \)

\( \Gamma' \vdash C \) by IH on \( A_1, A_2, \Gamma' \land E_1 \)

\( \Gamma' \vdash C \) by IH on \( A_1, A_2, \Gamma_1 \land E_1 \)

**To show:** \( \Gamma' \vdash C \)

\[\Gamma, x : A_1, y : A_2, M : C \]

\[\Gamma'', x : A_1, y : A_2 \vdash \text{let } y = \pi_1 x \text{ in } M : C \]
Linear Logic
GIRARD '87
\[ \Delta \]
\[ \{ A_1, \ldots, A_n \} + C \]
resources
must use each exactly once
in a proof

Unary "operator"
\[ - \]
\[ \circ \]
linear logic
traditional logic
\[ A \Rightarrow B \rightarrow A \lor B \]
\[ \neg (A \Rightarrow B) \land \neg (A \Rightarrow C) \]

\( \delta = " \text{tensor}" \)

\[ \Delta, A + B \]
\[ \Delta + A \rightarrow B \]
\[ \Delta + A \rightarrow C \]

\[ \Delta + A \rightarrow \Delta' \]
\[ \Delta' + C \]

\[ \delta \rightarrow \text{linear logic takes\ source}\]
\[ \neg \text{ weakening or contraction} \]
\[ \neg \text{ source} \]

\( \neg \text{ the i.o.r.m} \)
\( \neg \text{ i.o.r.m only once} \)